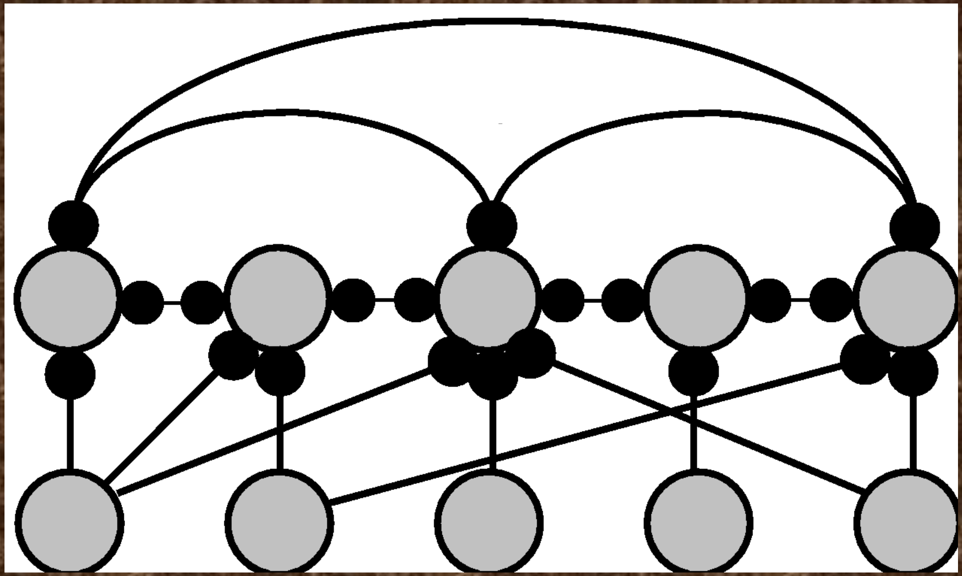
CSE/NB 528

Lecture 10: Recurrent Networks

(Chapter 7)

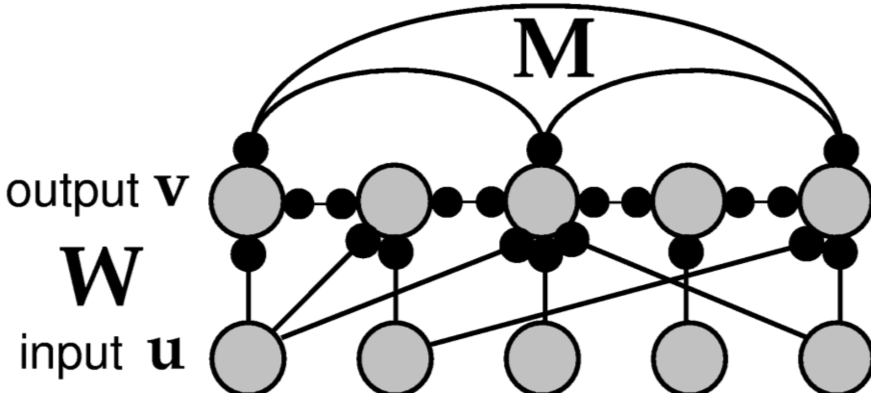


Lecture figures are from Dayan & Abbott's book
http://people.brandeis.edu/~abbott/book/index.html

What’s on our smörgåsbord today?

- Computation in Linear Recurrent Networks
  - Eigenvalue analysis
- Non-linear Recurrent Networks
  - Eigenvalue analysis
- Covered in:
  - Chapter 7 in Dayan & Abbott
  - Chapters 2, 4, and 12 in Anastasio (optional reading)

Linear Recurrent Networks



$$\tau \frac{d\mathbf{v}}{dt} = -\mathbf{v} + \mathbf{W}\mathbf{u} + \mathbf{M}\mathbf{v}$$

Output Decay Input Feedback

What can a Linear Recurrent Network do?

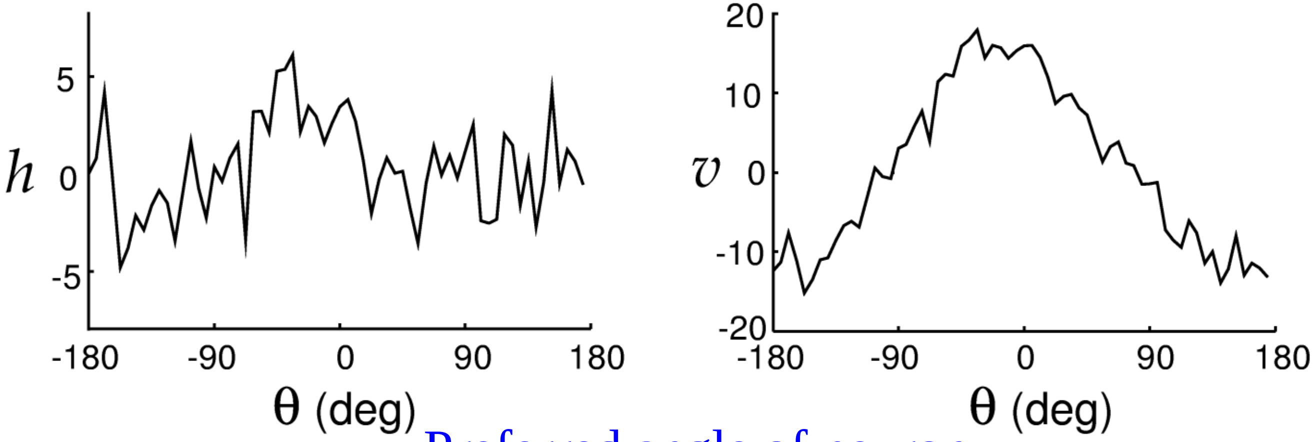
$$\tau \frac{d\mathbf{v}}{dt} = -\mathbf{v} + \mathrm{W}\mathbf{u} + \mathrm{M}\mathbf{v}$$

On-Board analysis based on eigenvectors of recurrent weight matrix M

Amplification in a Linear Recurrent Network

Input
(noisy cosine)

Output



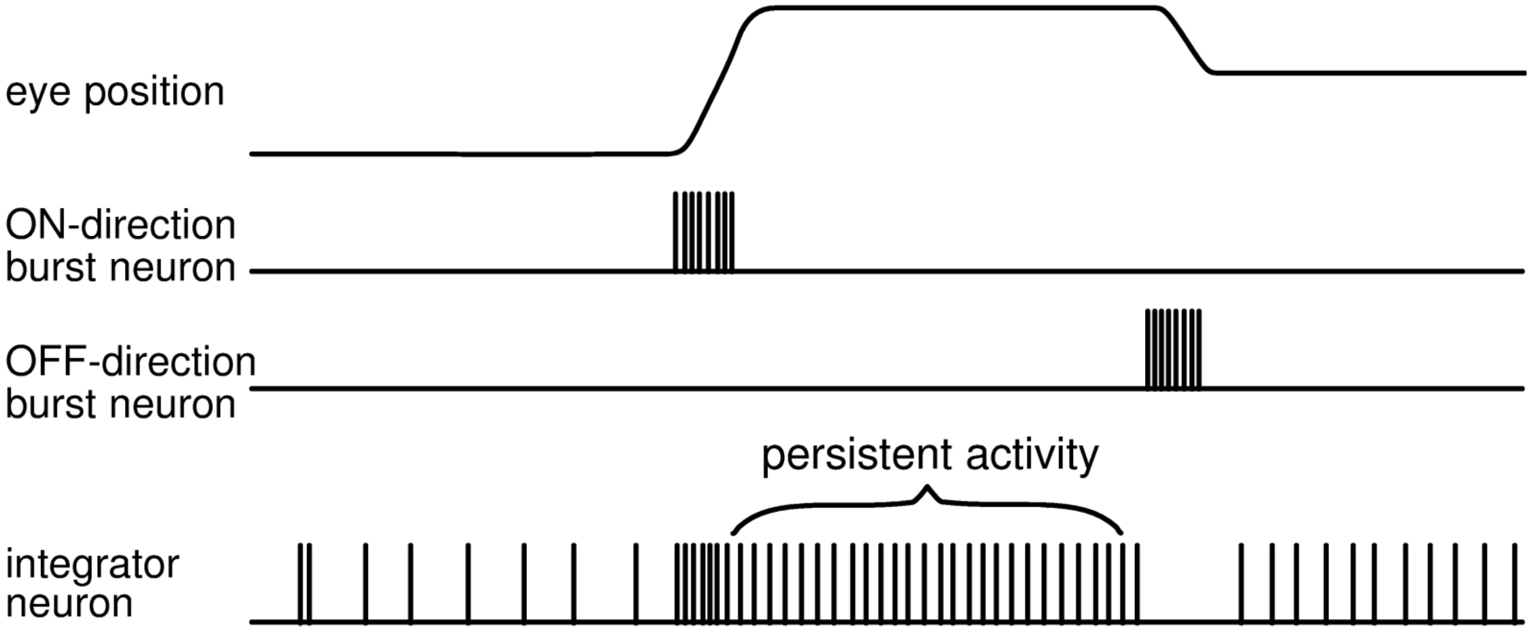
Preferred angle of neuron

$$M(\theta,\theta') = \frac{\lambda_1}{\pi}\cos(\theta-\theta')$$

All eigenvalues = 0 except $\lambda_1 = 0.9$ *i.e. amplification* $= \dfrac{1}{1-\lambda_1} = 10$

(See section 7.4 in Dayan & Abbott)

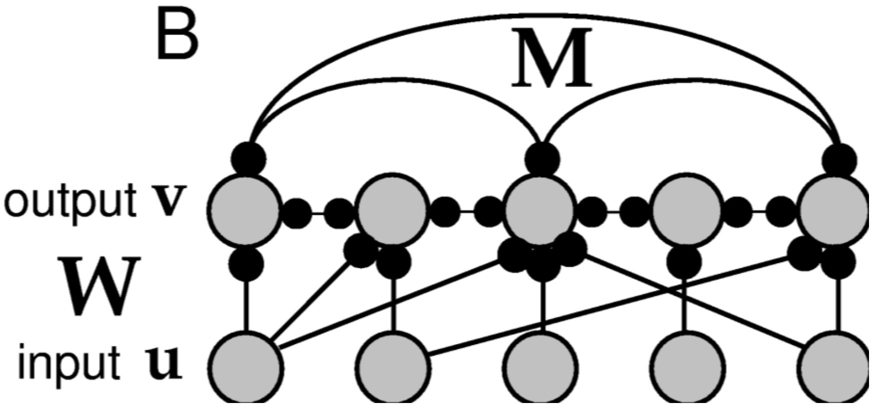
Memory for Maintaining Eye Position



Input: Bursts of spikes from brain stem oculomotor neurons

Output: Memory of eye position in medial vestibular nucleus

Nonlinear Recurrent Networks



$$\tau \frac{d\mathbf{v}}{dt} = -\mathbf{v} + \overset{\swarrow}{F}(\mathrm{W}\mathbf{u} + \mathrm{M}\mathbf{v})$$

Output Decay Input Feedback

(Convenient to use W**u** = **h**)

From Discrete to Continuous Nonlinear Recurrent Networks

$$\left.\begin{aligned}&\tau \frac{d\mathbf{v}}{dt} = -\mathbf{v} + F(\mathbf{h} + \mathrm{M}\mathbf{v}) \ \text{ or,}\\[2em]&\tau \frac{dv_i}{dt} = -v_i + F(h_i + \sum_j M_{ij} v_j)\end{aligned}\right\}$$

Discrete case
(small number of neurons)

Continuous case (in the limit of large numbers of neurons):

$$\tau \frac{dv(\theta)}{dt} = -v(\theta) + F(h(\theta) + \rho_\theta \int_{-\pi}^{\pi} M(\theta, \theta')v(\theta')d\theta')$$

θ = preferred stimulus of the neuron (e.g. orientation of input)

Example of a Continuous Recurrent Network

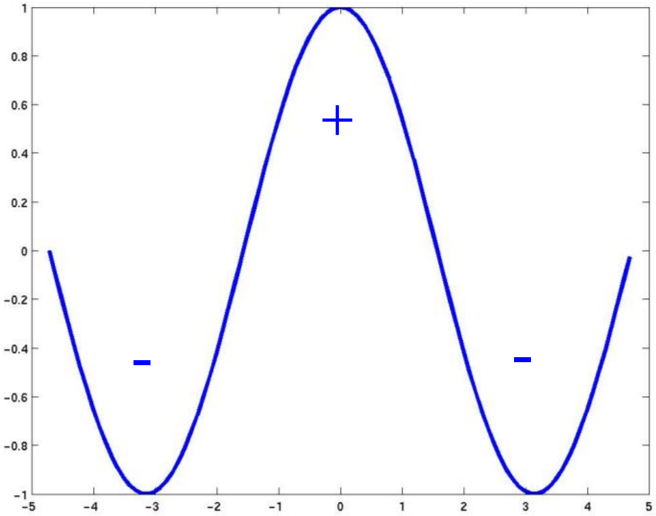
Choose *F* = rectification nonlinearity:

$$\tau \frac{dv(\theta)}{dt} = -v(\theta) + \left[h(\theta) + \int_{-\pi}^{\pi} M(\theta,\theta')v(\theta')d\theta' \right]^{+}$$

Choose recurrent connections *M* = cosine function of relative angle

$$M(\theta,\theta') = \frac{\lambda_1}{\pi}\cos(\theta - \theta')$$

Excitation nearby,
Inhibition further away

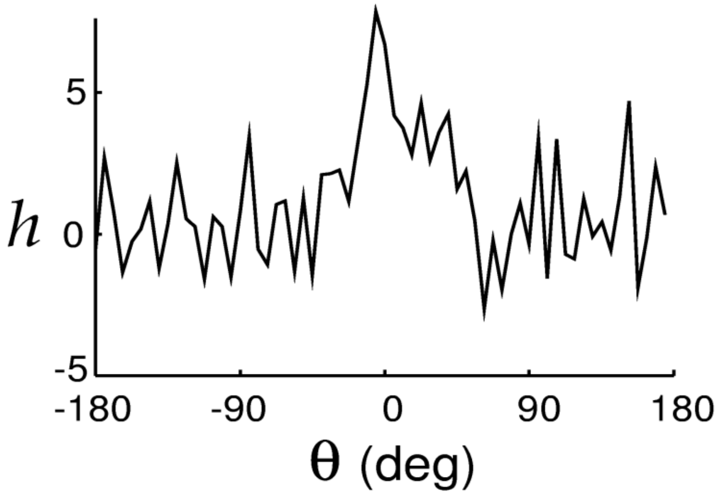


$(\theta - \theta')$

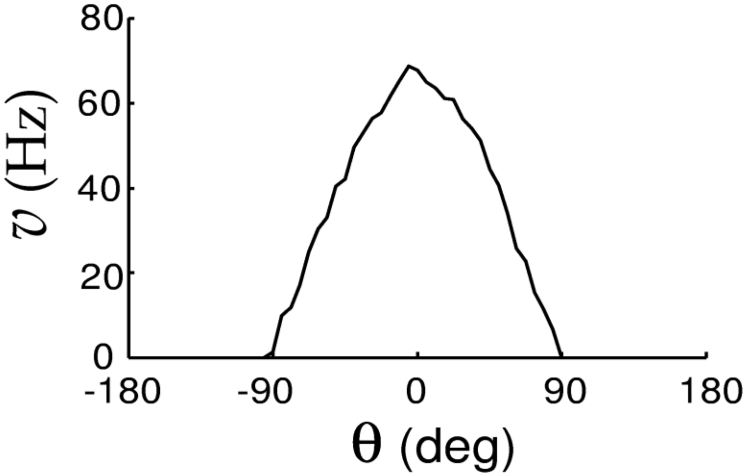
Is *M* symmetric? $M(\theta, \theta') = M(\theta', \theta')$?

Amplification in a Nonlinear Recurrent Network

Input



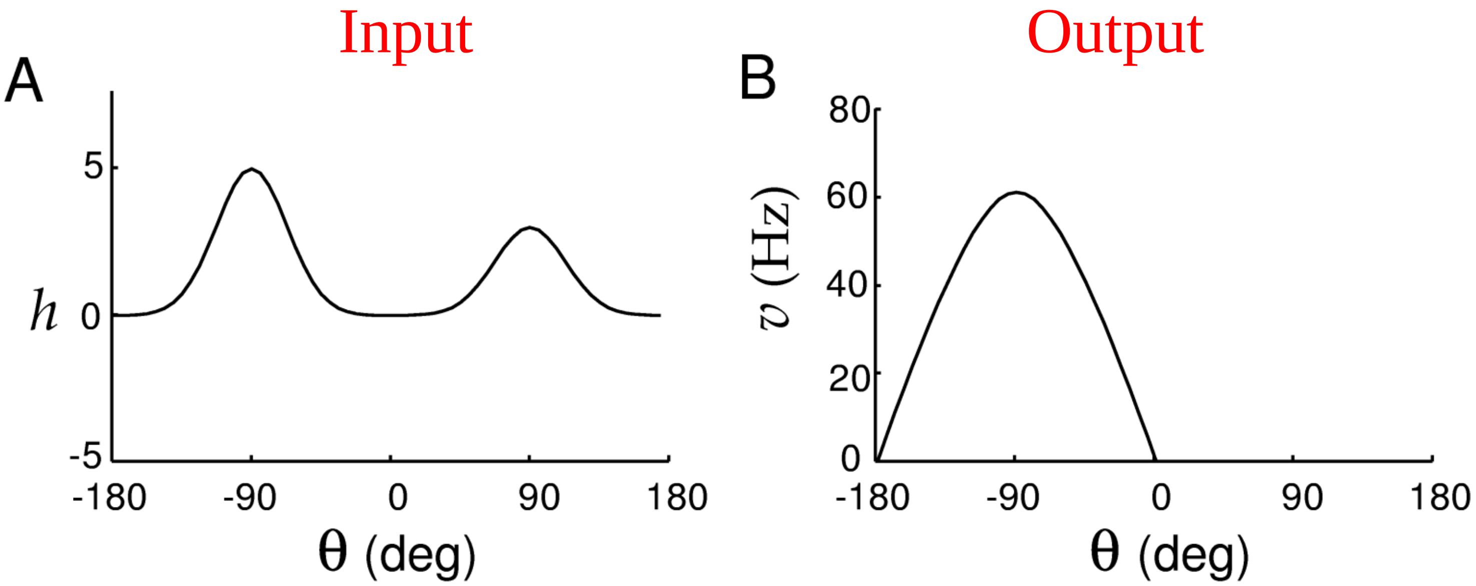
Output



$$\tau \frac{dv(\theta)}{dt} = -v(\theta) + \left[h(\theta) + \frac{\lambda_1}{\pi} \int_{-\pi}^{\pi} \cos(\theta - \theta')v(\theta')d\theta' \right]^+$$

$\lambda_1 = 1.9$ (but stable due to rectification)

Selective "Attention" in a Nonlinear Recurrent Network

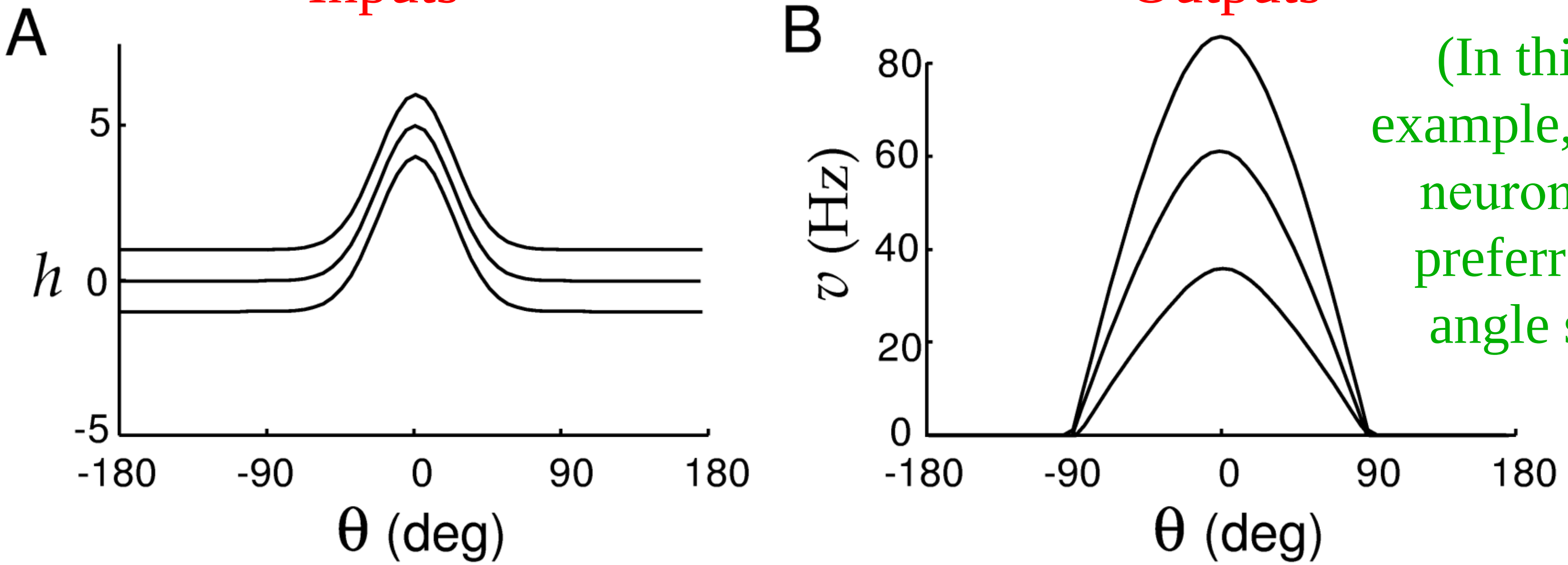


Network performs "winner-takes-all" input selection

Gain Modulation in a Nonlinear Recurrent Network

Inputs

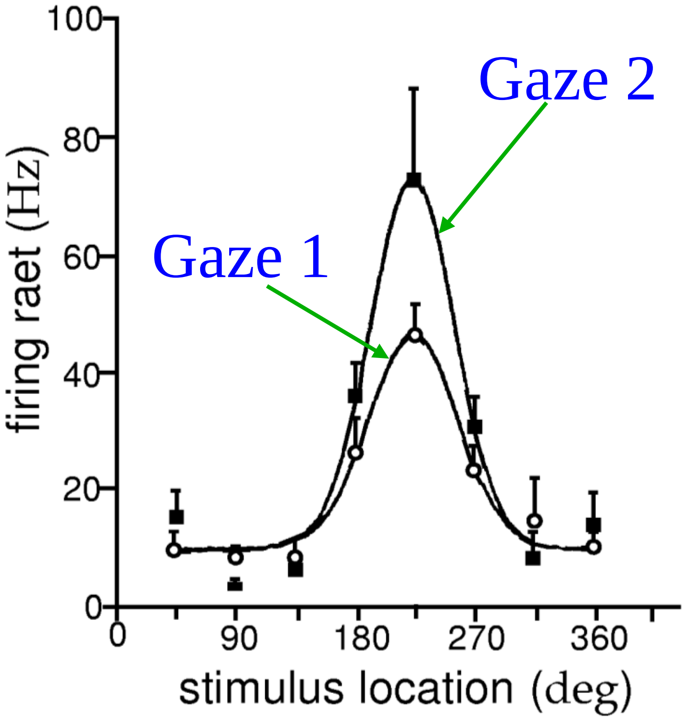
Outputs



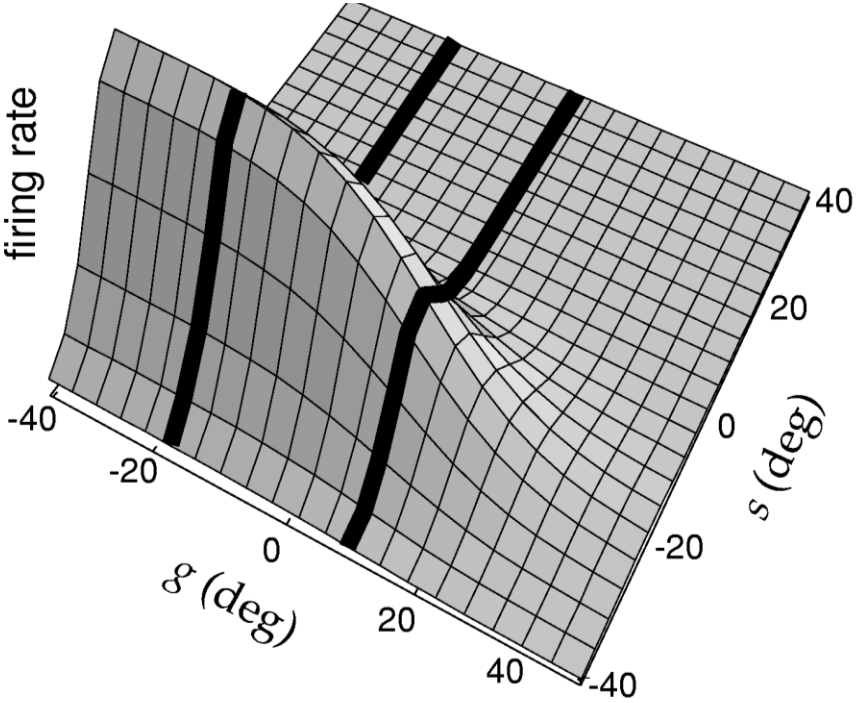
(In this example, θ = neuron's preferred angle s)

Changing the level of input by adding g multiplies the output

Gain Modulation in Parietal Cortex Neurons



Responses of Area 7a neuron

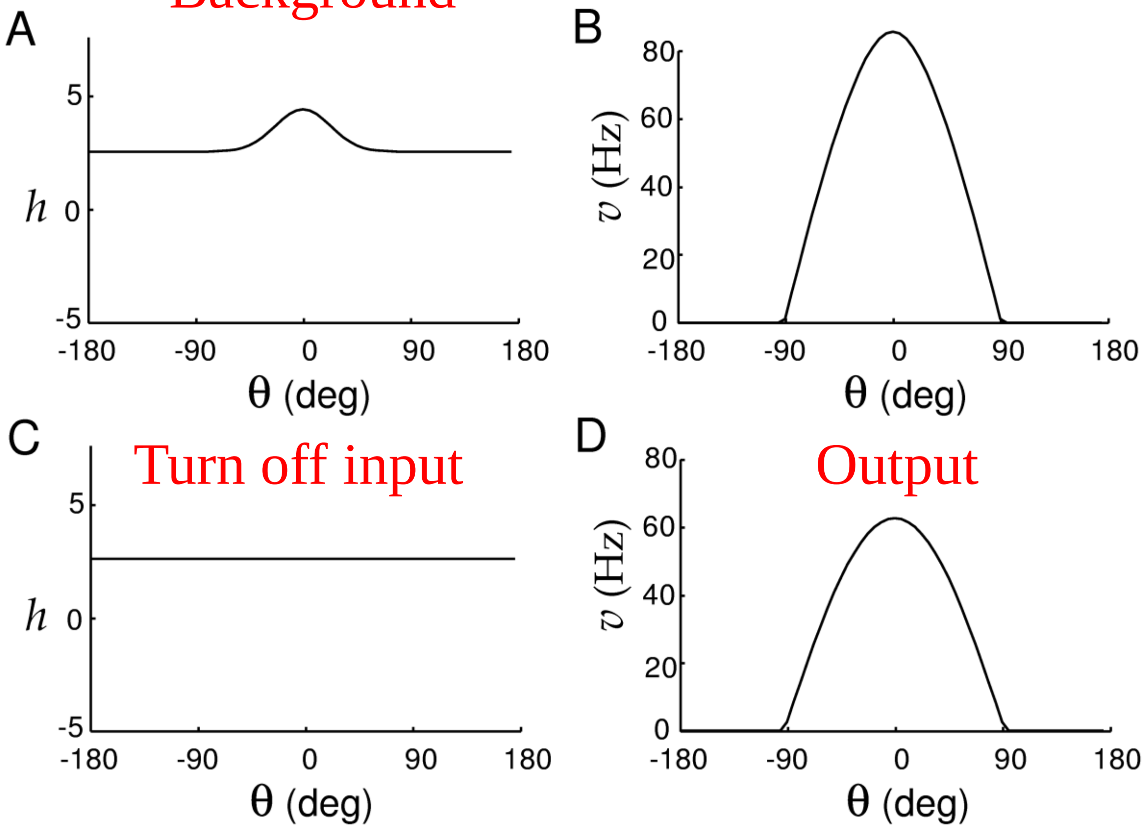


Example of a gain-modulated tuning curve

Short-Term Memory Storage in a Nonlinear Recurrent Network

Local Input + Background

Output



Turn off input

Output

Network maintains a *memory of previous activity* when input is turned off.

Similar to “short-term memory” or “working memory” in prefrontal cortex

Memory is maintained by recurrent activity

What about Non-Symmetric Recurrent Networks?

- Example: Network of Excitatory (E) and Inhibitory (I) Neurons
  - Connections can’t be symmetric: Why?

$$\tau_E \frac{dv_E}{dt} = -v_E + [M_{EE} v_E + M_{EI} v_I - \gamma_E]^+$$

$$\tau_I \frac{dv_I}{dt} = -v_I + [M_{II} v_I + M_{IE} v_E - \gamma_I]^+$$

Simple 2 neuron model for representing interacting populations
One excitatory neuron and one inhibitory neuron

Stability Analysis of Nonlinear Recurrent Networks

General case: $\frac{d\mathbf{v}}{dt} = \mathbf{f}(\mathbf{v})$

Suppose $\mathbf{v}_\infty$ is a fixed point (i.e., $\mathbf{f}(\mathbf{v}_\infty) = 0$)

Near $\mathbf{v}_\infty$, $\mathbf{v}(t) = \mathbf{v}_\infty + \boldsymbol{\varepsilon}(t)$ (i.e., $\frac{d\mathbf{v}}{dt} = \frac{d\boldsymbol{\varepsilon}}{dt}$)

Taylor expansion: $\mathbf{f}(\mathbf{v}(t)) = \mathbf{f}(\mathbf{v}_\infty) + \left.\frac{\partial \mathbf{f}}{\partial \mathbf{v}}\right|_{\mathbf{v}_\infty} \boldsymbol{\varepsilon}(t)$

$$i.e.\quad \frac{d\mathbf{v}}{dt} = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{v}}\right|_{\mathbf{v}_\infty} \boldsymbol{\varepsilon}(t) = J \cdot \boldsymbol{\varepsilon}(t) = \frac{d\boldsymbol{\varepsilon}}{dt}$$

J is the "Jacobian matrix"

Derive solution for **v**(*t*) based on eigen-analysis of *J*

Eigenvalues of *J* determine stability of network

(see Mathematical Appendix A.3 in textbook)

Example: Non-Symmetric Recurrent Networks

- Specific Network of Excitatory (E) and Inhibitory (I) Neurons:

$$\overset{10\text{ ms}}{\tau_E}\frac{dv_E}{dt} = -v_E + \left[\overset{1.25}{M_{EE}}v_E + \overset{-1}{M_{EI}}v_I - \overset{-10}{\gamma_E}\right]^+$$

$$\tau_I\frac{dv_I}{dt} = -v_I + \left[\underset{0}{M_{II}}v_I + \underset{1}{M_{IE}}v_E - \underset{10}{\gamma_I}\right]^+$$

Parameter
we will vary to
study the network

Linear Stability Analysis

$$\frac{dv_E}{dt} = \frac{-v_E + \left[M_{EE}v_E + M_{EI}v_I - \gamma_E\right]}{\tau_E}$$

$$\frac{dv_I}{dt} = \frac{-v_I + \left[M_{II}v_I + M_{IE}v_E - \gamma_I\right]}{\tau_I}$$

Take derivatives of right hand side with respect to both v_E and v_I

- Matrix of derivatives (the “Jacobian Matrix”):

$$J = \begin{bmatrix} \frac{(M_{EE}-1)}{\tau_E} & \frac{M_{EI}}{\tau_E} \\ \frac{M_{IE}}{\tau_I} & \frac{(M_{II}-1)}{\tau_I} \end{bmatrix}$$

Compute the Eigenvalues

- Jacobian Matrix:

$$J = \begin{bmatrix} \frac{(M_{EE}-1)}{\tau_E} & \frac{M_{EI}}{\tau_E} \\ \frac{M_{IE}}{\tau_I} & \frac{(M_{II}-1)}{\tau_I} \end{bmatrix}$$

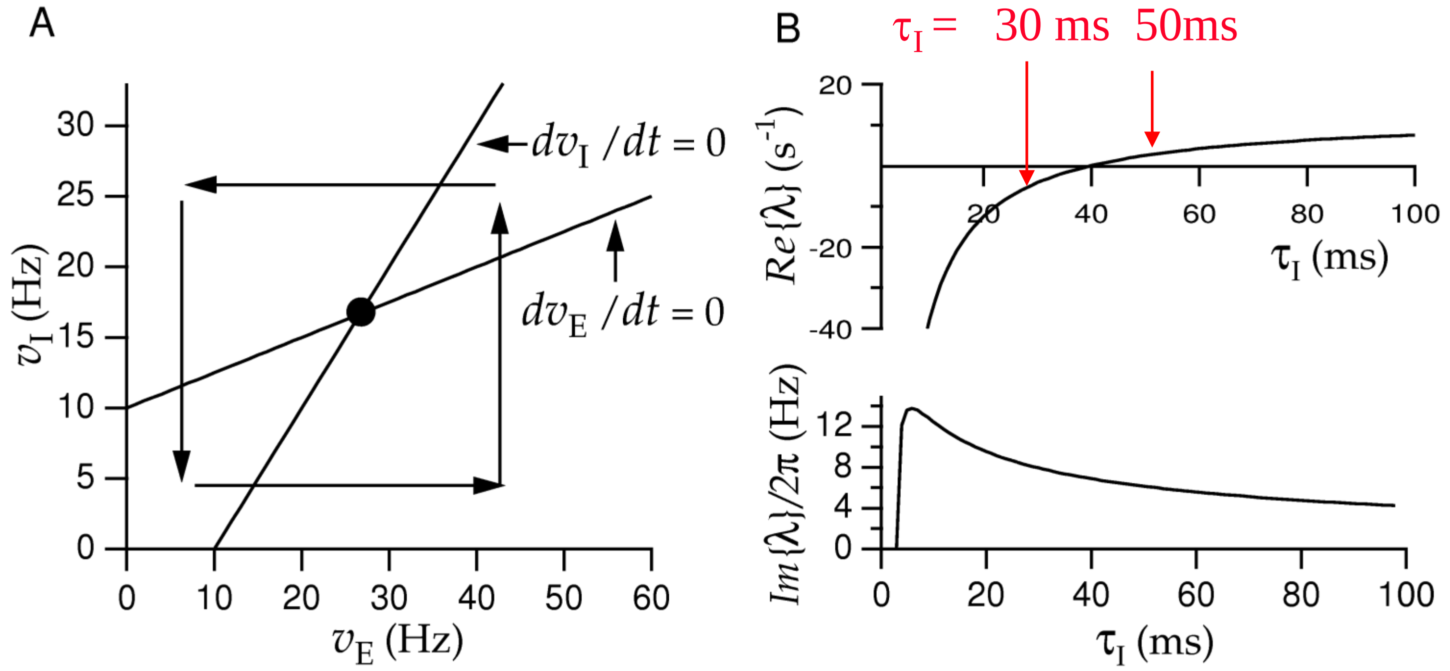
- Its two eigenvalues (obtained by solving *det*(J – λI) = 0):

$$\lambda = \frac{1}{2}\left(\frac{(M_{EE}-1)}{\tau_E} + \frac{(M_{II}-1)}{\tau_I} \pm \sqrt{\left(\frac{M_{EE}-1}{\tau_E} - \frac{M_{II}-1}{\tau_I} \right)^2 + 4\frac{M_{EI}M_{IE}}{\tau_E \tau_I}} \right)$$

Different dynamics depending on real and imaginary parts of λ

(see pages 410-412 of Appendix in Text)

Phase Plane and Eigenvalue Analysis



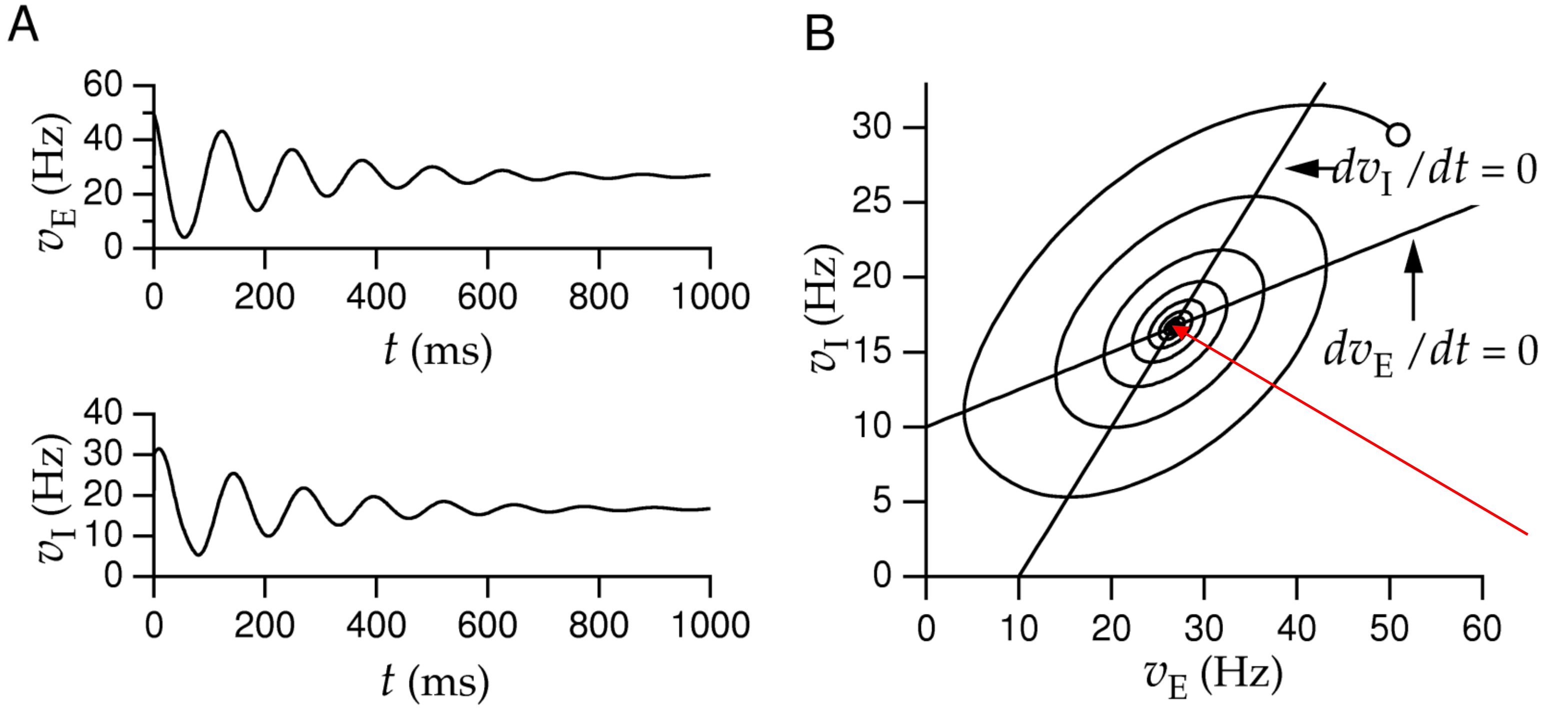
$\tau_I =$ 30 ms 50ms

$$10\frac{dv_E}{dt} = -v_E + [1.25v_E - v_I + 10]^+$$

$$\tau_I \frac{dv_I}{dt} = -v_I + [0 \cdot v_I + v_E - 10]^+$$

Damped Oscillations in the Network

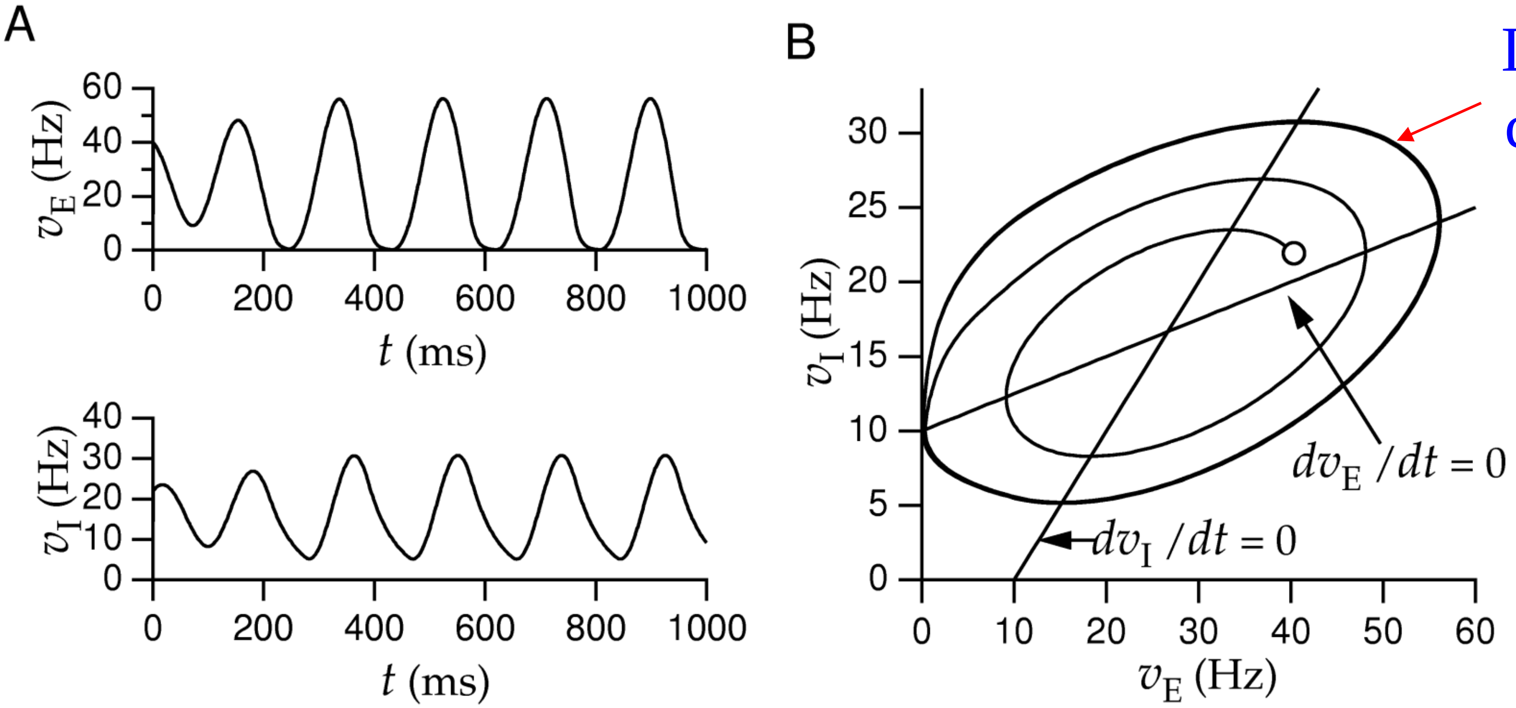
$\tau_I = 30$ ms (negative real eigenvalue)



Stable Fixed Point

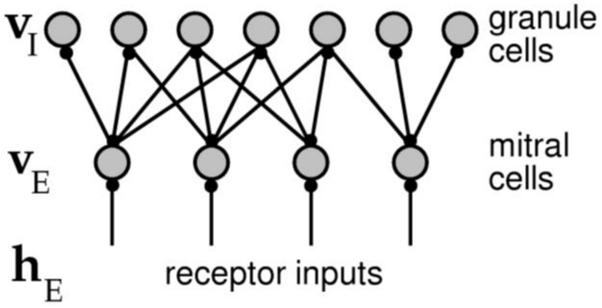
Unstable Behavior and Limit Cycle

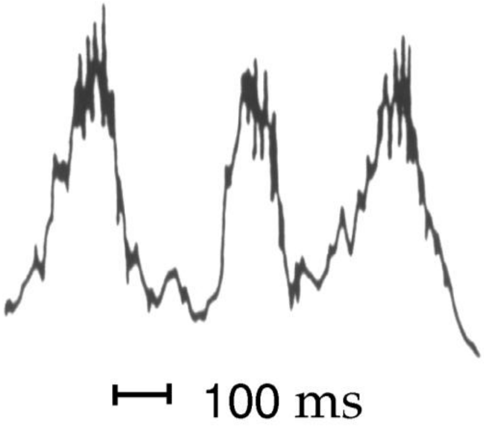
$\tau_I = 50$ ms (positive real eigenvalue)



Limit cycle

Oscillatory Activity in Real Networks





Activity in rabbit (or wabbit) olfactory bulb during 3 sniffs

(see Chapter 7 in textbook for details)

- Things to do:
  - Start reading Chapter 8 in D & A
  - Homework #2 due Sunday May 8
  - Start working on final project