Surfaces

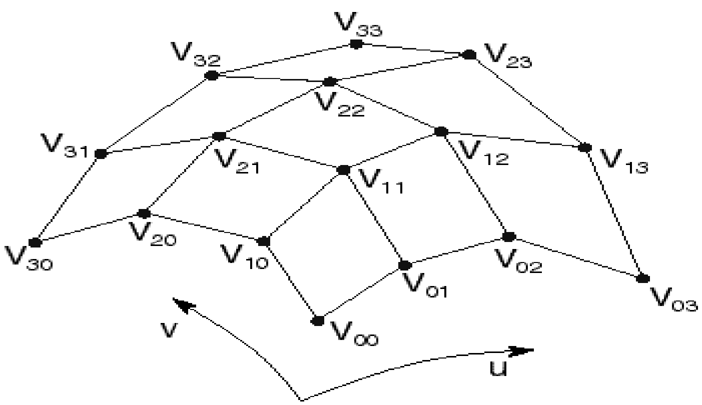
Reading

Foley et.al., Section 11.3

Recommended:

Bartels, Beatty, and Barsky. *An Introduction to Splines for use in Computer Graphics and Geometric Modeling*, 1987.

Tensor product Bézier surfaces



Given a grid of control points V_{ij}, forming a **control net**, contruct a surface $S(u,v)$ by:

- treating rows of V as control points for curves $V_0(u),\ldots, V_n(u)$.
- treating $V_0(u),\ldots, V_n(u)$ as control points for a curve parameterized by v.

Building surfaces from curves

Let the geometry vector vary by a second parameter v:

$$S(u,v) = \mathbf{U}\cdot\mathbf{M}\cdot\begin{bmatrix}\mathbf{G}_1(v)\\ \mathbf{G}_2(v)\\ \mathbf{G}_3(v)\\ \mathbf{G}_4(v)\end{bmatrix}$$

$$\mathbf{G}_i(v) = \mathbf{V}\cdot\mathbf{M}\cdot\mathbf{g}_i$$

$$\mathbf{g}_i = \begin{bmatrix}\mathbf{g}_{i1} & \mathbf{g}_{i2} & \mathbf{g}_{i3} & \mathbf{g}_{i4}\end{bmatrix}^T$$

Geometry matrices

By transposing the geometry curve we get:

$$\begin{aligned}\mathbf{G}_i(v)^T &= \left(\mathbf{V}\cdot\mathbf{M}\cdot\mathbf{g}_i\right)^T \\ &= \mathbf{g}_i^{\,T}\cdot\mathbf{M}^T\cdot\mathbf{V}^T \\ &= \left[\mathbf{g}_{i1}\quad \mathbf{g}_{i2}\quad \mathbf{g}_{i3}\quad \mathbf{g}_{i4}\right]\cdot\mathbf{M}^T\cdot\mathbf{V}^T\end{aligned}$$

Geometry matrices

Combining

$$\mathbf{G}_i(v) = \left[\mathbf{g}_{i1}\quad \mathbf{g}_{i2}\quad \mathbf{g}_{i3}\quad \mathbf{g}_{i4}\right]\cdot\mathbf{M}^T\cdot\mathbf{V}^T$$

And

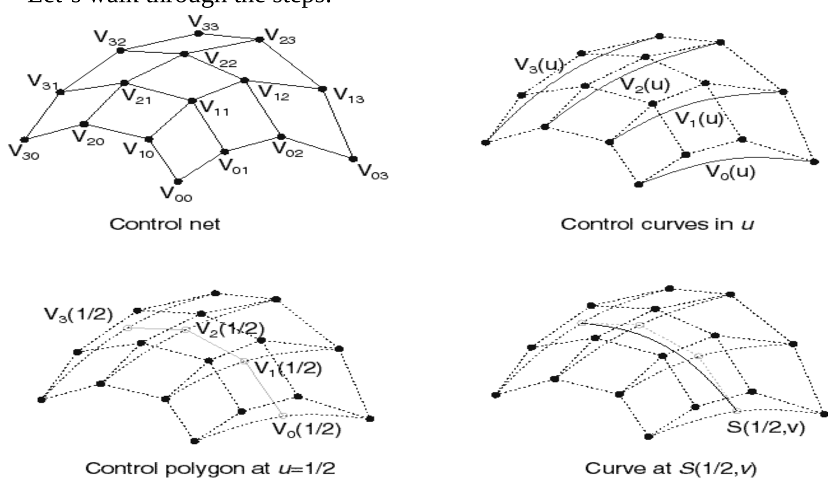
$$S(u,v) = \mathbf{U}\cdot\mathbf{M}\cdot\begin{bmatrix}\mathbf{G}_1(v)\\ \mathbf{G}_2(v)\\ \mathbf{G}_3(v)\\ \mathbf{G}_4(v)\end{bmatrix}^T$$

We get

$$S(u,v) = \mathbf{U}\cdot\mathbf{M}\cdot\begin{bmatrix}\mathbf{g}_{11} & \mathbf{g}_{12} & \mathbf{g}_{13} & \mathbf{g}_{14}\\ \mathbf{g}_{21} & \mathbf{g}_{22} & \mathbf{g}_{23} & \mathbf{g}_{24}\\ \mathbf{g}_{31} & \mathbf{g}_{32} & \mathbf{g}_{33} & \mathbf{g}_{34}\\ \mathbf{g}_{41} & \mathbf{g}_{42} & \mathbf{g}_{43} & \mathbf{g}_{44}\end{bmatrix}\mathbf{M}^T\cdot\mathbf{V}^T$$

Tensor product surfaces, cont.

Let's walk through the steps:

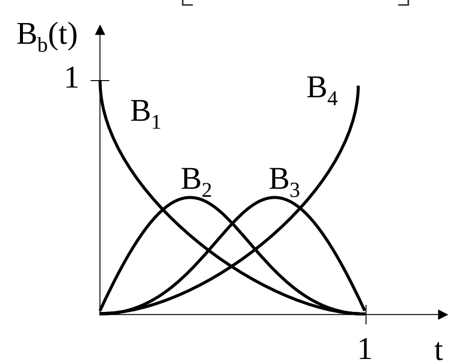


Which control points are interpolated by the surface?

Bezier Blending Functions

a.k.a. Bernstein polynomials

$$\mathbf{Q}(t) = \begin{bmatrix}t^3 & t^2 & t & 1\end{bmatrix}\begin{bmatrix}-1 & 3 & -3 & 1\\ 3 & -6 & 3 & 0\\ -3 & 3 & 0 & 0\\ 1 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\mathbf{P}_1\\ \mathbf{P}_2\\ \mathbf{P}_3\\ \mathbf{P}_4\end{bmatrix} = \mathbf{B}_b(t)\begin{bmatrix}\mathbf{P}_1\\ \mathbf{P}_2\\ \mathbf{P}_3\\ \mathbf{P}_4\end{bmatrix}$$



Matrix form

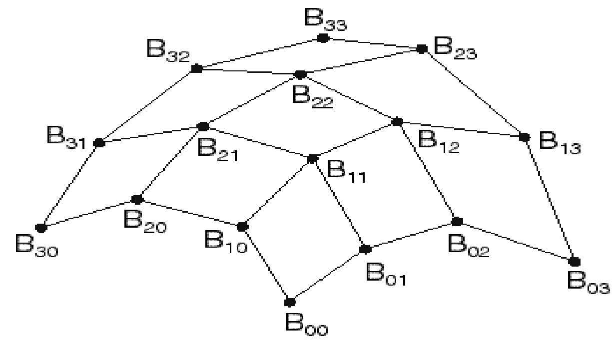
Tensor product surfaces can be written out explicitly:

$$S(u,v) = \sum_{i=0}^{n}\sum_{j=0}^{n} V_{ij} B_i^n(u) B_j^n(v)$$

$$= \begin{bmatrix} v^3 & v^2 & v & 1 \end{bmatrix} \mathbf{M}_{B\acute{e}zier} \ \mathbf{V} \ \mathbf{M}_{B\acute{e}zier}^T \begin{bmatrix} u^3 \\ u^2 \\ u \\ 1 \end{bmatrix}$$

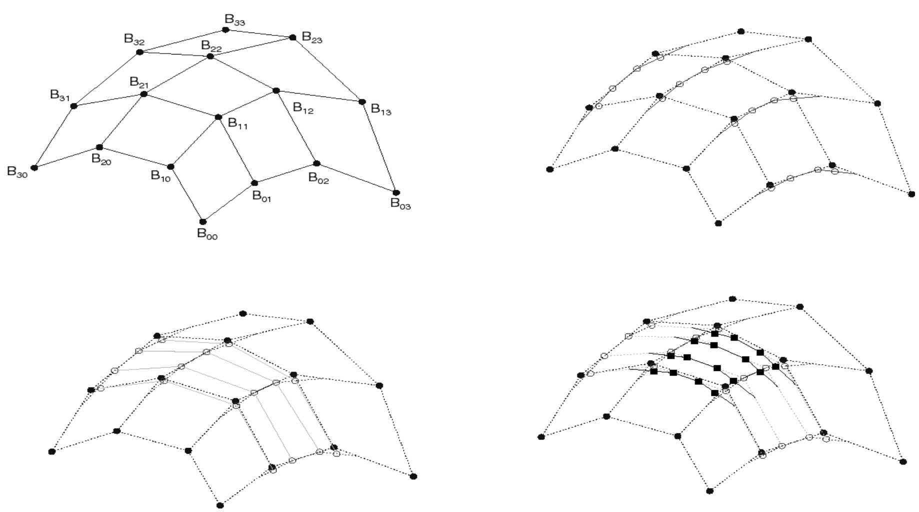
Tensor product B-spline surfaces

As with spline curves, we can piece together a sequence of Bézier surfaces to make a spline surface. If we enforce C2 continuity and local control, we get B-spline curves:



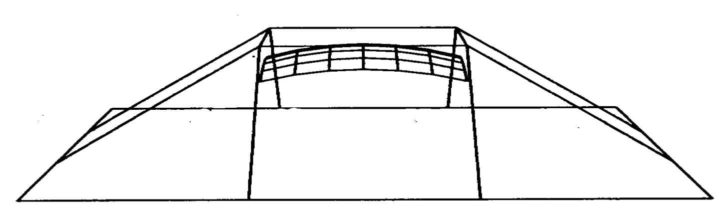
- treat rows of *B* as control points to generate Bézier control points in *u*.
- treat Bézier control points in *u* as B-spline control points in *v*.
- treat B-spline control points in *v* to generate Bézier control points in *u*.

Tensor product B-splines, cont.



Which B-spline control points are interpolated by the surface?

Tensor product B-splines, cont.

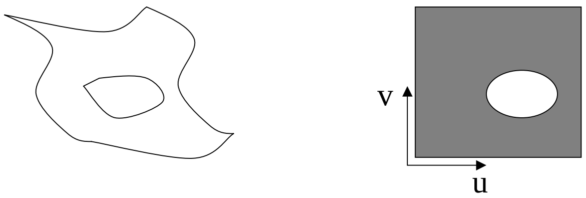


Trimmed NURBS surfaces

Uniform B-spline surfaces are a special case of NURBS surfaces.

Sometimes, we want to have control over which parts of a NURBS surface get drawn.

For example:

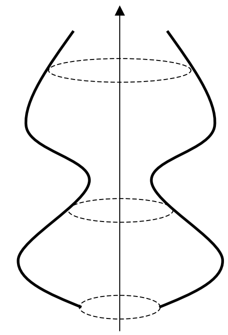


We can do this by **trimming** the *u*-*v* domain.

- Define a closed curve in the *u*-*v* domain (a **trim curve**)
- Do not draw the surface points inside of this curve.

It's really hard to maintain continuity in these regions, especially while animating.

Surfaces of revolution



Idea: rotate a 2D **profile curve** around an axis.

What kinds of shapes can you model this way?

Variations

Several variations are possible:

- Scale $C(u)$ as it moves, possibly using length of $T(v)$ as a scale factor.
- Morph $C(u)$ into some other curve $C'(u)$ as it moves along $T(v)$.
- …

Constructing surfaces of revolution

Given: A curve $C(u)$ in the *yz*-plane:

$$C(u) = \begin{bmatrix} 0 \\ c_y(u) \\ c_z(u) \\ 1 \end{bmatrix}$$

Let $R_x(\theta)$ be a rotation about the *x*-axis.

Find: A surface $S(u,v)$ which is $C(u)$ rotated about the *z*-axis.

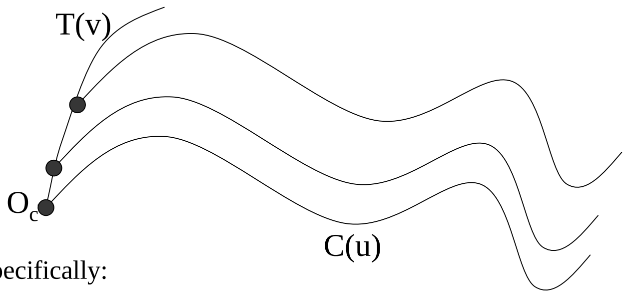
$$S(u,v) = \mathbf{R_x}(v) \cdot C(u)$$

General sweep surfaces

The **surface of revolution** is a special case of a **swept surface**.

Idea: Trace out surface $S(u,v)$ by moving a **profile curve** $C(u)$ along a **trajectory curve** $T(v)$.

$$S(u,v) = \mathbf{T}(T(v)) \cdot C(u)$$



More specifically:

- Suppose that $C(u)$ lies in an (x_c, y_c) coordinate system with origin O_c.
- For every point along $T(v)$, lay $C(u)$ so that O_c coincides with $T(v)$.

Orientation

The big issue:

- How to orient $C(u)$ as it moves along $T(v)$?

Here are two options:

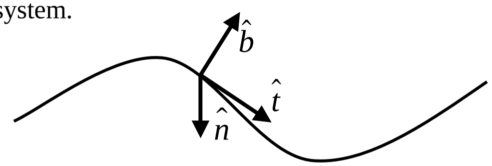
1. **Fixed** (or **static**): Just translate O_c along $T(v)$.

2. Moving. Use the **Frenet frame** of $T(v)$.

- Allows smoothly varying orientation.
- Permits surfaces of revolution, for example.

Frenet frames

Motivation: Given a curve $T(v)$, we want to attach a smoothly varying coordinate system.



To get a 3D coordinate system, we need 3 independent direction vectors.

$$\hat{t}(v) = normalize(T'(v))$$
$$\hat{b}(v) = normalize(T'(v) \times T''(v))$$
$$\hat{n}(v) = \hat{b}(v) \times \hat{t}(v)$$

As we move along $T(v)$, the Frenet frame (t,b,n) varies smoothly.

Frenet swept surfaces

Orient the profile curve $C(u)$ using the Frenet frame of the trajectory $T(v)$:

1. Put $C(u)$ in the **normal plane** *nb*.
2. Place O_c on $T(v)$.
3. Align x_c for $C(u)$ with $-n$.
4. Align y_c for $C(u)$ with b.

If $T(v)$ is a circle, you get a surface of revolution exactly?

Summary

What to take home:

- How to construct tensor product Bézier surfaces
- How to construct tensor product B-spline surfaces
- Surfaces of revolution
- Construction of swept surfaces from a profile and trajectory curve
  - With a fixed frame
  - With a Frenet frame