CSE 527
Lecture 11

Relative entropy
Convergence of EM
Weight matrix motif models

Talks this week

- 590CB Today, 3:30 MEB 243, Maynard Olson, "The Imprint of Balancing Selection on the Human Genome"
- Dr. James Kent
  "The Gene Family Browser and other Recent Research at genome.ucsc.edu"
  Wednesday 3:30, Hitchcock 132
- 590CB Next Week Bob Waterston

Relative Entropy

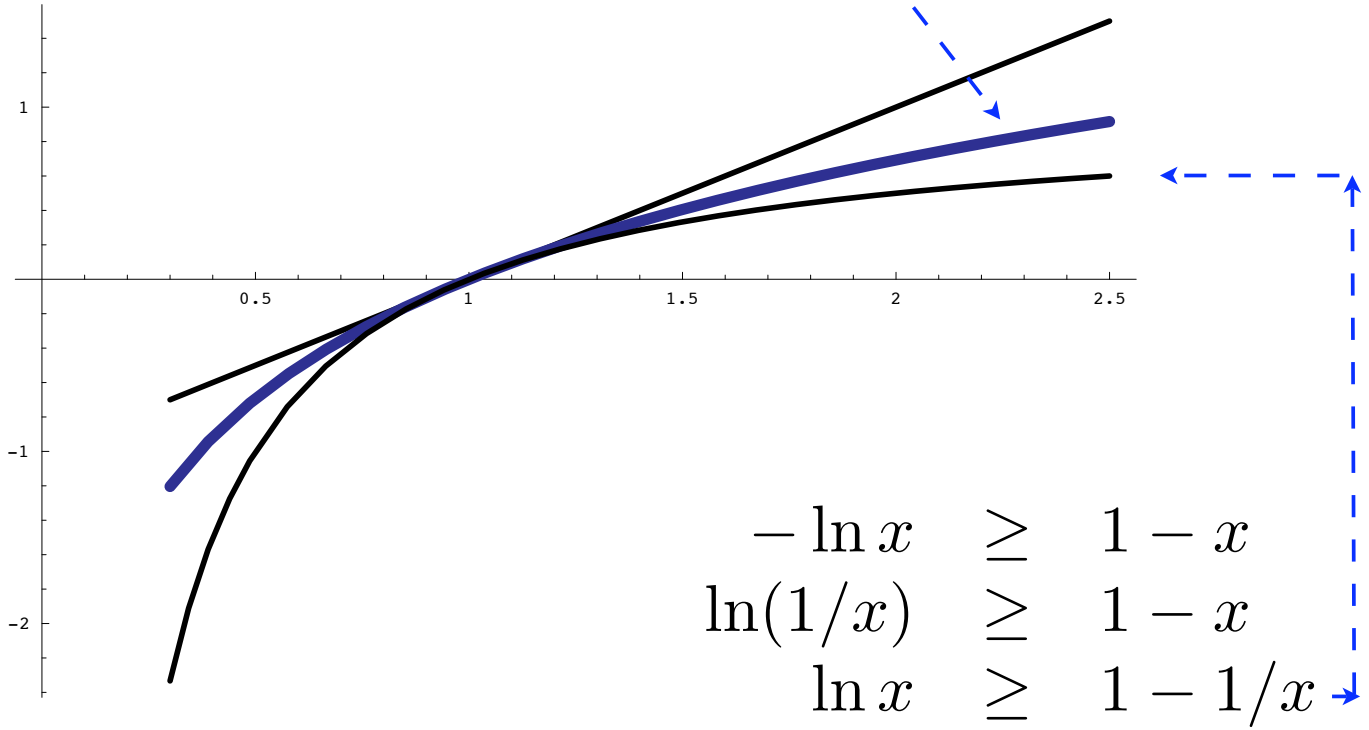
- AKA Kullback-Liebler Distance/Divergence, Information Content
- Given distributions P, Q

$$H(P||Q) = \sum_{x \in \Omega} P(x) \log \frac{P(x)}{Q(x)}$$

Notes:

$$\text{Let } P(x) \log \frac{P(x)}{Q(x)} = 0 \text{ if } P(x) = 0 \text{ [since } \lim_{y \to 0} y \log y = 0]$$

$$\text{Undefined if } 0 = Q(x) < P(x)$$


$\ln x \quad \leq \quad x - 1$
1
0.5
1
1.5
2
2.5
-1
-2
$-\ln x \quad \geq \quad 1 - x$
$\ln(1/x) \quad \geq \quad 1 - x$
$\ln x \quad \geq \quad 1 - 1/x$

Theorem: $H(P||Q) \geq 0$

$$
\begin{aligned}
H(P||Q) &= \textstyle\sum_x P(x) \log \frac{P(x)}{Q(x)} \\
&\geq \textstyle\sum_x P(x) \left(1 - \frac{Q(x)}{P(x)}\right) \\
&= \textstyle\sum_x (P(x) - Q(x)) \\
&= \textstyle\sum_x P(x) - \sum_x Q(x) \\
&= 1 - 1 \\
&= 0
\end{aligned}
$$

Furthermore: H(P||Q) = 0 if and only if P = Q

EM Convergence

Visible x

hidden y

parameters θ

Goal Maximum likelihood estimate of θ

i.e. Find θ maximizing $\Pr(X|\theta)$ (or $\log P(x|\theta)$)

$P(y|x) = P(x,y)/P(x)$ so $P(x) = P(x,y)/P(y|x)$

$\forall y$:

$$\log P(x|\theta) = \log P(x,y|\theta) - \log P(y|x,\theta)$$

$$\log P(x|\theta) =$$

$$\underbrace{\sum_y P(y|x,\theta^t) \cdot \log P(x,y|\theta)}_{Q(\theta|\theta^t)} - \sum_y P(y|x,\theta^t) \cdot \log P(y|x,\theta)$$

$$\log P(x|\theta) = Q(\theta|\theta^t) - \Sigma_y P(y|x,\theta^t) \cdot \log P(y|x,\theta)$$

A key trick: Q is easier to optimize than whole thing

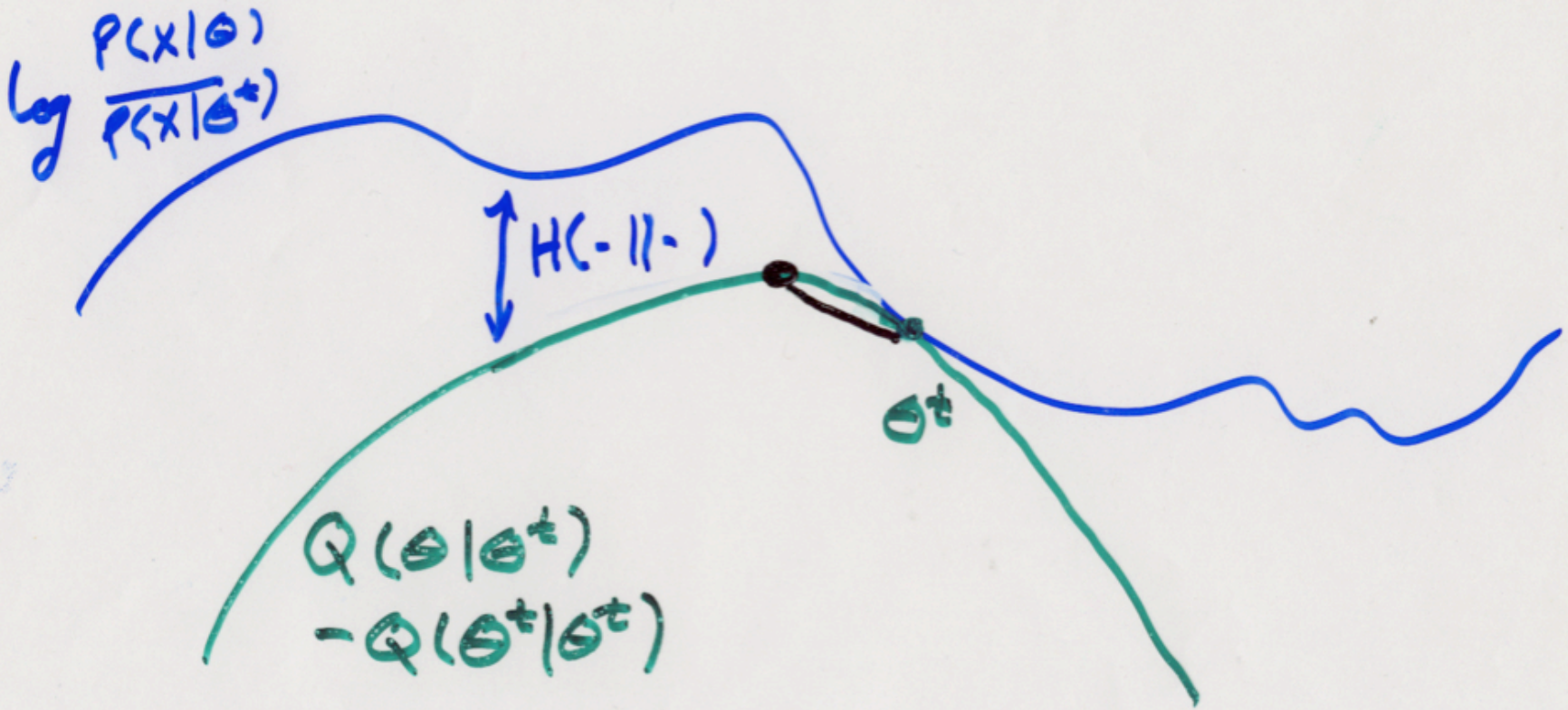
① $\log P(x|\theta) - \log P(x|\theta^t) =$

② $Q(\theta|\theta^t) - Q(\theta^t|\theta^t)$

$$+ \sum_y P(y|x,\theta^t) \log \frac{P(y|x,\theta^t)}{P(y|x,\theta)}$$

$$H(P(y|x,\theta^t) \,\|\, P(y|x,\theta)) \geq 0$$

$\therefore$ ① ≥ 0 if ② ≥ 0


log P(x|θ) / P(x|θ^t)
H(·||·)
θ^t
Q(θ|θ^t)
-Q(θ^t|θ^t)

Sequence Motifs

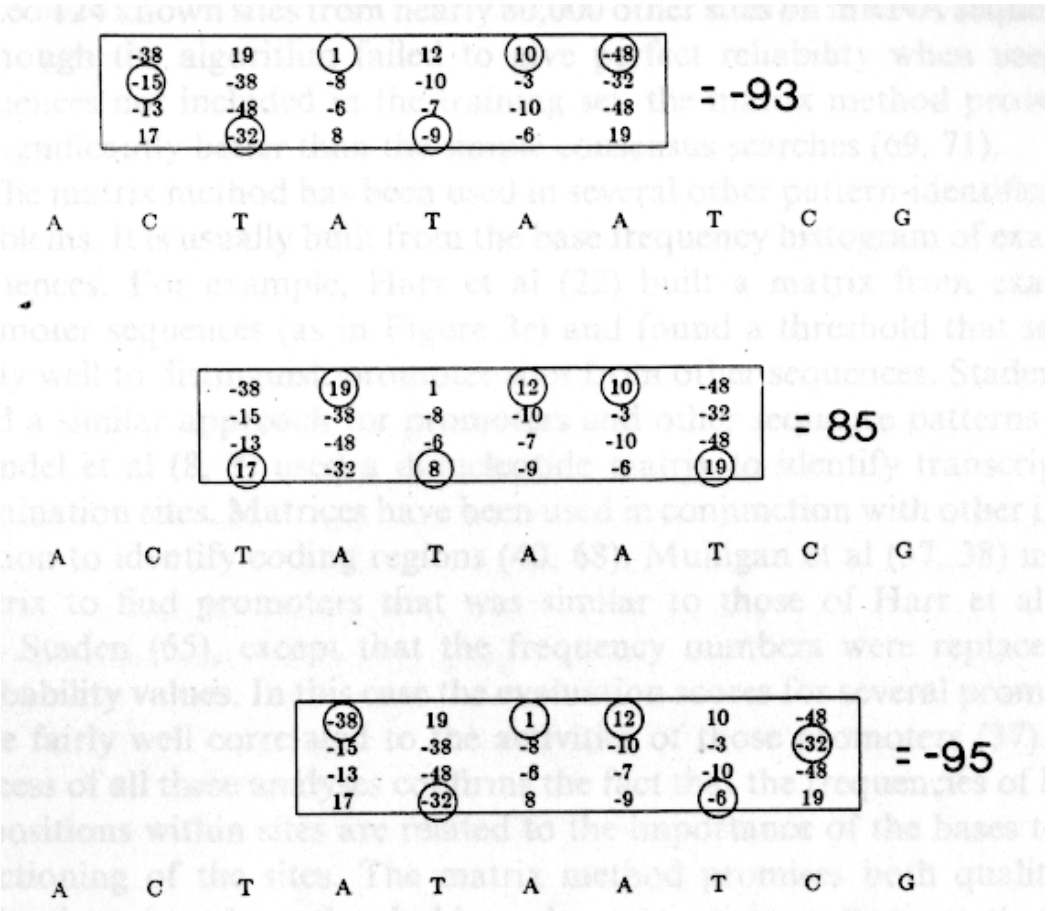
E. coli Promoters

- "TATA Box" - consensus TATAAT ~ 10bp upstream of transcription start
- *Not* exact: of 168 studied
  - nearly all had 2/3 of TAxyzT
  - 80-90% had all 3
  - 50% agreed in each of x,y,z
  - no perfect match
- Other common features at -35, etc.

TATA Box Frequencies

pos / base	1	2	3	4	5	6
A	2	95	26	59	51	1
C	9	2	14	13	20	3
G	10	1	16	15	13	0
T	79	3	44	13	17	96

Scanning for TATA



Stormo, Ann. Rev. Biophys. Biophys Chem, 17, 1988, 241-263

Weight Matrices: Statistics

- Assume:

  $f_{b,i}$ = frequency of base b in position i

  f_b = frequency of base b in all sequences

- Log likelihood ratio, given S = $B_1B_2...B_6$:

$$\log\left[\frac{P(S|\text{“promoter”})}{P(S|\text{“nonpromoter”})}\right] = \log\left[\frac{\prod_{i=1}^{6} f_{B_i,i}}{\prod_{i=1}^{6} f_{B_i}}\right] = \sum_{i=1}^{6} \log\left[\frac{f_{B_i,i}}{f_{B_i}}\right]$$

Weight Matrices: Chemistry

- Experiments show ~80% correlation of log likelihood weight matrix scores to measured binding energy of RNA polymerase to variations on TATAAT consensus