

Parametric surfaces

Brian Curless
CSE 557
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Reading

Required:

- ♦ Shirley, 2.5

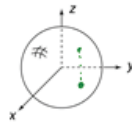
Optional

- ♦ Bartels, Beatty, and Barsky. *An Introduction to Splines for use in Computer Graphics and Geometric Modeling*, 1987.

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Mathematical surface representations

- ♦ Explicit $z=f(x,y)$ (a.k.a. a "height field")
 - what if the curve isn't a function, like a sphere?

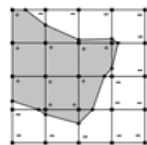


$$x^2 + y^2 + z^2 = 1$$

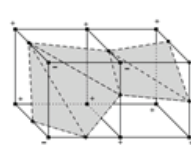
$$g(x,y,z) = x^2 + y^2 + z^2 - 1$$

- ♦ Implicit $g(x,y,z) = 0$

$N \sim \nabla g$



Isocontour from "marching squares"



Isocontour from "marching cubes"

$$\nabla g = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix}$$

- ♦ Parametric $S(u,v) = (x(u,v), y(u,v), z(u,v))$

• For the sphere:

$$x(u,v) = r \cos 2\pi v \sin \pi u$$

$$y(u,v) = r \sin 2\pi v \sin \pi u$$

$$z(u,v) = r \cos \pi u$$



As with curves, we'll focus on parametric surfaces.

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Surfaces of revolution

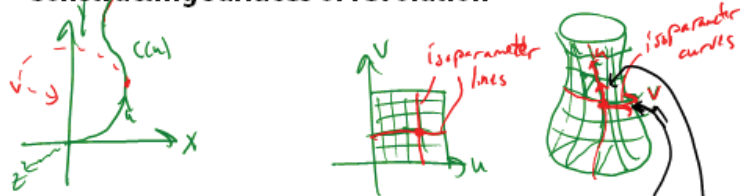


Idea: rotate a 2D **profile curve** around an axis.

What kinds of shapes can you model this way?

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Constructing surfaces of revolution



Given: A curve $C(u)$ in the xy -plane:

$$C(u) = \begin{bmatrix} c_x(u) \\ c_y(u) \\ 0 \\ 1 \end{bmatrix}$$

Let $R_x(\theta)$ be a rotation about the x -axis.

Find: A surface $S(u,v)$ which is $C(u)$ rotated about the x -axis.

Solution:

$$S(u,v) = R_y(2\pi v)C(u)$$

$$\frac{\partial S(u,v)}{\partial v} = t_v$$

$$\frac{\partial S(u,v)}{\partial u} = t_u$$

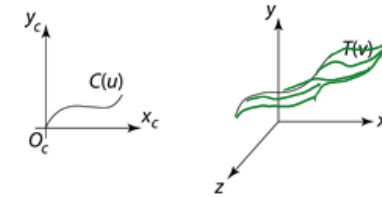
$$N = t_v \times t_u$$

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General sweep surfaces

The **surface of revolution** is a special case of a **swept surface**.

Idea: Trace out surface $S(u,v)$ by moving a **profile curve** $C(u)$ along a **trajectory curve** $T(v)$.



More specifically:

- Suppose that $C(u)$ lies in an (x_c, y_c) coordinate system with origin O_c .
- For every point along $T(v)$, lay $C(u)$ so that O_c coincides with $T(v)$.

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Orientation

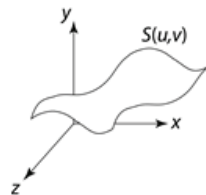


The big issue:

- How to orient $C(u)$ as it moves along $T(v)$?

Here are two options:

1. **Fixed** (or **static**): Just translate O_c along $T(v)$.



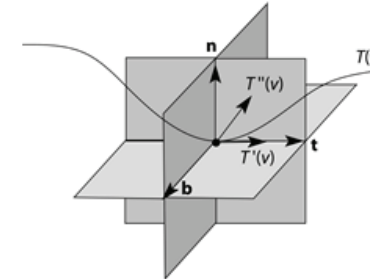
2. **Moving**. Use the **Frenet frame** of $T(v)$.

- Allows smoothly varying orientation.
- Permits surfaces of revolution, for example.

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Frenet frames

Motivation: Given a curve $T(v)$, we want to attach a smoothly varying coordinate system.



To get a 3D coordinate system, we need 3 independent direction vectors.

Tangent: $\mathbf{t}(v) = \text{normalize}[T'(v)]$

Binormal: $\mathbf{b}(v) = \text{normalize}[T'(v) \times T''(v)]$

Normal: $\mathbf{n}(v) = \mathbf{b}(v) \times \mathbf{t}(v)$

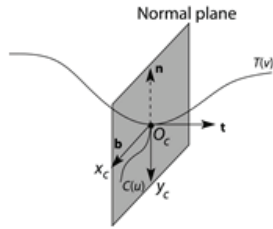
As we move along $T(v)$, the Frenet frame (t, b, n) varies smoothly.

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Frenet swept surfaces

Orient the profile curve $C(u)$ using the Frenet frame of the trajectory $T(v)$:

- Put $C(u)$ in the **normal plane**.
- Place O_c on $T(v)$.
- Align x_c for $C(u)$ with \mathbf{b} .
- Align y_c for $C(u)$ with $-\mathbf{n}$.



If $T(v)$ is a circle, you get a surface of revolution exactly!

Where might these frames be ambiguous or undetermined?



Problem: self-intersection after transport \Rightarrow oriented manifold

0: not const in a deriv.

$$1. T'(v) = 0 \Rightarrow \|T'(s)\| = 1 \text{ (arc length)}$$

$$2. T''(v) = 0 \Rightarrow \|T''(s)\| \stackrel{?}{=} 0$$

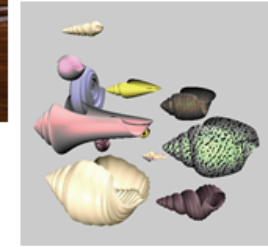
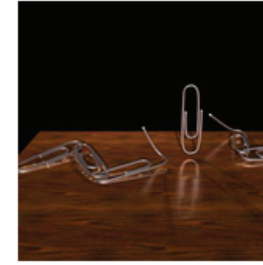


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Variations

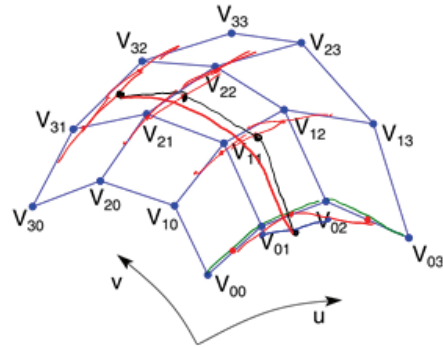
Several variations are possible:

- Scale $C(u)$ as it moves, possibly using length of $T(v)$ as a scale factor.
- Morph $C(u)$ into some other curve $\tilde{C}(u)$ as it moves along $T(v)$.
- ...



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Tensor product Bézier surfaces



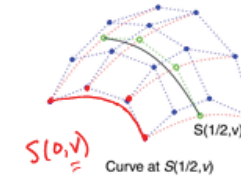
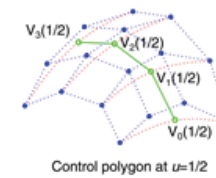
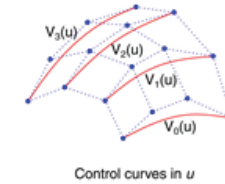
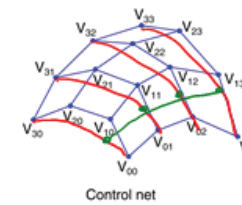
Given a grid of control points V_{ij} forming a **control net**, construct a surface $S(u, v)$ by:

- treating rows of V (the matrix consisting of the V_{ij}) as control points for curves $V_0(u), \dots, V_n(u)$.
- treating $V_0(u), \dots, V_n(u)$ as control points for a curve parameterized by v .

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Tensor product Bézier surfaces, cont.

Let's walk through the steps:



Which control points are interpolated by the surface?

4 corners

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Matrix form of Bézier curves and surfaces

Recall that Bézier curves can be written in terms of the Bernstein polynomials:

$$Q(u) = \sum_{i=0}^n V_i b_i(u)$$

They can also be written in a matrix form:

$$Q^T(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_0^T \\ V_1^T \\ V_2^T \\ V_3^T \end{bmatrix}$$

$$= \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \mathbf{M}_{\text{Bézier}} \mathbf{V}_{\text{curve}}$$

$$-u^3 + 3u^2 - 3u + 1 = (1-u)^3 = b_0(u)$$

Tensor product surfaces can be written out similarly:

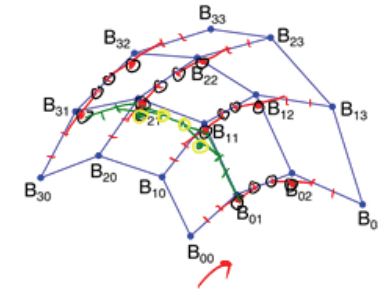
$$S(u, v) = \sum_{i=0}^n \sum_{j=0}^n V_i^T b_i(u) b_j(v)$$

$$= \begin{bmatrix} u^2 & u^2 & u & 1 \end{bmatrix} \mathbf{M}_{\text{Bézier}} \mathbf{V}_{\text{surface}} \mathbf{M}_{\text{Bézier}}^T \begin{bmatrix} v^2 \\ v^2 \\ v \\ 1 \end{bmatrix}$$

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Tensor product B-spline surfaces

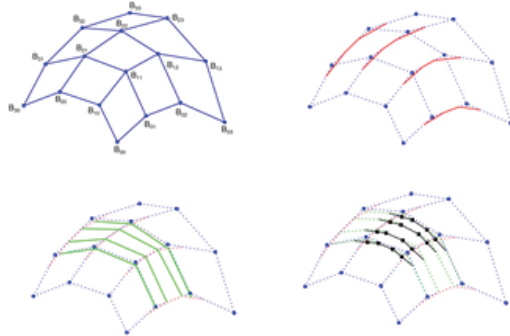
As with spline curves, we can piece together a sequence of Bézier surfaces to make a spline surface. If we enforce C^2 continuity and local control, we get B-spline curves:



- ♦ treat rows of B as control points to generate Bézier control points in u .
- ♦ treat Bézier control points in u as B-spline control points in v .
- ♦ treat B-spline control points in v to generate Bézier control points in u .

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Tensor product B-spline surfaces, cont.



Which B-spline control points are interpolated by the surface?

None

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Matrix form of B-spline surfaces

For curves, we can write a matrix that generates Bézier control points from B-spline control points:

$$\begin{bmatrix} V_0^T \\ V_1^T \\ V_2^T \\ V_3^T \end{bmatrix} = \begin{bmatrix} 1/6 & 2/3 & 1/6 & 0 \\ 0 & 2/3 & 1/3 & 0 \\ 0 & 1/3 & 2/3 & 0 \\ 0 & 1/6 & 2/3 & 1/6 \end{bmatrix} \begin{bmatrix} B_0^T \\ B_1^T \\ B_2^T \\ B_3^T \end{bmatrix}$$

$$\mathbf{V}_{\text{curve}} = \mathbf{M}_{\text{B-spline}} \mathbf{B}_{\text{curve}}$$

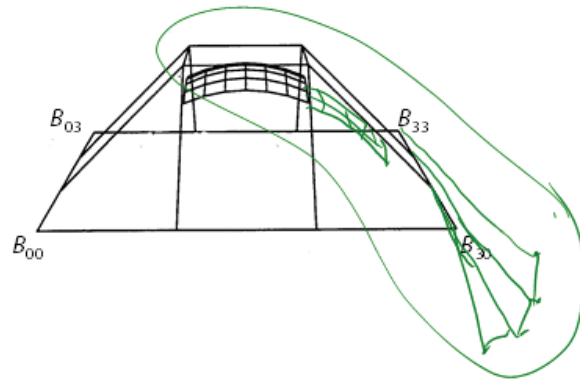
We can arrive at a similar form for tensor product B-spline surfaces:

$$\mathbf{V}_{\text{surface}} = \mathbf{M}_{\text{B-spline}} \mathbf{B}_{\text{surface}} \mathbf{M}_{\text{B-spline}}^T$$

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Tensor product B-splines, cont.

Another example:



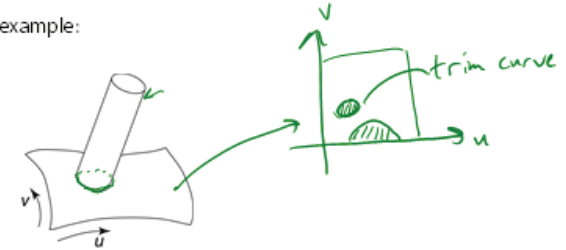
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Trimmed NURBS surfaces

Uniform B-spline surfaces are a special case of NURBS surfaces.

Sometimes, we want to have control over which parts of a NURBS surface get drawn.

For example:



We can do this by **trimming** the $u-v$ domain.

- ◆ Define a closed curve in the $u-v$ domain (a **trim curve**)
- ◆ Do not draw the surface points inside of this curve.

It's really hard to maintain continuity in these regions, especially while animating.

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