CSE 546 Review Problems

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1 Probability and Statistics

Many of these are borrowed from or inspired by problems and examples in *All of Statistics* by Wasserman.

1. *Ross, Ch 5, problem 29.* Let X be a random variable with a continuous cumulative distribution function F. Define the random variable Y by $Y = F(X)$. Show that Y is uniformly distributed over $(0, 1)$.

2. *Ross, Ch 5, problem 30.* Let X have probability density f. Find the probability density function of the random variable Y defined by $Y = aX + b$.

3. Let X be a positive random variable with probability density function f so that $\mathbb{P}(X > 0) = 1$ and $\mathbb{E}[X] = \int_0^\infty xf(x)dx$. Show that $\mathbb{E}[X] = \int_0^\infty \mathbb{P}(X \geq x)dx$.

4. Let $X \sim$ unifom$(0, 1)$. Let $0 < a < b < 1$. Let

$$Y = \begin{cases} 1 & \text{if } 0 < X < b \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad Z = \begin{cases} 1 & \text{if } a < X < 1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Are Y and Z independent? Why or why not?

   (b) Find $\mathbb{E}[Y|Z = z]$

5. Let $X_1, \ldots, X_n \sim$ uniform$(0, 1)$ and let $Y = \max\{X_1, \ldots, X_n\}$. Find $\mathbb{E}[Y_n]$

6. Let $X_1, \ldots, X_n$ be independent random variables expectation $\mathbb{E}[X_i] = \mu_i$ and variance $\mathbb{V}(X_i) = \mathbb{E}[(X_i - \mu_i)^2] = \sigma_i^2$. For scalars $a_1, \ldots, a_n$ define $Z = \sum_{i=1}^n a_i X_i$. What is $\mathbb{E}[Z]$ and $\mathbb{V}(Z) = \mathbb{E}[(Z - \mathbb{E}[Z])^2]$?

7. For $i = 1, \ldots, n$ let $X_i \overset{iid}{\sim} \mathcal{N}(\mu, \sigma^2)$. Let $\widehat{\mu} = \frac{1}{n}\sum_{i=1}^n X_i$. What is the distribution of $\widehat{\mu}$?

8. For any two random variables X, Y define $\text{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]$.

   (a) If $\mathbb{E}[Y|X = x] = x$ show that $\text{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])^2]$.

   (b) If X, Y are independent show that $\text{Cov}(X, Y) = 0$.

2 Linear Algebra

1. An *Orthogonal matrix* U is a matrix whose rows (and columns) are orthogonal vectors of unit norm. This means that $U^T U = UU^T = I$.

   (a) Show that orthogonal matrices preserve the dot product; i.e. if U is orthogonal, then

$$\langle u, v\rangle = \langle Uu, Uv\rangle$$

   where $\langle u, v\rangle = u^T v$.

   (b) If P and Q are orthogonal matrices, show that their product PQ is also orthogonal.

2. Let C and B be square matrices, and let C be invertible. Show that, for $k = 1, 2, \ldots,$

$$(CBC^{-1})^k = C(B^k)C^{-1}$$

   Hint: Begin by proving this for $k = 2$.

3. Prove that if A is a symmetric matrix with n distinct eigenvalues, then its eigenvectors are orthogonal.

4. Suppose that A is a symmetric matrix. Prove, without appealing to calculus, that the solution to $\arg\max_x x^T Ax$ s.t. $||x||_2 = 1$ is the eigenvector x_1 corresponding to the largest eigenvalue λ_1 of A.

5. The *trace* of a matrix is the sum of the diagonal entries; $Tr(A) = \sum_i A_{ii}$. If $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{m\times n}$, show that $Tr(AB) = Tr(BA)$.

6. Let $v_1, \ldots, v_n$ be a set of non-zero vectors in $\mathbb{R}^d$. Let $V = [v_1, \ldots, v_n]$ be the vectors concatenated.
   (a) What is the minimum and maximum rank of $\sum_{i=1}^n v_i v_i^T$?
   (b) What is the minimum and maximum rank of V?
   (c) Let $A \in \mathbb{R}^{D\times d}$ for $D > d$. What is the minimum and maximum rank of $\sum_{i=1}^n (Av_i)(Av_i)^T$?
   (d) What is the minimum and maximum rank of AV? What if V is rank d?