Lecture 14

CSE 522: Advanced Algorithms

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1 Max-Flow

In the last class, we discussed max-flow problem in terms of total-unimodularity. Note that primal has integer optimal solution implies dual also has integer optimal solution. Define Dual max-flow by the following linear program:

$$\begin{aligned} \max \quad & \sum c_{uv} d_{uv} \\ s.t. \quad & d_{st} = 1 \\ & d_{uv} \geq 0, \ \forall\, u, v \\ & d_{uv} + d_{vw} \geq d_{uvw}, \quad \forall\, u, v, w \end{aligned}$$

where c_{uv} is the cost of edge (u, v).

The optimal solution should be in the following way: Assume $V = \{s\} \cup S \cup T \cup \{t\}$, where $S \cap T = \emptyset$. Then for each edge $e \in \{(u, v) \mid u \in S, v \in T\}$, the flow across e is unit.

2 Submodularity

Definition 2.1 *Given a finite base set U, function $f : 2^U \to R$ is said to be* submodular *if*

$$f(S) + f(T) \geq f(S \cup T) + f(S \cap T),$$

for any $S, T \subseteq U$.

Definition 2.2 *Given a finite base set U, function $f : 2^U \to R$ is said to be* supermodular *if*

$$f(S) + f(T) \leq f(S \cup T) + f(S \cap T),$$

for any $S, T \subseteq U$.

Definition 2.3 *Given a finite base set U, function $f : 2^U \to R$ is said to be* modular *if*

$$f(S) + f(T) = f(S \cup T) + f(S \cap T),$$

for any $S, T \subseteq U$.

Note that the addition of two submodular functions is also submodular. In the following we give a few examples for submodularity.

Example 2.1 *$U = \{a_1, \ldots, a_m\}$, where each $a_i \in R^n$. Define $f(S) = rank(S)$. Then f is submodular.*

Example 2.2 *Given graph $G = (V, E)$, let $U = V$. Define*

$$f(S) = |\{(u, v) \mid u \in S, v \in V - S\}|.$$

Then f is submodular.

Example 2.3 *Given graph $G = (V, E)$ and two specific vertices s and t, let $U = V - \{s, t\}$. Define*

$$f(S) = |\{(u, v) \mid u \in S \cup \{s\}, v \in V - S\}|.$$

Then f is submodular.

Example 2.4 *$U = \{P_1, \ldots, P_m\}$, where $P_1, \ldots, P_m$ are distinct subsets. Define $f(S) = |\cup_{i \in S} P_i|$, and $g(S) = f(S) - |S|$. Both f and g are submodular.*

Theorem 2.1 *The following definition is equivalent to submodularity: for any $S \subseteq T \subseteq U$, $i \notin S, T$, $f(S \cup \{i\}) - f(S) \geq f(T \cup \{i\}) - f(T)$.*

Proof. If function f is submodular, then

$$\begin{aligned} f(S \cup \{i\}) + f(T) &\geq f((S \cup \{i\}) \cup T) + f((S \cup \{i\}) \cap T) \\ &= f(T \cup \{i\}) + f(S). \end{aligned}$$

On the other hand, to prove submodularity, we need to show

$$f(S) - f(S \cap T) \geq f(S \cup T) - f(T),$$

which is equivalent to

$$f((S \cap T) \cup (S - T)) - f(S \cap T) \geq f(T \cup (S - T)) - f(T).$$

We can prove this by induction on $|S - T|$. □

3 Minimizing Submodular Functions

Submodular function minimization problem means given a finite set U and a submodular function $f : 2^U \to R$, we are required to find a subset $X \subseteq U$ with $f(X)$ minimum.

Theorem 3.1 *(Lovász'80) A submodular function f can be minimized in polynomial time.*

Assume for each $u \in U$, we have a valuation $x_u \geq 0$, where for any $P \subseteq U$, we have

$$\begin{aligned} f(P) &\geq \sum_{u \in P} x_u, \text{ for } \forall\ P \subseteq U \\ f(P) &= \sum_{u \in P} x_u, \text{ if } P = S, T \end{aligned}$$

Then

$$\begin{aligned} f(S) + f(T) &\geq f(S \cap T) + f(S \cup T) \\ &\geq \sum_{u \in S \cap T} x_u + \sum_{u \in S \cup T} x_u \\ &= \sum_{u \in S} x_u + \sum_{u \in T} x_u \\ &= f(S) + f(T). \end{aligned}$$

If $f(S) + f(T)$ is minimized, $f(S \cap T)$ and $f(S \cup T)$ are also minimized.

4 Application

Let $U = \{P_1, \ldots, P_m\}$, $f(S) = \Sigma_{i \in S} \xi(S, i)$, where $\xi(S, i) \geq 0$ denotes the payment of player i when S is selected. We require that

$$\xi(T, i) \leq \xi(S, i), \quad \text{for any } S \subseteq T. \tag{1}$$

Fix S, let $\xi(S, i) = g_i(t)$, where initially, $t = 0$ and $g_i(t) = 0$ for each $i \in S$. As t increases, $g_i(t)$ increases continuously. Note that when t increases, at some point some subset S' of S becomes tight, that is, $\sum_{i \in S'} \xi(S', i) = f(S')$. Remove S' from S, and increase $g_i(t)$ for other $i \notin S'$ further until S becomes tight. Note that for $S_1, S_2 \subseteq S$, if S_1 becomes tight at time t_1 and S_2 becomes tight at t_2, where $t_1 \leq t_2$, then $S_1 \cup S_2$ becomes tight at time t_2.

Theorem 4.1 *$\xi(S,i)$ constructed above satisfies $\xi(S,i) \geq 0$ and $\xi(T,i) \leq \xi(S,i)$, for any $S \subseteq T$.*

Proof. For any $S \subseteq T \subseteq U$, assume at time t, the tight set for S and T is F_S and F_T, respectively. Due to submodularity,

$$f(F_S^t) + f(F_T^t) \geq f(F_S^t \cap F_T^t) + f(F_S^t \cup F_T^t),$$

which implies

$$\sum_{i \in F_S^t} x_i' + \sum_{i \in F_T^t} x_i \geq \sum_{i \in F_S^t \cap F_T^t} x_i' + \sum_{i \in F_S^t \cup F_T^t} x_i.$$

And thus,

$$\sum_{i \in F_S^t \cup F_T^t} x_i \geq \sum_{i \in F_S^t \cup F_T^t} x_i'.$$

□