Maximum Likelihood and Expectation Maximization

Probability Basics

	Ex	Ex
Sample Space	$\{1, \dots, 6\}$	$\mathbb{R}$
Distribution	$p_1 \dots p_6 \geq 0, \sum p_i = 1$	p.d.f $f(x) \geq 0, \int_{\mathbb{R}} f(x)dx = 1$
	eg $p_1 = \dots = p_6 = \frac{1}{6}$	$\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

Population vs Sample

population mean	$\mu = \sum i p_i$	$\mu = \int x f(x) dx$
population Variance	$\sigma^2 = \sum (i-\mu)^2 p_i$	$\sigma^2 = \int (x-\mu)^2 f(x) dx$
sample mean		$\sum x_i / n$
Sample Variance		$\sum (x_i - \bar{x})^2 / n$

Parameter Estimation

- Assuming sample $x_1, x_2, \dots, x_n$ is from a parametric distributionf(x| theta), estimate theta.
- E.g.:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\theta = (\mu, \sigma^2)$$

Maximum Likelihood Estimation

One (of many) approaches to parameter est.

Likelihood of $x_1 \cdots x_n$ =

$$L(x_1, \dots, x_n | \theta) = \underbrace{\prod_{i=1}^{n}}_{\text{assume indp.}} f(x_i | \theta)$$

View this as a function of θ

what θ maximizes the likelihood

Typical approach: $\frac{\partial}{\partial \theta} L(x|\theta) = 0$

or $\frac{\partial}{\partial \theta} \ln L(x|\theta) = 0$

Example

$X_1 \cdots X_n$ coin flips; Θ = prob of heads

n_0 tails, n_1 heads, $n_0 + n_1 = n$

$$L(x_1 \cdots x_n | \Theta) = (1-\Theta)^{n_0} (\Theta)^{n_1}$$

$$\ln L = n_0 \ln(1-\Theta) + n_1 \ln \Theta$$

$$\frac{d}{d\Theta} \ln L = \frac{-n_0}{1-\Theta} + \frac{n_1}{\Theta} = 0$$

$$n_0 \Theta = n_1 (1-\Theta)$$

$$(n_0 + n_1)\Theta = n_1$$

$$\boxed{\Theta = \frac{n_1}{n}}$$

And verify it's max, not min
& not better on boundary

Example

$x_i \sim N(\mu, \sigma^2)$, $\sigma^2 = 1$, μ unknown

$$L(x_1 \cdots x_n | \Theta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x_i - \Theta)^2}{2}}$$

$$\ln L(x_1 \cdots x_n | \Theta) = \sum -\frac{1}{2} \ln 2\pi - \frac{(x_i - \Theta)^2}{2}$$

$$\frac{dL}{d\Theta} = -\sum (x_i - \Theta) = 0$$

$$\left(\sum x_i\right) - n\Theta = 0$$

$$\boxed{\Theta = \sum x_i / n}$$

And verify it's max, not min
& not better on boundary

Example

$x_i \sim N(\mu, \sigma^2)$, both unknown

...

$$\ln L(x_1 \cdots x_n | \Theta_1, \Theta_2) = \sum -\frac{1}{2} \ln 2\pi\Theta_2 - \frac{(x_i - \Theta_1)^2}{2\Theta_2}$$

$$\frac{\partial}{\partial \Theta_1} \ln L(x_1 \cdots x_n | \Theta_1, \Theta_2) = \sum \frac{x_i - \Theta_1}{\Theta_2} = 0$$

$$\Rightarrow \boxed{\Theta_1 = \sum x_i / n}$$

$$\frac{\partial}{\partial \Theta_2} \ln L(x_1 \cdots x_n | \Theta_1, \Theta_2) = \sum -\frac{2\pi}{2 \cdot 2\pi \Theta_2} + \frac{(x_i - \Theta_1)^2}{2\Theta_2^2}$$

$$\sum -\frac{1}{2\Theta_2} + \frac{(x_i - \Theta_1)^2}{2\Theta_2^2} = 0$$

$$\sum (x_i - \Theta_1)^2 = n\Theta_2$$

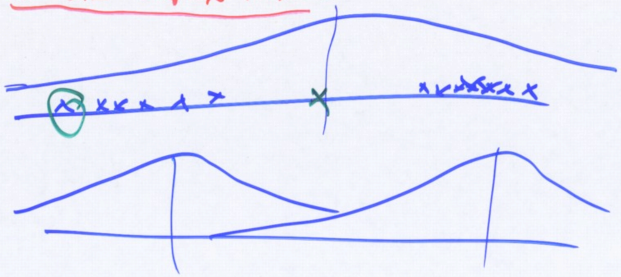
$$\boxed{\Theta_2 = \sum (x_i - \Theta_1)^2 / n}$$

A Biased (but consistent) estimate of population variance

An Example of Overfitting

Unbiased estimate: $\sum_{i=1}^{n} \frac{(x_i - \theta_1)^2}{n-1}$

A More Complex Problem



2 distributions $f_1(x)$, $f_2(x)$

$f_1(x|\theta_1)$, $f_2(x|\theta_2)$

Mixing parameter τ_1 τ_2

$\tau_1 + \tau_2 = 1$

Likelihood

$$\Theta$$

$$L(x_1 \dots x_4 \mid \tau_1 \tau_2 \mu_1 \mu_2 \sigma_1^2 \sigma_2^2 \dots)$$

$$= \prod_{i=1}^{n} \sum_{j=1}^{2} \tau_j f_j(x_i|\theta)$$

Probably too messy for closed-form solution

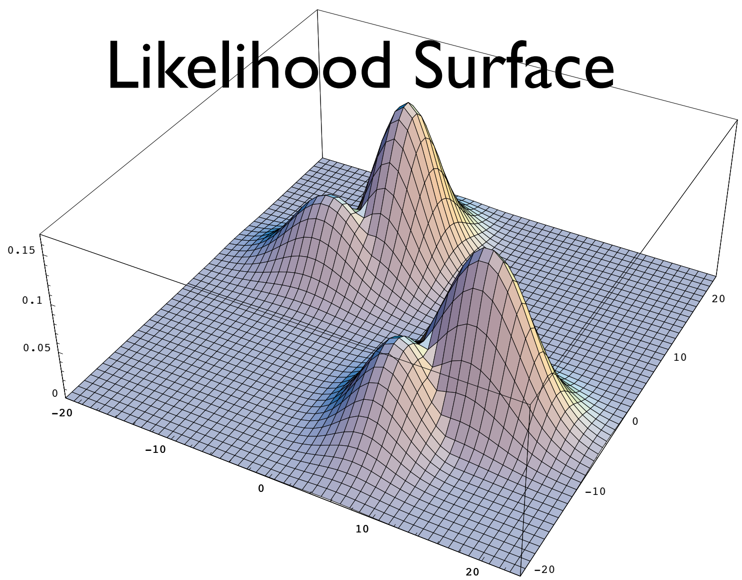
Full data

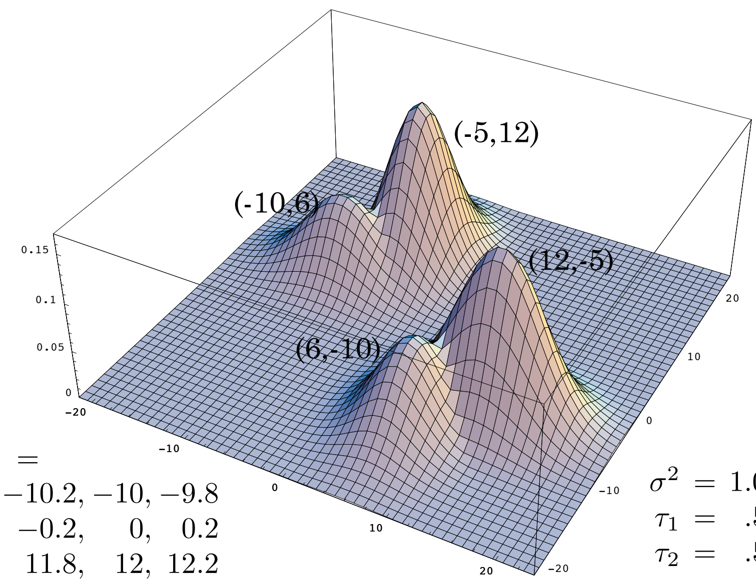
$$\begin{array}{ccc} x_1 & z_{11} & z_{12} \\ x_2 & z_{21} & z_{22} \\ x_3 & z_{31} & z_{32} \end{array}$$

$\underbrace{z_{i1} \quad z_{i2}}_{\text{Hidden Variables}}$

z_{ij}: $\begin{cases} 0 \\ 1 \text{ iff } x_i \text{ comes from distribution } j \end{cases}$

Likelihood Surface





$x_i =$
$-10.2, -10, -9.8$
$-0.2, \quad 0, \quad 0.2$
$11.8, \quad 12, \quad 12.2$

$\sigma^2 = 1.0$
$\tau_1 = .5$
$\tau_2 = .5$

assume τ_j θ_j fixed

A event that x_i drawn from f_1

B . . . . f_2

D "data" x_i is observed

$P(A \mid D)$

$P(D \mid A)$

$$P(A \mid D) = \frac{P(D \mid A)\, P(A)}{P(D)}$$ Bayes rule

$$P(D) = \underset{f_1(x_i \mid \theta_1)}{P(D \mid A)} \cdot \underset{\tau_1}{P(A)} + \underset{f_2(x_i \mid \theta_2)}{P(D \mid B)} \cdot \underset{\tau_2}{P(B)}$$

Expected value of z_{i1}

M step

$L(x_1 z_{11} z_{12} x_2 z_{21} z_{22} \cdots \mid \theta \tau)$

x_i's known

if z_{ij} known, then MLE $\theta\tau$ easy

But we don't.

Instead maximize expected likelihood of visible data

$E(L(x_1, x_2, \dots x_n \mid \theta\tau))$

where Expectation is over distribution of hidden values (z_{ij}'s)

$L(\vec{x}, \vec{z} \mid \theta \tau)$

$$\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2} \sum_{j=1}^{2} z_{ij} (x_i - \mu_j)^2}$$

$E(\ln L(x z \mid \theta\tau)) =$

$$E\left[\sum_{i=1}^{n} -\tfrac{1}{2} \ln 2\pi\sigma^2 - \frac{1}{2\sigma^2} \sum_{j=1}^{2} z_{ij} (x_i - \mu_j)^2 \right]$$

$$= \sum_{i=1}^{n} -\tfrac{1}{2} \ln 2\pi\sigma^2 \underbrace{- \frac{1}{2\sigma^2} \sum_{j=1}^{2} E(z_{ij}) (x_i - \mu_j)^2}$$

Find μ_j maximizing ↑ using $E(z_{ij})$ From E-step.