Lecture 7

Prize Collecting Steiner Forest Problem

Oct 23, 2004
Lecturer: Kamal Jain
Notes: Neva Cherniavsky

7.1 Problem description

$G = (V, E)$, $C : E \rightarrow \mathbb{R}^+$ cost function.
$\pi : V \times V \rightarrow \mathbb{R}^+$
$\pi(i, j) = \pi(j, i)$
$\pi(i, i) = 0$
$H \subseteq G$
If you don't have a path between i and j, then you pay a penalty of $\pi(i, j)$. $\sum_{e \in H} C_e + \sum_{i,j} \pi_{i,j}$ where i, j are in different components.

Last time:

$$\min \sum_{e \in H} c_e x_e + \sum_{i,j} A_{i,j} z_{i,j}$$

We had $x_e \in \{0, 1\}$, which we relaxed to $x_e \geq 0$, and $z_{ij} \in \{0, 1\}$, which we relaxed to $z_{ij} \geq 0$.

Definition 7.1. $S \odot ij$ means S is a set that separates i and j. That is, $i \in S, j \notin S$, or $i \notin S, j \in S$.

The equation $z_{ij} + \sum_{e \in \delta(S)} x_e \geq 1$ means either we pay the penalty z_{ij} or a path exists. The dual variable corresponds to $Y_{S,ij}$ where S separates i and j.

7.2 Dual

$$\max \sum_{S,ij} Y_{S,ij}$$

$$\begin{aligned} \forall e \sum_{S: e \in \delta(S), ij: S \odot ij} Y_{S,ij} &\leq c_e \\ \forall ij \sum_{S: S \odot ij} Y_{S,ij} &\leq \pi_{ij} \\ Y_{S,ij} &\geq 0 \end{aligned}$$

Note that there are lots of duals. For dual $Y_{S,ij}$, who should be paying for this dual? If we make the wrong one pay, we may not get a factor.

7.2.1 Example

Using Goemans-Williams, if we have three sets, with i, j, and k each in their own set, and $\pi_{ij} = 10$, $\pi_{ik} = 100$, and everyone else is 0.

At some point the penalty for ij will go down. At time 6.67, j and k are 6.67, and i gets 3.3 each. But now we can't raise j's dual anymore. Because of this we can't prove any factor. We will not attack this problem head on, though that's an interesting thing to do. Instead we'll avoid it.

7.3 Goemans-Williamson form

$$\max \sum_S Y_S$$
$$\begin{aligned}
\forall S \;\; Y_S &= \sum_{S \odot ij} Y_{S,ij} \\
\forall e \sum_{S: e \in \delta(S)} Y_S &\leq c_e \\
\forall ij \sum_{S: S \odot ij} Y_{S,ij} &\leq \pi_{ij} \\
Y_S &\geq 0 \\
Y_{S,ij} &\geq 0
\end{aligned}$$

Run the same algorithm, i.e., increase Y_S's until we can find some feasible $Y_{S,ij}$'s.

7.3.1 Equality of linear programs

Lemma 7.1. *These two programs are the same.*

Proof. Take a feasible solution in one and it is feasible in the other. □

The only problem arises with the equality constraint, because Farka's lemma will make it negative. So change it to $\forall S \;\; Y_S \leq \sum_{S \odot ij} Y_{S,ij}$ and add $Y_S \geq 0$.

Lemma 7.2. *The programs are still the same with this modified constraint.*

Proof. One direction is still the same. Take $\sum_{S,ij} Y_{S,ij} = Y_S$ and then $\max \sum_S Y_S$ works.

Going the other way, decrease $Y_{S,ij}$'s until $Y_S = \sum_{S \odot ij} Y_{S,ij}$. Then the $Y_{S,ij}$'s are a feasible solution for the other dual. □

Call this new dual Dual1.

7.4 Dual1

$$\begin{aligned} \max \sum_S Y_S & & \\ \forall S \;\; Y_S & \leq & \sum_{S \odot ij} Y_{S,ij} \\ \forall e \sum_{S: e \in \delta(S)} Y_S & \leq & c_e \\ \forall ij \sum_{S: S \odot ij} Y_{S,ij} & \leq & \pi_{ij} \\ Y_S & \geq & 0 \\ Y_{S,ij} & \geq & 0 \end{aligned}$$

$\text{OPT}_I \geq \text{OPT}_{LP} \geq \text{DUAL}_{LP}$

Now we use Farka's lemma to prove feasibility, because Farka's will give proof of infeasibility if the program is infeasible, and if no such proof exists we know the LP is feasible.

We want to avoid the problem of the $Y_{S,ij}$'s. To do so, we note that the algorithm gives the Y_S's, so constraints become constants:

$$\begin{aligned} \forall S : \sum_{S \odot ij} Y_{S,ij} & \geq & Y_S \\ \forall ij : -\sum_{S: S \odot ij} Y_{S,ij} & \geq & -\pi_{ij} \\ \forall S, ij : \;\; Y_{S,ij} & \geq & 0 \end{aligned}$$

If the program is infeasible, there exist coefficients $\alpha_S \geq 0$, $\alpha_{ij} \geq 0$, and $\beta_{S,ij} \geq 0$ such that the corresponding coefficients of the $Y_{S,ij}$'s are 0

$$\alpha_S - \alpha_{ij} + \beta_{S,ij} = 0$$

Summing the equations:

$$\alpha_S \sum_{S \odot ij} Y_{S,ij} - \alpha_{ij} \sum_{S: S \odot ij} Y_{S,ij} + \beta_{S,ij} Y_{S,ij} \geq \sum_S \alpha_S Y_S - \sum_{ij} \alpha_{ij} \pi_{ij}$$

Therefore we want to prove that $\forall \alpha_S \geq 0$, $\alpha_{ij} \geq 0$, and $\beta_{S,ij} \geq 0$ such that $\alpha_S - \alpha_{ij} + \beta_{S,ij} = 0 \forall S, i, j$, we must have $\sum_S \alpha_S Y_S - \sum_{ij} \alpha_{ij} \pi_{ij} \leq 0$.

We can get rid of the $\beta_{S,ij}$ and make the inequality: $\forall \alpha_S, \alpha_{ij}$ such that $\alpha_S - \alpha_{ij} \leq 0 \forall S, i, j$, we must have $\sum_S \alpha_S Y_S - \sum_{ij} \alpha_{ij} \pi_{ij} \leq 0$.

Any feasible solution in Dual1 is feasible in this new program. But any feasible solution in the new program is not necessarily feasible in Dual1; we have to use Farka's lemma.

$$\alpha_S \leq \alpha_{ij}$$

If α_{ij} is very big, then the second inequality is easy; so the second inequality is most powerful when α_{ij} is minimized. $\alpha_{ij} = \max_{ij \odot S} \alpha_S$ (min value).

$$\begin{aligned} \forall \alpha_S \geq 0 \ \ \sum_S \alpha_S Y_S - \sum_{ij} \max_{ij \odot S}(\alpha_S)\pi_{ij} &\leq 0 \\ \sum_S \alpha_S Y_S &\leq \sum_{ij} \max_{ij \odot S}(\alpha_S)\pi_{ij} \end{aligned}$$

We can also go the other way, and look at α_S maximized instead of α_{ij} minimized. $\alpha_S = \min_{ij \odot S} \alpha_{ij}$ (max value). Then

$$\forall \alpha_{ij} \geq 0 \ \ \sum_S \min_{ij \odot S}(\alpha_i j) Y_S \leq \sum_{ij} \alpha_{ij}\pi_{ij}$$

We can convert to either equation. We will use the first because there's more constraints and hence more flexibility.

7.5 Solve LP

Our LP is now

$$\begin{aligned} \max \sum_S Y_S & \\ \forall \alpha_S \geq 0 \ \ \sum_S \alpha_S Y_S &\leq \sum_{ij} \max_{ij \odot S}(\alpha_S)\pi_{ij} \\ \forall e \ \ \sum_{S: e \in \delta(S)} Y_S &\leq c_e \\ Y_S &\geq 0 \end{aligned}$$

Lemma 7.3. *$\overline{\alpha}$ vectors which have at most one non-zero value in their range are sufficient. I.e. $\overline{\alpha} = \{0, 0, 1, 1, 1, 1.5, 2.3, 0.9, 0.8\}$ is not allowed, but $\overline{\alpha} = \{0, 0, 0.8, 0.8, 0, 0, 0.8, 0\}$ is okay.*

This is to get rid of the infinite number of constraints problem, since $\overline{\alpha} \geq 0$ can be anything.

Proof. $\alpha_{\max} = \max_S \alpha_S$. $\alpha_{2nd\max} = \text{second}\max_S \alpha_S$. Both are nonzero. We can prove this trivially, because if both are 0, there is no constant, and if one is 0 and the other is not, then the lemma is proved. $\alpha_{\text{adjust}} = \alpha_{\max} - \alpha_{2nd\max}$.

$$\begin{aligned} \alpha^1 : \alpha^1_S &= \min(\alpha_S, \alpha_{2nd\max}) \\ \alpha^2 : \alpha^2_S &= \alpha_S - \alpha^1_S \end{aligned}$$

α^2 has only one value in its range, which is α_{adjust}. The constraint corresponding to α is the same as the constraint corresponding to the sum of α^1 and α^2. By induction, we can continue to do this, because α^1 has one less value.

To prove cons(α) = cons(α^1) + cons(α^2), prove $\alpha_S = \alpha_S^1 + \alpha_S^2$.

$$\max_{S \odot ij}(\alpha_S^1) + \max_{S \odot ij}(\alpha_S^2) \geq \max_{S \odot ij}(\alpha_S^1 + \alpha_S^2)$$

We would need to prove $\leq$ also to get equality, and would do case analysis to prove this. $\square$

Wlog, we can just make $\max_{ij \odot S}(\alpha_S) = 1$ (normalization). Then the LP is:

$$\begin{aligned} \max \sum_S Y_S & \\ \forall \alpha_S \in \{0,1\} \quad \sum_S \alpha_S Y_S &\leq \sum_{ij} \max_{ij \odot S}(\alpha_S)\pi_{ij} \\ \forall e \quad \sum_{S: e \in \delta(S)} Y_S &\leq c_e \\ Y_S &\geq 0 \end{aligned}$$

Notation: $\mathfrak{S} = \{S_1, \ldots, S_k\}$. $\mathfrak{S} \odot ij$ iff $\exists S \in \mathfrak{S}$ such that $S \odot ij$. $f(\mathfrak{S}) = \sum_{ij \odot S} \pi_{ij}$.

7.6 New LP

$$\begin{aligned} \max \sum_S Y_S & \\ \forall \mathfrak{S} \quad \sum_{S \in \mathfrak{S}} Y_S &\leq f(\mathfrak{S}) \\ \forall e \quad \sum_{S: e \in \delta(S)} Y_S &\leq c_e \\ Y_S &\geq 0 \end{aligned}$$

U universe. $f : 2^U \to \mathbb{R}$ such that $f(S) + f(T) \geq f(S \cup T) + f(S \cap T) \forall S, T \subseteq U$
$\Leftrightarrow$
$\forall S \subseteq U, T \subseteq S, i \notin T, i \notin S$:

$$f(S \cup \{i\}) - f(S) \leq f(T \cup \{i\}) - f(T)$$

Nice utility function, it gives more stuff with less marginal gain from new stuff.

$U = 2^V$ base family. $f : 2^{2^V} \to \mathbb{R}$. $\mathfrak{S}_1, \mathfrak{S}_2$:

$$f(\mathfrak{S}_1) + f(\mathfrak{S}_2) \geq f(\mathfrak{S}_1 \cup \mathfrak{S}_2) + f(\mathfrak{S}_1 \cap \mathfrak{S}_2)$$

If π_{ij} contributes on rhs, it will be added to lhs:

$$f(\mathfrak{S}_1 \cup \mathfrak{S}_2) \in \mathfrak{S}_1 \cup \mathfrak{S}_2 \text{ such that } S \odot ij$$

If $S \in \mathfrak{S}_1$, π_{ij} contributes to $\mathfrak{S}_1$.
If $S \in \mathfrak{S}_2$, π_{ij} contributes to $\mathfrak{S}_2$.
$S \in \mathfrak{S}_1 \cap \mathfrak{S}_2, S \odot ij$ will contribute to both.

Lemma 7.4. *Feasible dual Y_S. Suppose the constraints $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are tight. We claim that the constraint for $\mathfrak{S}_1 \cup \mathfrak{S}_2$ and $\mathfrak{S}_1 \cap \mathfrak{S}_2$ is also tight.*

Proof. Tight constraints:
LHS($\mathfrak{S}_1$) = $f(\mathfrak{S}_1)$
LHS($\mathfrak{S}_2$) = $f(\mathfrak{S}_2)$
LHS($\mathfrak{S}_1 \cup \mathfrak{S}_2) \leq f(\mathfrak{S}_1 \cup \mathfrak{S}_2)$
LHS($\mathfrak{S}_1 \cap \mathfrak{S}_2) \leq f(\mathfrak{S}_1 \cap \mathfrak{S}_2)$
LHS($\mathfrak{S}_1$) + LHS($\mathfrak{S}_2) \geq f(\mathfrak{S}_1 \cup \mathfrak{S}_2) + f(\mathfrak{S}_1 \cap \mathfrak{S}_2)$ since
LHS($\mathfrak{S}_1$) + LHS($\mathfrak{S}_2$) = LHS($\mathfrak{S}_1 \cap \mathfrak{S}_2$) + LHS($\mathfrak{S}_1 \cup \mathfrak{S}_2$)
We can see this with a Venn diagram. Therefore, they are equal and the constraints are tight, □