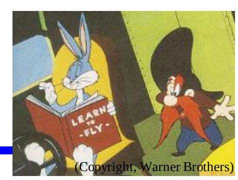
CSE/NEURO 528
Lecture 10: Plasticity and Learning
(Chapter 8)

Gameplan

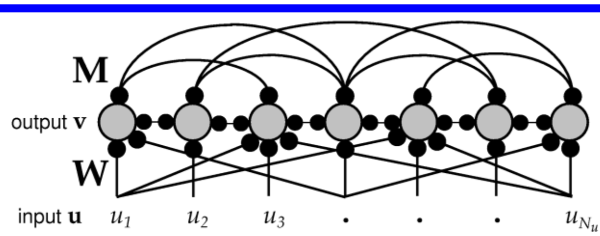


- Plasticity and Learning
  - Types: Unsupervised, Supervised, and Reinforcement learning
- Today: Unsupervised Learning
  - Hebb rule and its variants (Covariance, Oja rule)
  - Mathematical formulation
  - Stability analysis of learning rules

So far, we have been analyzing networks with *fixed* sets of synaptic weights W and M
(based on eigenvalues of M etc.)

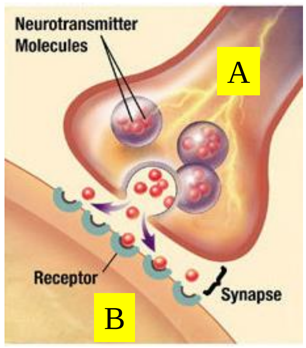
Can synaptic weights be adapted in response to inputs?

Plasticity and Learning: Adapting the Connections

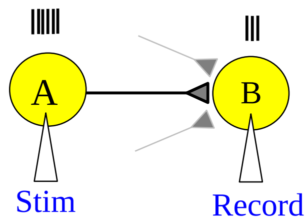


- Question 1: How do we adapt the synaptic weights W and M to solve useful tasks?
- Question 2: How does the brain do it?

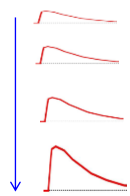
Synaptic Plasticity in the Brain



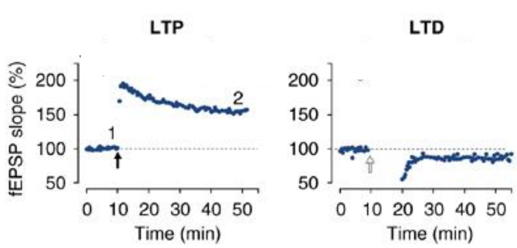
How can we measure plasticity using electrodes?



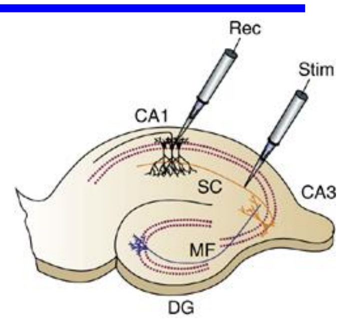
Increase in EPSP size for same input over time



Data from an Experiment



LTP = Long Term Potentiation
LTD = Long Term Depression

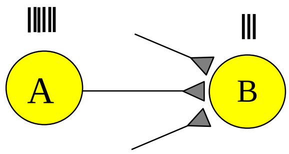


Hippocampus

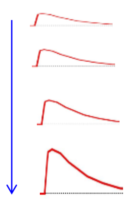
(From: Citri and Malenka, Neuropsychopharmacology, 2008)

Long Term Potentiation (LTP)

LTP = Experimentally observed *increase* in synaptic strength that lasts for hours or days



Increase in EPSP size for same input over time



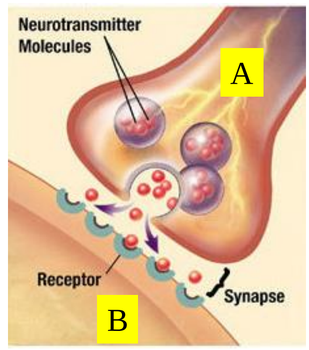
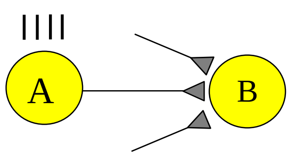


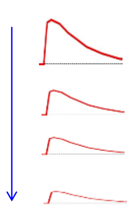
Image Source: Wikimedia Commons

Long Term Depression (LTD)

LTD = Experimentally observed *decrease* in synaptic strength that lasts for hours or days



Decrease in EPSP size for same input over time



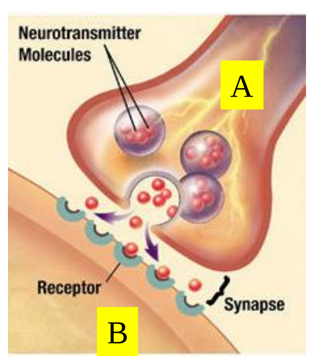
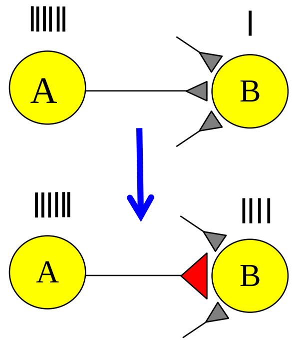
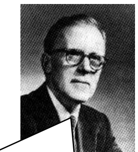


Image Source: Wikimedia Commons

Hebb's Learning Rule





If neuron A repeatedly takes part in firing neuron B, then the synapse from A to B is strengthened

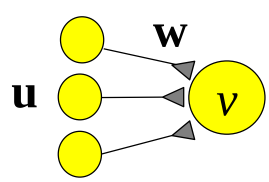
"Neurons that fire together wire together!"

Image Source: Wikimedia Commons

Formalizing Hebb's Rule

- Consider a single linear neuron with steady state output:

  $v = \mathbf{w} \cdot \mathbf{u} = \mathbf{w}^T \mathbf{u} = \mathbf{u}^T \mathbf{w}$



- Basic Hebb Rule: $\tau_w \frac{d\mathbf{w}}{dt} = \mathbf{u}v$

  Discrete Implementation:

$$\tau_w \frac{\mathbf{w}(t+\Delta t) - \mathbf{w}(t)}{\Delta t} = \mathbf{u}v \quad (\text{or } \mathbf{w}(t+\Delta t) = \mathbf{w}(t) + \frac{\Delta t}{\tau_w}\mathbf{u}v))$$

$$\mathbf{w}_{i+1} = \mathbf{w}_i + \varepsilon \cdot \mathbf{u}v \quad (\text{or } \Delta \mathbf{w} = \varepsilon \cdot \mathbf{u}v)$$

What is the average effect of the Hebb rule?

- Hebb Rule: $\tau_w \frac{d\mathbf{w}}{dt} = \mathbf{u}v$
- Average effect of the rule:

$$\tau_w \frac{d\mathbf{w}}{dt} = \langle \mathbf{u}v \rangle_{\mathbf{u}} = \langle \mathbf{u}\mathbf{u}^T\mathbf{w} \rangle_{\mathbf{u}} = \langle \mathbf{u}\mathbf{u}^T \rangle_{\mathbf{u}} \mathbf{w} = Q\mathbf{w}$$

- Q is the input correlation matrix: $Q = \langle \mathbf{u}\mathbf{u}^T \rangle_{\mathbf{u}}$

Covariance Rule

- Hebb rule only increases synaptic weights (LTP)
  - ➪ What about LTD?
- Covariance rule:

$$\tau_w \frac{d\mathbf{w}}{dt} = \mathbf{u}(v - \langle v \rangle)$$

(Note: LTD for low or no output given some input)

- Average effect of the rule:

$$\tau_w \frac{d\mathbf{w}}{dt} = \langle \mathbf{u}(v - \langle v \rangle) \rangle_{\mathbf{u}} = \langle \mathbf{u}(\mathbf{u}^T - \langle \mathbf{u} \rangle^T)\mathbf{w} \rangle_{\mathbf{u}} = \left(\langle \mathbf{u}\mathbf{u}^T \rangle - \langle \mathbf{u} \rangle \langle \mathbf{u} \rangle^T \right) \mathbf{w}$$

$$= C\mathbf{w} \quad (C \text{ is the input covariance matrix } \langle \mathbf{u}\mathbf{u}^T \rangle - \langle \mathbf{u} \rangle \langle \mathbf{u} \rangle^T)$$

Are these learning rules stable?

- Does **w** converge to a stable value or explode?
  - Look at what happens to the length of **w** over time
- Hebb rule: $\tau_w \frac{d\mathbf{w}}{dt} = \mathbf{u}v$

$$\frac{d\|\mathbf{w}\|^2}{dt} = 2\mathbf{w}^T \frac{d\mathbf{w}}{dt} = 2\mathbf{w}^T(\mathbf{u}v/\tau_w) = \frac{2}{\tau_w}v^2 > 0$$

w grows without bound!

- Covariance rule: $\tau_w \frac{d\mathbf{w}}{dt} = \mathbf{u}(v - \langle v \rangle)$

$$\frac{d\|\mathbf{w}\|^2}{dt} = 2\mathbf{w}^T \frac{d\mathbf{w}}{dt} = 2\mathbf{w}^T(\mathbf{u}(v - \langle v \rangle)/\tau_w) = \frac{2}{\tau_w}(v^2 - v\langle v \rangle)$$

Averaging RHS, $\frac{d\|\mathbf{w}\|^2}{dt} = \frac{2}{\tau_w}(\langle v^2 \rangle - \langle v \rangle^2) = \frac{2}{\tau_w}\sigma_v{}^2 > 0$

w grows without bound!

Oja's Rule for Hebbian Learning

- Oja's rule: $\tau_w \frac{d\mathbf{w}}{dt} = \mathbf{u}v - \alpha v^2 \mathbf{w}$ $\quad (\alpha > 0)$
- Stable?

$$\frac{d\|\mathbf{w}\|^2}{dt} = 2\mathbf{w}^T \frac{d\mathbf{w}}{dt} = \frac{2}{\tau_w}\mathbf{w}^T(\mathbf{u}v - \alpha v^2 \mathbf{w}) = \frac{2}{\tau_w}(v^2 - \alpha v^2 \mathbf{w}^T \mathbf{w})$$

i.e., $\tau_w \frac{d\|\mathbf{w}\|^2}{dt} = 2v^2(1 - \alpha\|\mathbf{w}\|^2)$

At steady state: $\|\mathbf{w}\|^2 = \frac{1}{\alpha}$ i.e., $\|\mathbf{w}\| = \frac{1}{\sqrt{\alpha}}$

w does not grow without bound, i.e.,
Oja's rule is stable!

Summary: Hebbian Learning

- Hebb rule:

  $$\tau_w \frac{d\mathbf{w}}{dt} = \mathbf{u}v$$ Unstable (unless constraint on $\|\mathbf{w}\|$ is imposed)

- Covariance rule:

  $$\tau_w \frac{d\mathbf{w}}{dt} = \mathbf{u}(v - \langle v \rangle)$$ Unstable (unless constraint on $\|\mathbf{w}\|$ is imposed)

- Oja's rule:

  $$\tau_w \frac{d\mathbf{w}}{dt} = \mathbf{u}v - \alpha v^2 \mathbf{w}$$ Stable $\|\mathbf{w}\| \to \frac{1}{\sqrt{\alpha}}$

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What does Hebbian Learning do anyway?

- Start with the averaged Hebb rule: $\tau_w \frac{d\mathbf{w}}{dt} = Q\mathbf{w}$
- How do we solve this equation to find $\mathbf{w}(t)$?
  ➪ Eigenvectors to the rescue (again)!
- Write $\mathbf{w}(t)$ in terms of *eigenvectors* of Q: $\mathbf{w}(t) = \sum_i c_i(t)\mathbf{e}_i$
- Substitute in Hebb rule diff. eq. and simplify as before:

  $$\tau_w \frac{dc_i}{dt} = \lambda_i c_i \quad \text{i.e.,} \quad c_i(t) = c_i(0)\exp(\lambda_i t / \tau_w)$$

  $$\mathbf{w}(t) = \sum_i c_i(t)\mathbf{e}_i = \sum_i c_i(0)\exp(\lambda_i t / \tau_w)\mathbf{e}_i$$

  For large t, largest eigenvalue term dominates: $\mathbf{w}(t) \propto \mathbf{e}_1$

  (For Oja's rule: $\mathbf{w}(t) = \frac{\mathbf{e}_1}{\sqrt{\alpha}}$)

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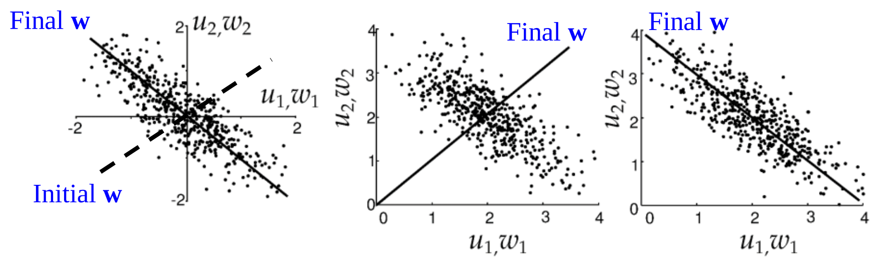
The Brain can do Statistics!*

Hebbian Learning implements *Principal Component Analysis* (PCA)

Hebb Rule	Hebb Rule	Covariance Rule
Input mean = (0,0)	Input mean = (2,2)	Input mean = (2,2)



Hebbian learning learns a weight vector aligned with the principal eigenvector of input correlation/covariance matrix (i.e., direction of maximum variance)

*See a previous lecture for "The Brain can do Calculus!"

Image Source: Dayan & Abbott textbook

Next Class: Unsupervised to Supervised Learning

- Things to do:
  - Finish Chapter 8 and Start Chapter 10
  - Homework 3 due on Sunday, February 19
  - Start mini-project