CSE/NB 528

Lecture 10: Recurrent Networks

(Chapter 7)

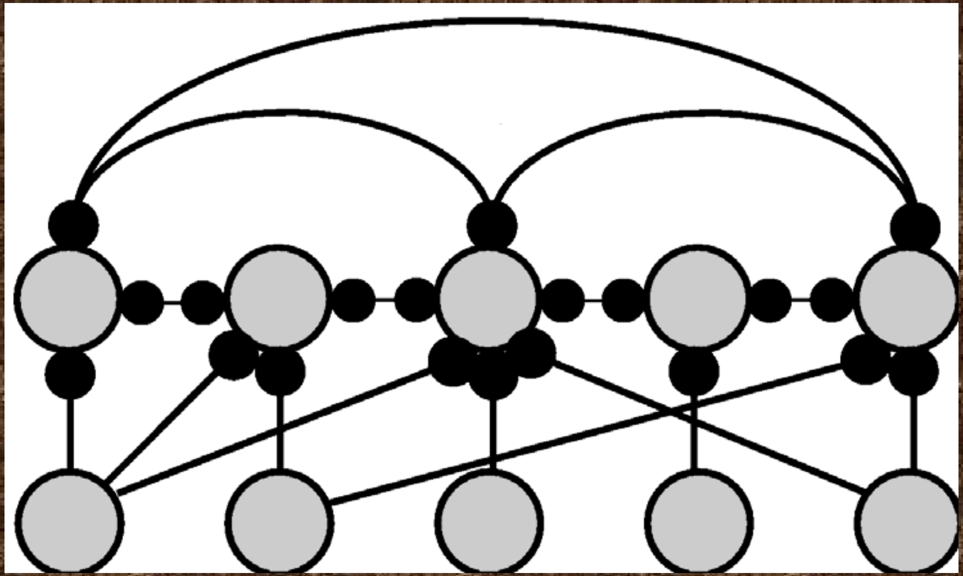


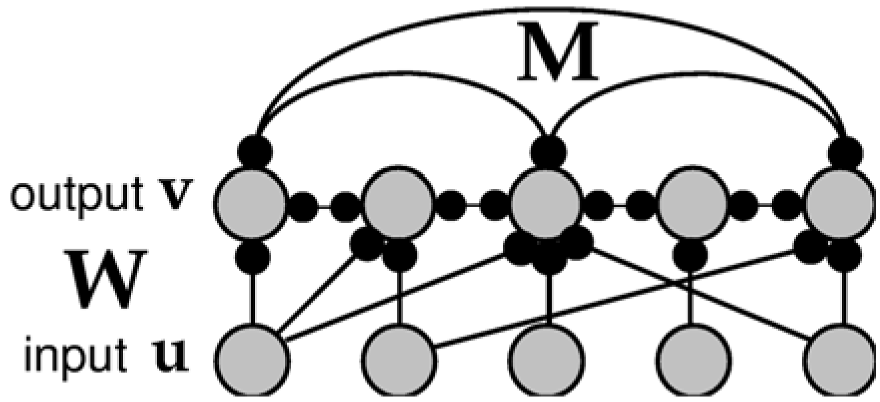
Image from http://clasdean.la.asu.edu/news/images/ubep2001/neuron3.jpg

Lecture figures are from Dayan & Abbott's book
http://people.brandeis.edu/~abbott/book/index.html

What's on our platter today?

- Computation in Recurrent Networks
  - Linear Recurrent Networks
    - Stability analysis using eigenvalues
    - Can amplify inputs
    - Can store short-term memory
  - Associative Memory (Hopfield net)
    - Showing Stability via Lyapunov function

Recurrent Networks



$$\tau \frac{d\mathbf{v}}{dt} = -\mathbf{v} + F(\mathrm{W}\mathbf{u} + \mathrm{M}\mathbf{v})$$

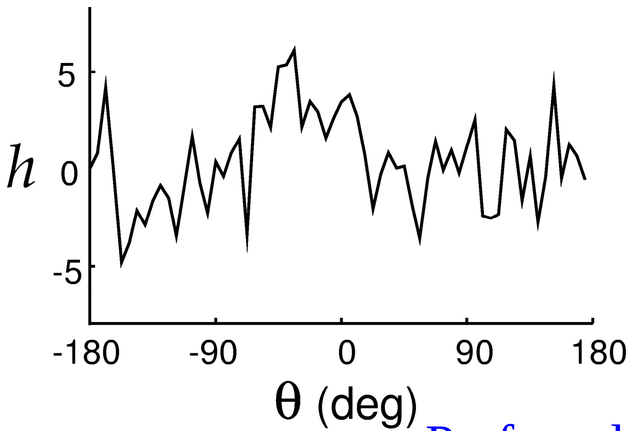
Output Decay Input Feedback

What can a Linear Recurrent Network do?

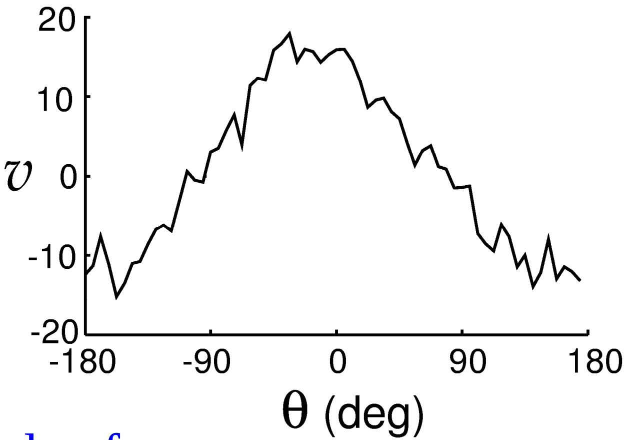
Analysis based on eigenvectors of recurrent weight matrix

Amplification in a Linear Recurrent Network

Input (noisy cosine)



Output

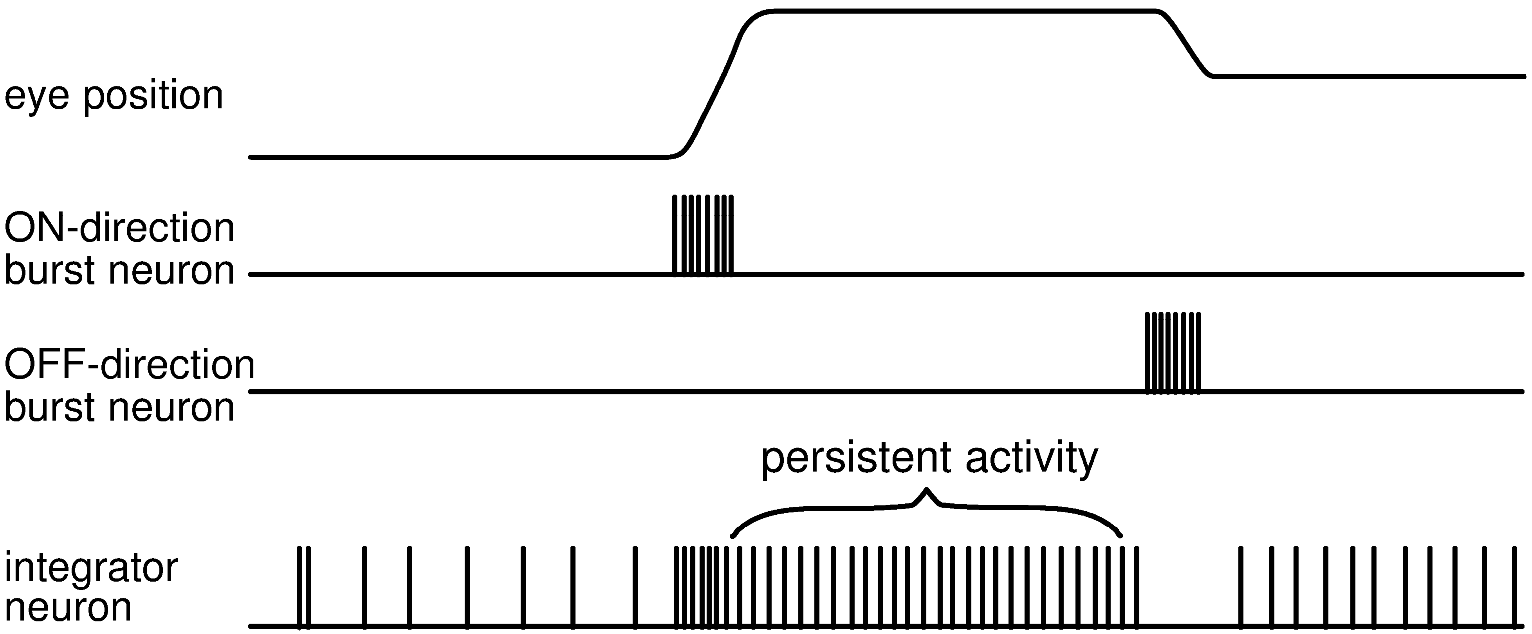


Preferred angle of neuron

$$M(\theta,\theta') = \frac{\lambda_1}{\pi}\cos(\theta - \theta')$$

All eigenvalues = 0 except $\lambda_1 = 0.9$ *i.e. amplification* $= \frac{1}{1-\lambda_1} = 10$

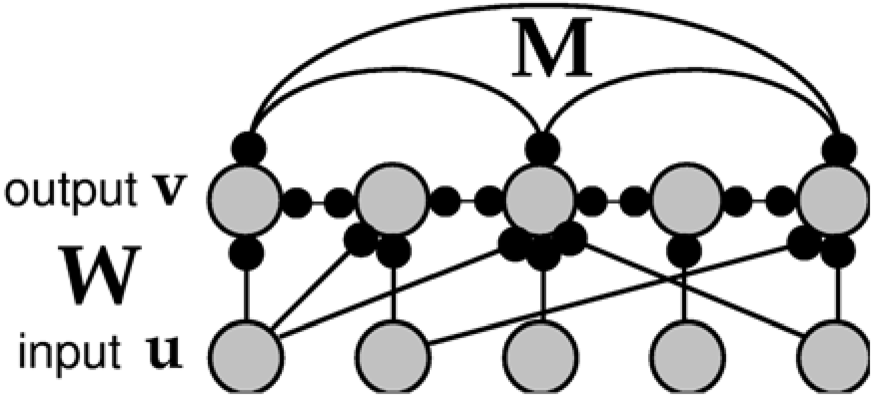
Input Integration for Maintaining Eye Position



Input: Bursts of spikes from brain stem oculomotor neurons

Output: Memory of eye position in medial vestibular nucleus

Nonlinear Recurrent Networks



Two types of firing-rate models

$$\tau \frac{d\mathbf{I}}{dt} = -\mathbf{I} + \mathrm{W}\mathbf{u} + \mathrm{M} \cdot F(\mathbf{I})$$

Current Dynamics (firing rate v = F(I))

$$\tau \frac{d\mathbf{v}}{dt} = -\mathbf{v} + F(\mathrm{W}\mathbf{u} + \mathrm{M}\mathbf{v})$$

Firing-Rate Dynamics

Output Decay Input Feedback

(Convenient to use W**u** = **h**)

Continuous Nonlinear Recurrent Networks

$$\tau \frac{d\mathbf{v}}{dt} = -\mathbf{v} + F(\mathbf{h} + \mathrm{M}\mathbf{v}) \text{ or,}$$

$$\tau \frac{dv_i}{dt} = -v_i + F(h_i + \sum_j M_{ij} v_j)$$

Discrete case (small number of neurons)

Continuous case (in the limit of large numbers of neurons):

$$\tau \frac{dv(\theta)}{dt} = -v(\theta) + F(h(\theta) + \rho_\theta \int_{-\pi}^{\pi} M(\theta, \theta') v(\theta') d\theta')$$

θ = preferred stimulus of the neuron (e.g. orientation of input)

Example of a Continuous Recurrent Network

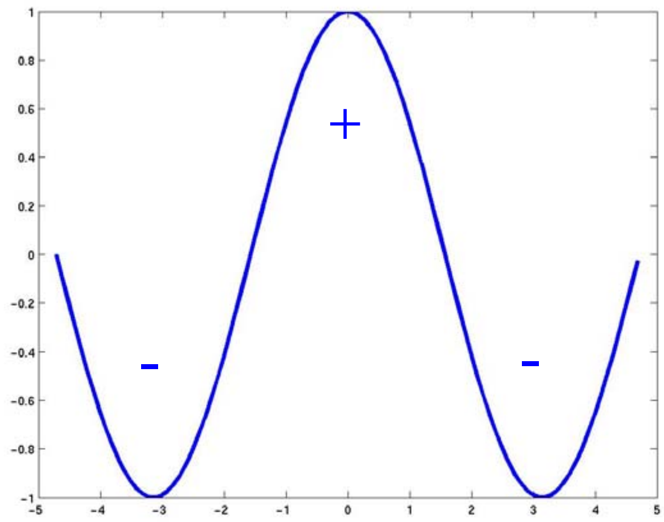
Choose F = rectification nonlinearity:

$$\tau \frac{dv(\theta)}{dt} = -v(\theta) + \left[h(\theta) + \int_{-\pi}^{\pi} M(\theta,\theta')v(\theta')d\theta' \right]^{+}$$

$$M(\theta,\theta') = \frac{\lambda_1}{\pi}\cos(\theta - \theta')$$

Choose recurrent connections = cosine function of relative angle

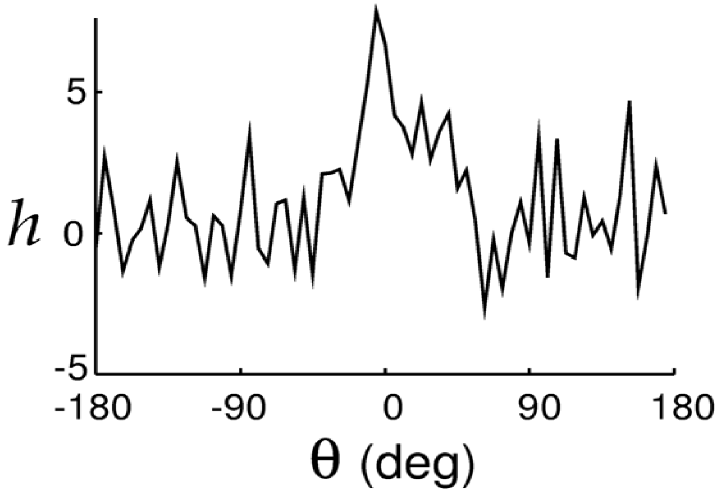
Excitation nearby,
Inhibition further away



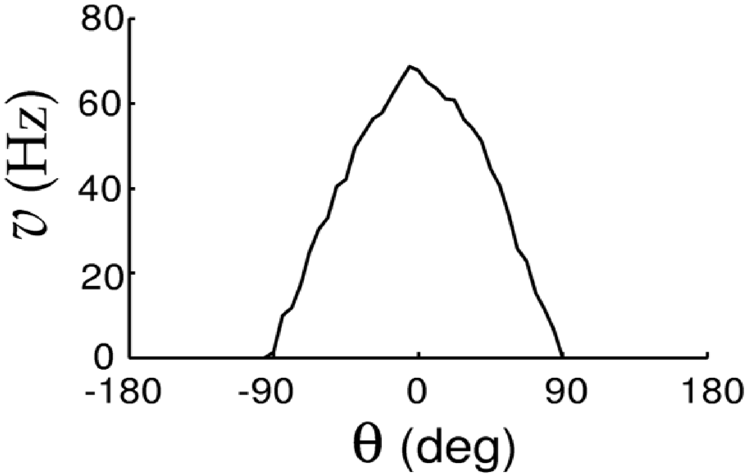
$(\theta - \theta')$

Amplification in a Nonlinear Recurrent Network

Input



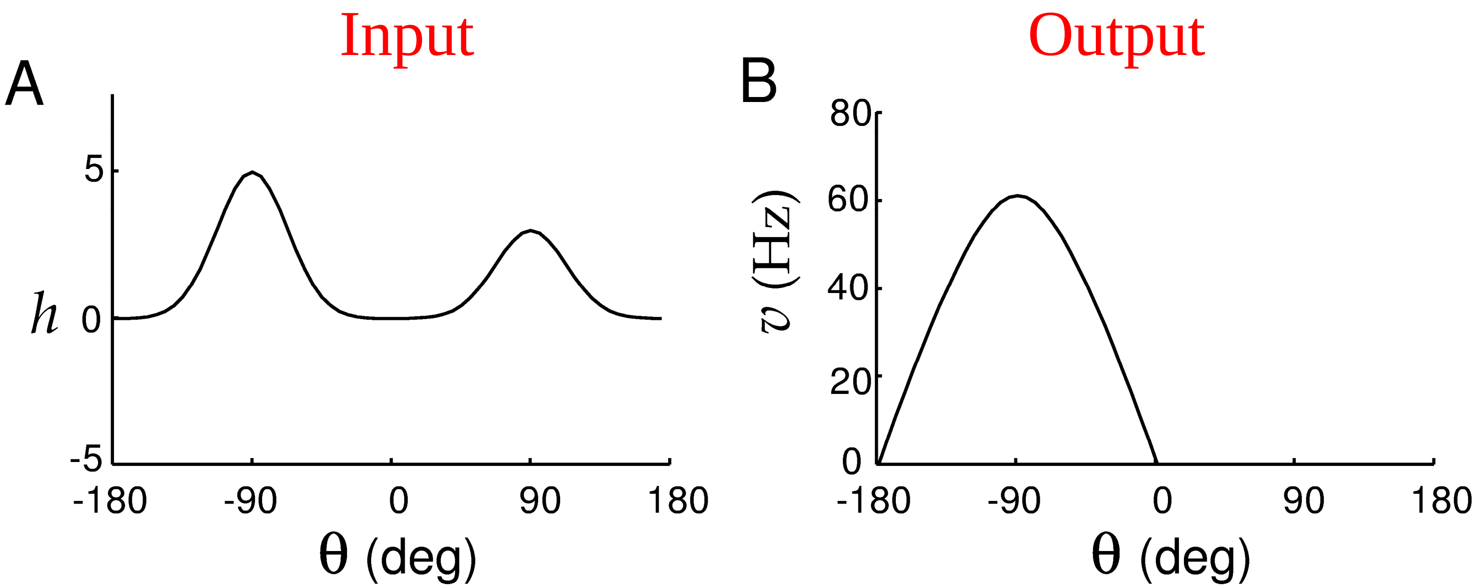
Output



$$\tau \frac{dv(\theta)}{dt} = -v(\theta) + \left[h(\theta) + \frac{\lambda_1}{\pi} \int_{-\pi}^{\pi} \cos(\theta - \theta')v(\theta')d\theta' \right]^+$$

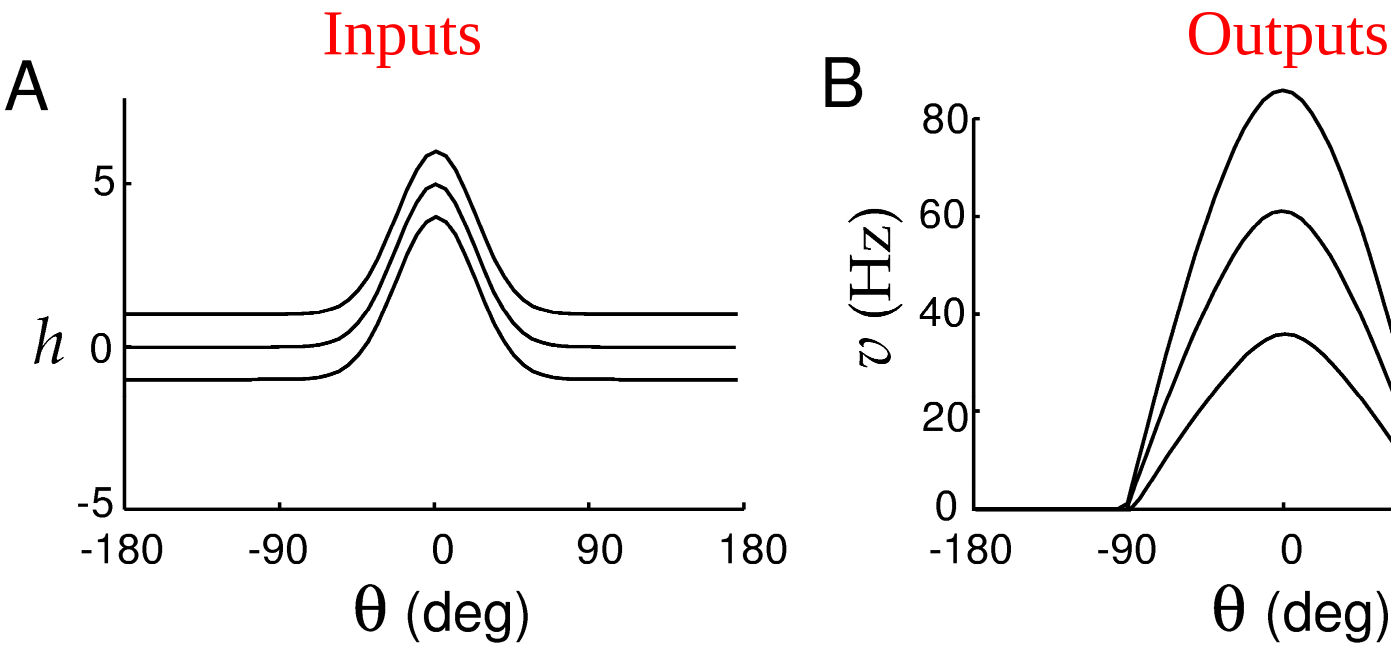
$\lambda_1 = 1.9$ (but stable due to rectification)

Selective "Attention" in a Nonlinear Recurrent Network



Network performs "winner-takes-all" input selection

Gain Modulation in a Nonlinear Recurrent Network



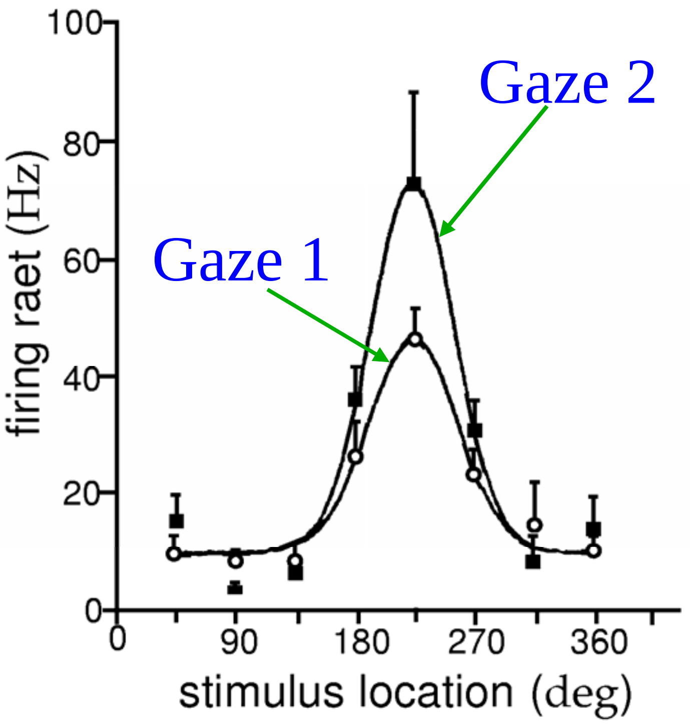
(In this example, θ = neuron's preferred angle s)

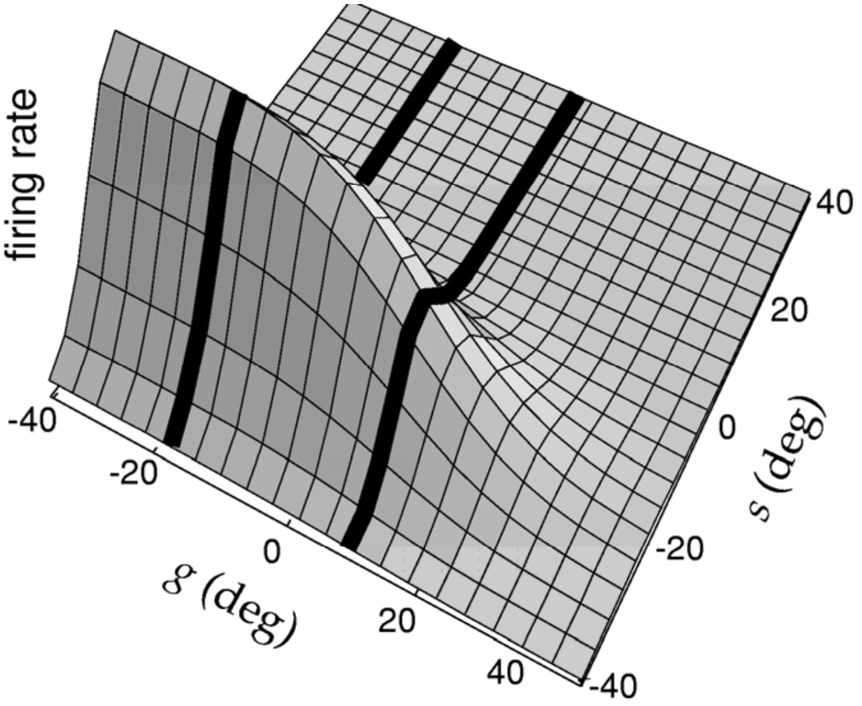
Changing the level of input by adding g multiplies the output

If h = s + g (s = stimulus angle on retina, g = gaze angle), then network output is gain-modulated similar to parietal cortex neurons

(Width determined by recurrent matrix *M*)

Gain Modulation in Parietal Cortex Neurons

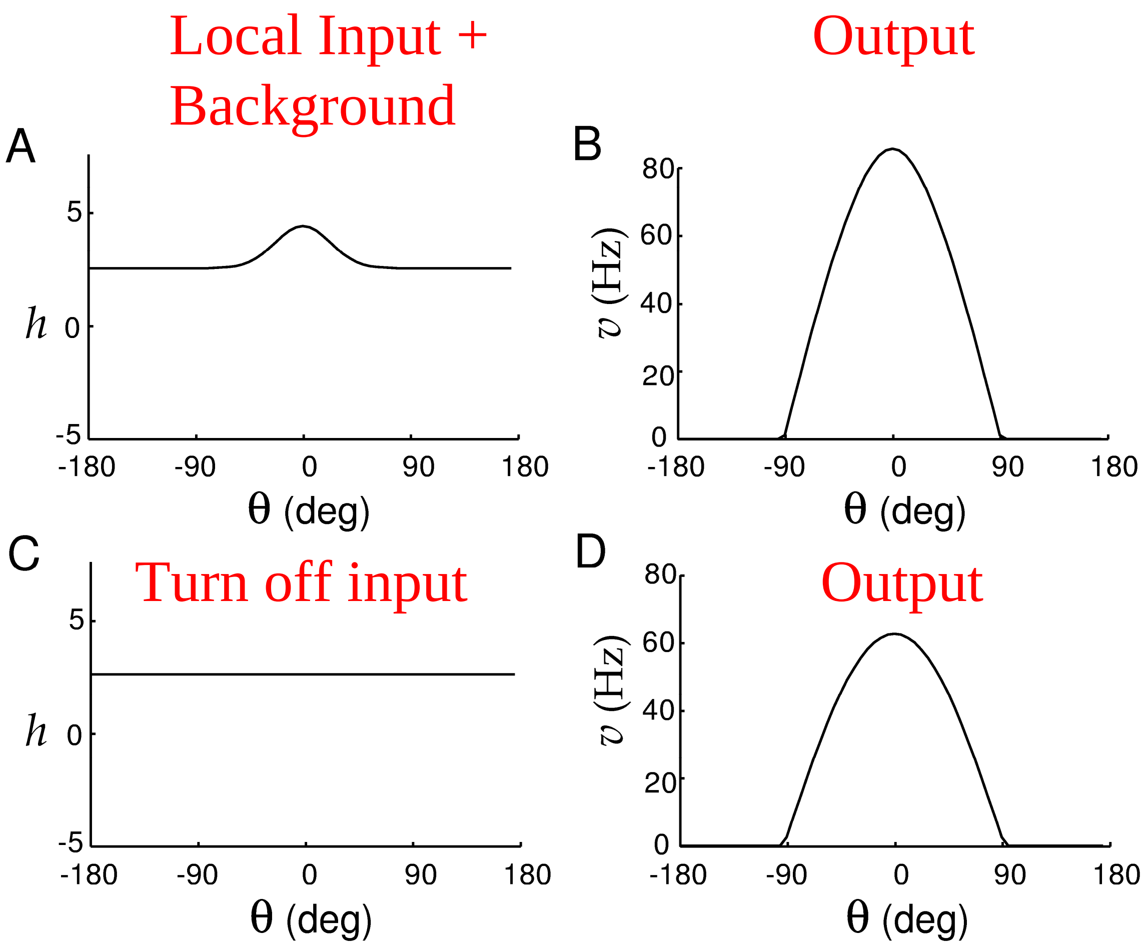




Responses of Area 7a neuron

Example of a gain-modulated tuning curve

Short-Term Memory Storage in a Nonlinear Recurrent Network



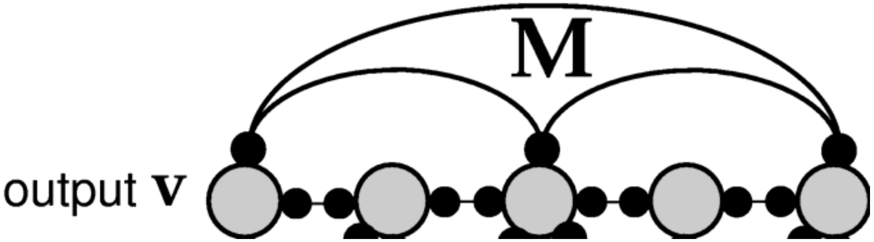
Network maintains a *memory of previous activity* when input is turned off.

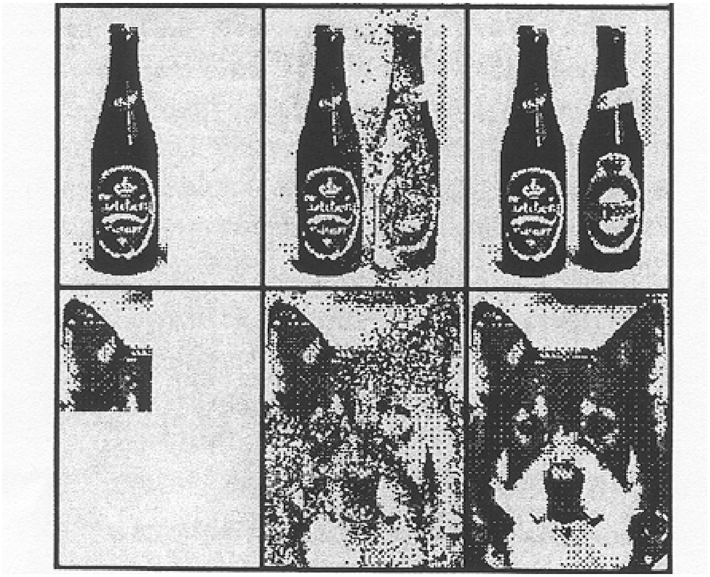
Similar to "short-term memory" or "working memory" in prefrontal cortex

Memory is maintained by recurrent activity

Associative Memories (Hopfield Networks)

- Fully connected, no feedforward inputs



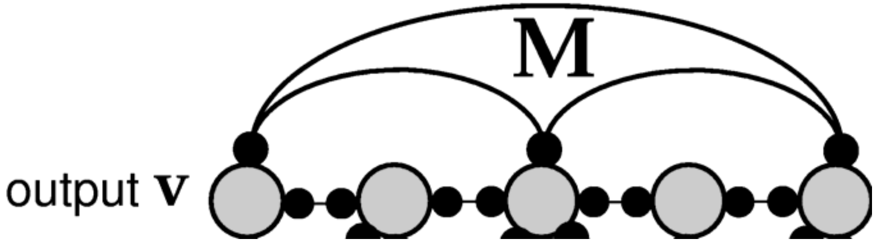


Idea: Store patterns as *fixed points* of this network

Can the network complete noisy or incomplete patterns?

Associative Memories (Hopfield Networks)

- Fully connected, no feedforward inputs



$$\tau \frac{d\mathbf{I}}{dt} = -\mathbf{I} + \mathrm{M} \cdot g(\mathbf{I}) \quad \text{or,}$$

$$\tau \frac{dI_i}{dt} = -I_i + \sum_j M_{ij} v_j \quad \text{where } v_j = g(I_j)$$

g = sigmoid function

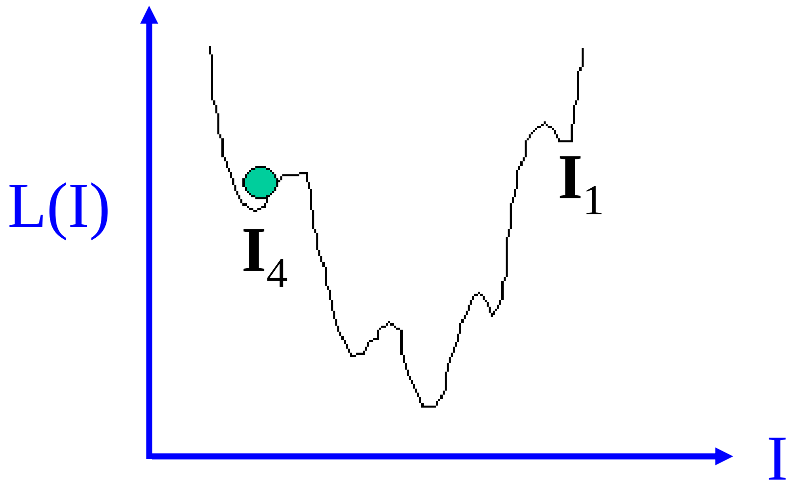
Idea: Store patterns as *fixed points* of this network

Question: Will **I** always converge to a fixed point?

Enter…Lyapunov Functions

- Idea: If dI/dt causes some function L(I) to always decrease or remain constant (i.e. $dL/dt \leq 0$) and L has a lower bound (with $dL/dt = 0$ only if $dI/dt = 0$), *then $dI/dt = 0$ eventually*
  - ➪ **Network converges to a fixed point**

- L also called "energy" function or "cost" function



Lyapunov for Hopfield networks

- What is a good Lyapunov function L(I) for Hopfield nets?
- What constraints are required on the recurrent weights **M**?

Lyapunov for Hopfield networks

Given: $\tau \frac{dI_i}{dt} = -I_i + \sum_j M_{ij} v_j$ where $v_j = g(I_j) = \tanh(\beta I_j)$

Define: $L(I) = -\frac{1}{2}\sum_{ij} M_{ij} v_i v_j + \sum_i \int_0^{v_i} g^{-1}(v)dv$

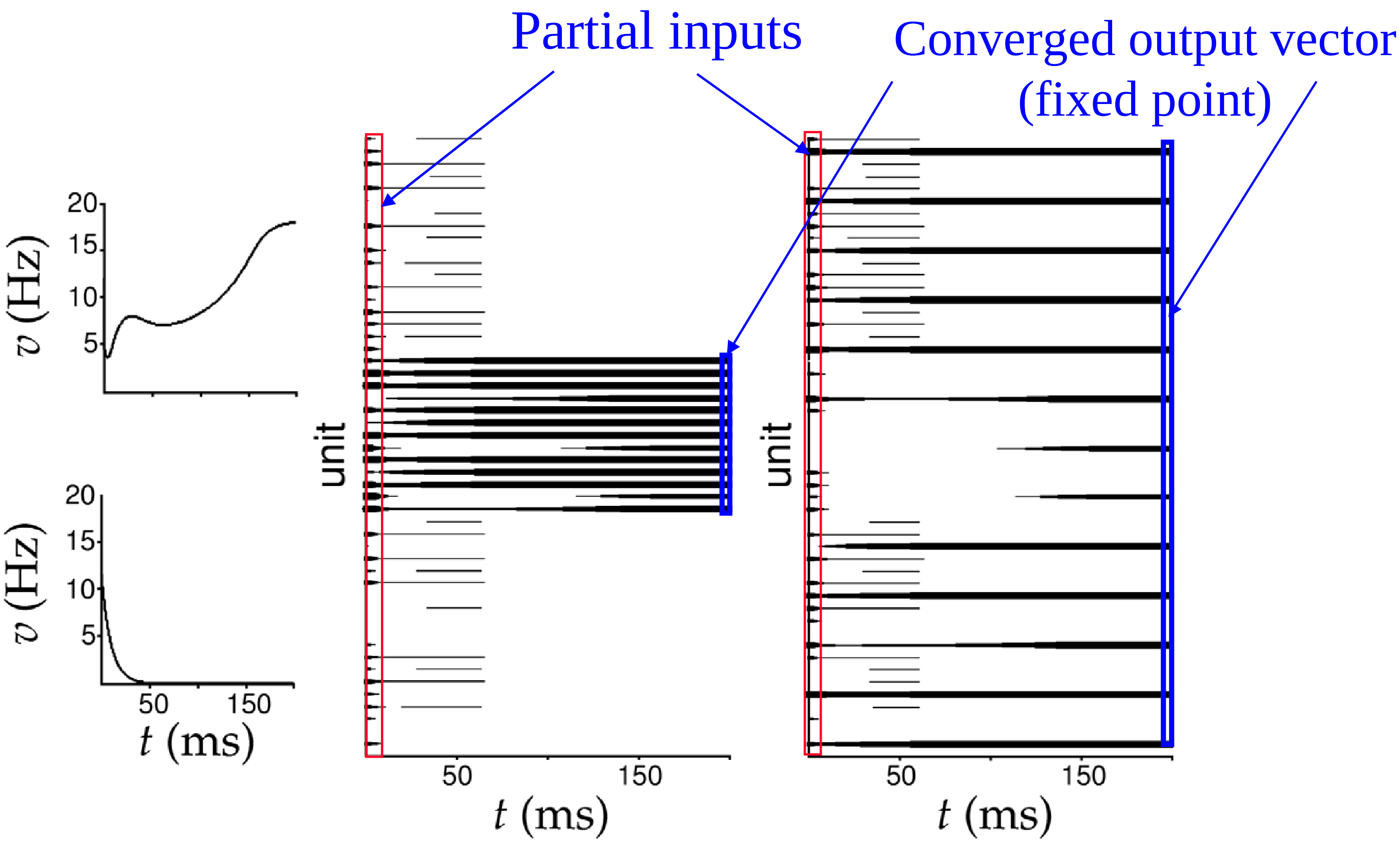
If M is symmetric ($M_{ij} = M_{ji}$), we can show:

$$\frac{dL}{dt} = -\tau \sum_i g'(I_i)\left(\frac{dI_i}{dt}\right)^2 \leq 0$$

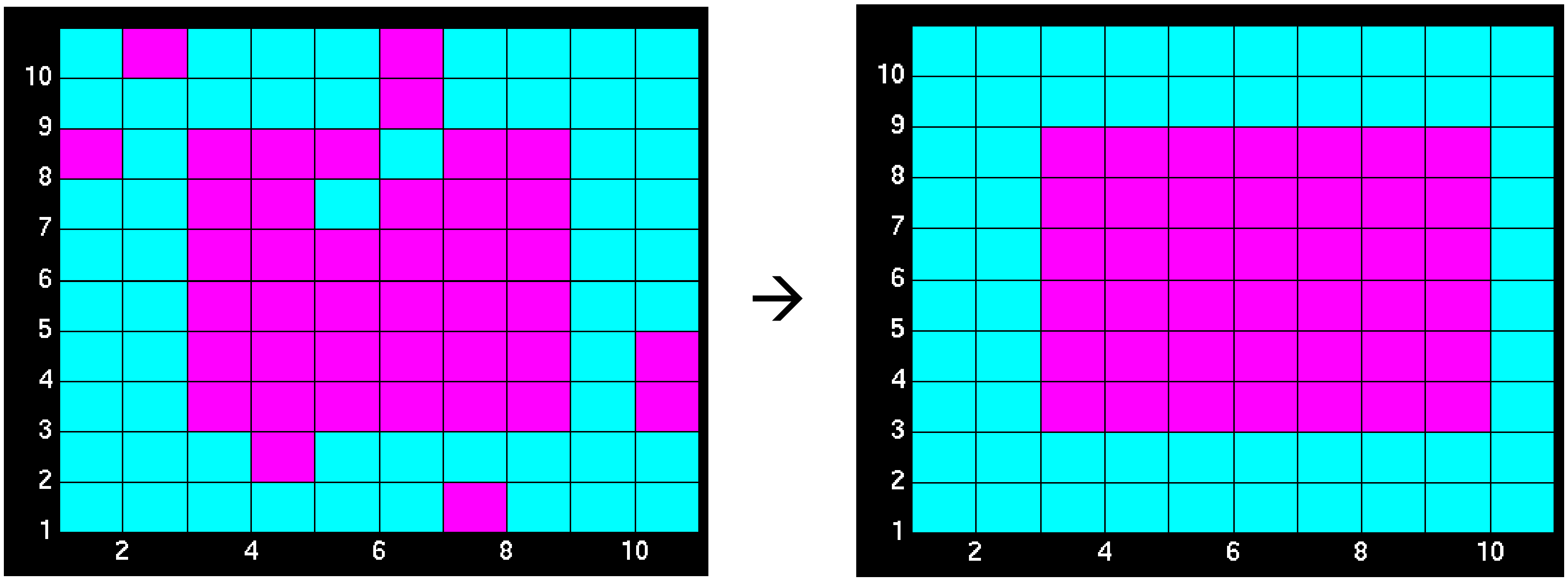
[Try to show this!]

Since L is bounded from below and $dL/dt = 0$ only if $dI_i/dt = 0$, L cannot decrease forever and $dI_i/dt = 0$ eventually for all i

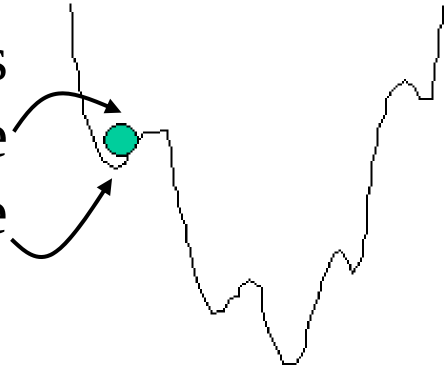
Example of Auto-Associative Memory



Pattern Completion in a Hopfield Network



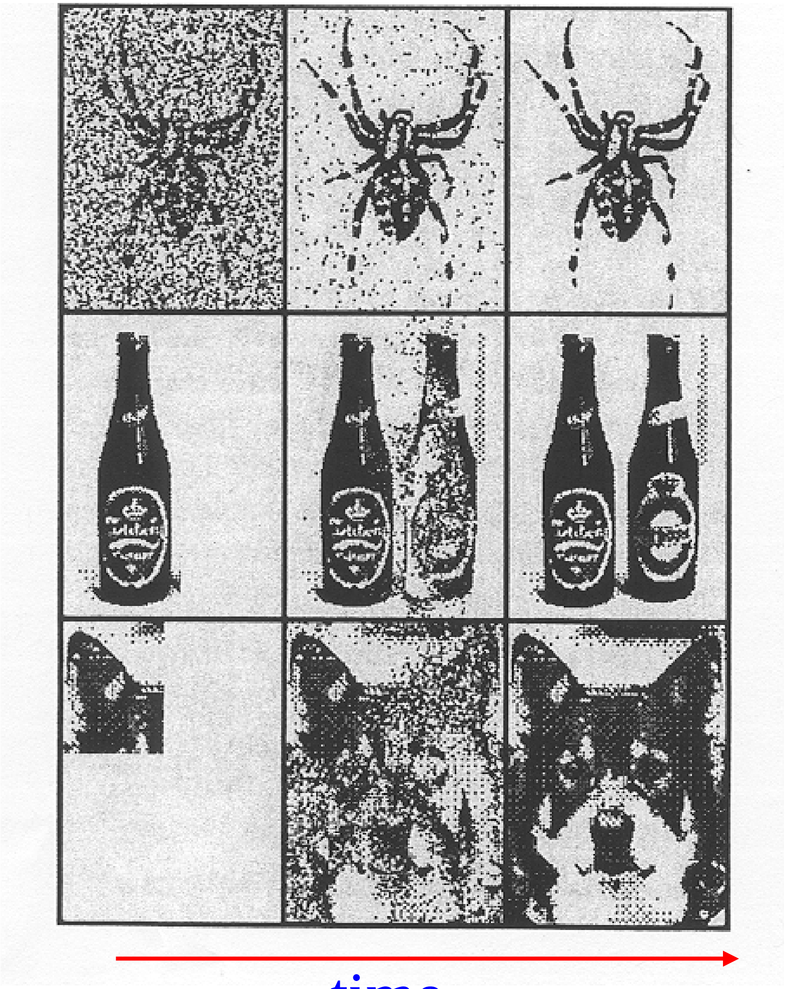
Network converges
from here
to here



Local minimum
("attractor") of cost
(or "energy") function
stores pattern

Pattern Recall in Hopfield Nets

Initial states



Stable states (fixed points)

time

Next Class: Plasticity and Learning

- Things to do:
  - Start reading Chapter 8
  - Homework #3 due next Thursday May 14
  - Start working on mini-project