

Fourier analysis and sampling theory

Brian Curless
CSE 557
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Reading

Required:

- ♦ Shirley, Ch. 9

Recommended:

- ♦ Ron Bracewell, The Fourier Transform and Its Applications, McGraw-Hill.
- ♦ Don P. Mitchell and Arun N. Netravali, "Reconstruction Filters in Computer Computer Graphics," Computer Graphics, (Proceedings of SIGGRAPH 88). 22 (4), pp. 221-228, 1988.

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What is an image?

We can think of an **image** as a function, f , from \mathbb{R}^2 to \mathbb{R} :

- ♦ $f(x,y)$ gives the intensity of a channel at position (x,y)
- ♦ Realistically, we expect the image only to be defined over a rectangle, with a finite range:
 - $f: [a,b] \times [c,d] \rightarrow [0,1]$

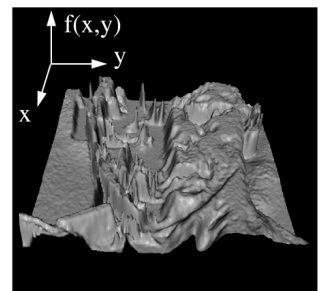
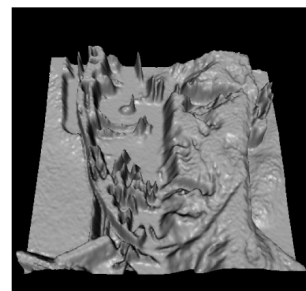
A color image is just three functions pasted together. We can write this as a "vector-valued" function:

$$f(x, y) = \begin{bmatrix} r(x, y) \\ g(x, y) \\ b(x, y) \end{bmatrix}$$

We'll focus in grayscale (scalar-valued) images for now.

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Images as functions



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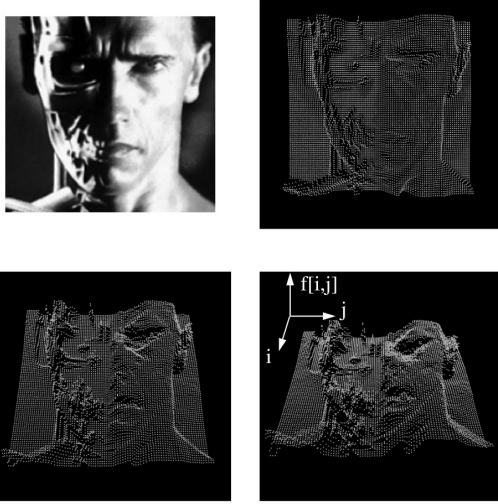
Digital images

In computer graphics, we usually create or operate on **digital (discrete)** images:

- ◆ **Sample** the space on a regular grid
- ◆ **Quantize** each sample (round to nearest integer)

If our samples are Δ apart, we can write this as:

$$f[n, m] = \text{Quantize}\{f(n \Delta, m \Delta)\}$$



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Motivation: filtering and resizing

What if we now want to:

- ◆ smooth an image?
- ◆ sharpen an image?
- ◆ enlarge an image?
- ◆ shrink an image?

In this lecture, we will explore the mathematical underpinnings of these operations.

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Convolution

One of the most common methods for filtering a function, e.g., for smoothing or sharpening, is called **convolution**.

In 1D, convolution is defined as:

$$\begin{aligned} g(x) &= f(x) * h(x) \\ &= \int_{-\infty}^{\infty} f(x') h(x - x') dx' \\ &= \int_{-\infty}^{\infty} f(x') \tilde{h}(x' - x) dx' \end{aligned}$$

where $\tilde{h}(x) \equiv h(-x)$.

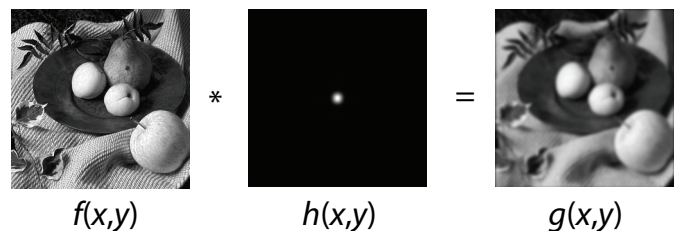
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Convolution in 2D

In two dimensions, convolution becomes:

$$\begin{aligned} g(x, y) &= f(x, y) * h(x, y) \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x', y') h(x - x', y - y') dx' dy' \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x', y') \tilde{h}(x' - x, y' - y) dx' dy' \end{aligned}$$

where $\tilde{h}(x, y) = h(-x, -y)$.



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Fourier transforms

Convolution, while a bit cumbersome looking, actually has a beautiful structure when viewed in terms of **Fourier analysis**.

We can represent functions as a weighted sum of sines and cosines.

We can think of a function in two complementary ways:

- ♦ **Spatially** in the **spatial domain**
- ♦ **Spectrally** in the **frequency domain**

The **Fourier transform** and its inverse convert between these two domains:

$$\begin{array}{ccc}
 \boxed{\text{Spatial domain}} & \begin{array}{c} \longrightarrow \\ \\ \longleftarrow \end{array} & \begin{array}{c} F(s) = \int_{-\infty}^{\infty} f(x)e^{-i2\pi sx} dx \\ \\ f(x) = \int_{-\infty}^{\infty} F(s)e^{i2\pi sx} ds \end{array} & \begin{array}{c} \longrightarrow \\ \\ \longleftarrow \end{array} & \boxed{\text{Frequency domain}}
 \end{array}$$

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Fourier transforms (cont'd)

$$\begin{array}{ccc}
 \boxed{\text{Spatial domain}} & \begin{array}{c} \longrightarrow \\ \\ \longleftarrow \end{array} & \begin{array}{c} F(s) = \int_{-\infty}^{\infty} f(x)e^{-i2\pi sx} dx \\ \\ f(x) = \int_{-\infty}^{\infty} F(s)e^{i2\pi sx} ds \end{array} & \begin{array}{c} \longrightarrow \\ \\ \longleftarrow \end{array} & \boxed{\text{Frequency domain}}
 \end{array}$$

$f(x)$ is usually a real signal, but $F(s)$ is generally complex:

$$F(s) = A(s) + iB(s) = |F(s)|e^{i2\pi\theta(s)}$$

where magnitude $|F(s)|$ and phase $\theta(s)$ are:

$$\begin{aligned}
 |F(s)| &= \sqrt{A^2(s) + B^2(s)} \\
 \theta(s) &= \tan^{-1}[B(s)/A(s)]
 \end{aligned}$$

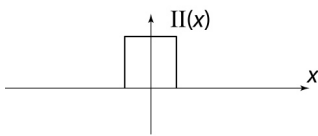
Where do the sines and cosines come in?

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Fourier transform example

Spatial domain

Frequency domain

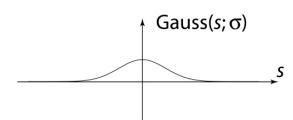
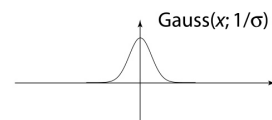
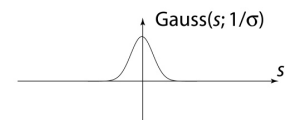
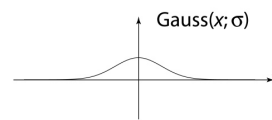
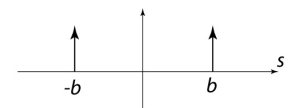
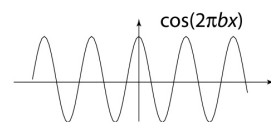
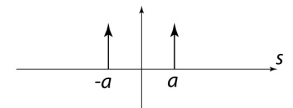
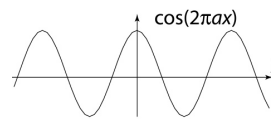


$$\Pi(x) = \begin{cases} 1 & |x| < 1/2 \\ 1/2 & |x| = 1/2 \\ 0 & |x| > 1/2 \end{cases}$$

More 1D Fourier examples

Spatial domain

Frequency domain



I derived the cosine version, but actually $\int_{-\infty}^{\infty} \cos() dx$ is not well-defined, so problematic.

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Some properties of FT's

$$F(s) = \int_{-\infty}^{\infty} f(x)e^{-i2\pi sx} dx$$

Amplitude scaling:

Additivity:

Domain scaling:

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2D Fourier transform

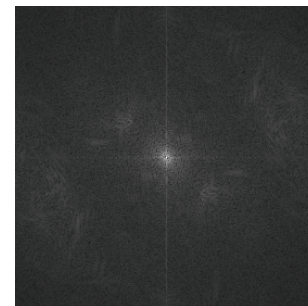
$$F(s_x, s_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)e^{-i2\pi(s_x x + s_y y)} dx dy$$

Spatial domain Frequency domain

$$f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(s_x, s_y)e^{i2\pi(s_x x + s_y y)} ds_x ds_y$$

Spatial domain

Frequency domain

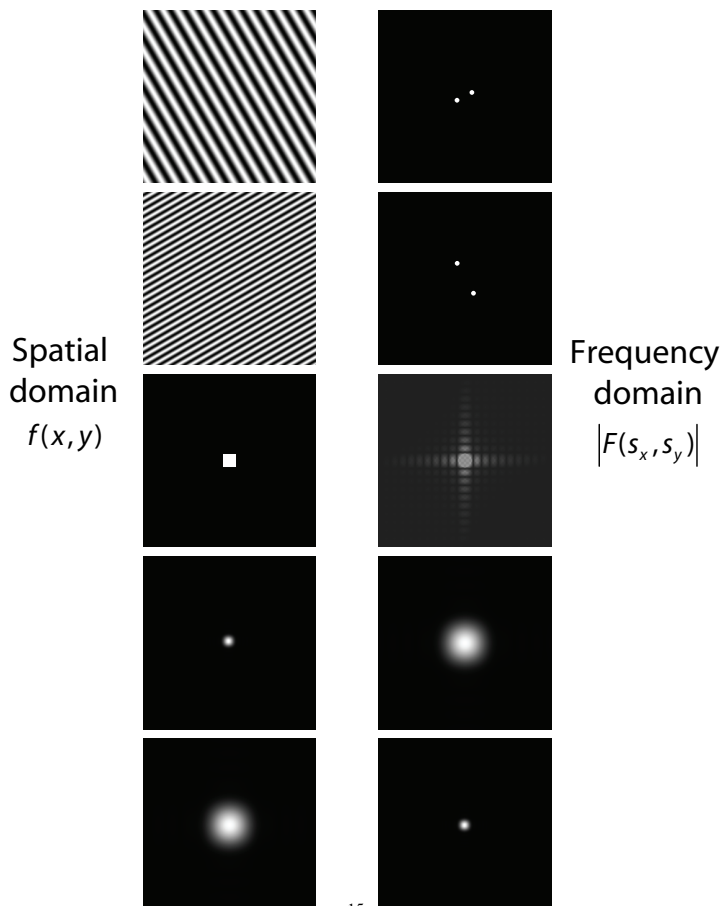


$f(x, y)$

$|F(s_x, s_y)|$

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2D Fourier examples



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Fourier transforms and convolution

What is the Fourier transform of the convolution of two functions? (The answer is very cool!)

$$f * h \longleftrightarrow ??$$

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Convolution theorems

Convolution theorem: Convolution in the spatial domain is equivalent to multiplication in the frequency domain.

$$f * h \longleftrightarrow F \cdot H$$

Symmetric theorem: Convolution in the frequency domain is equivalent to multiplication in the spatial domain.

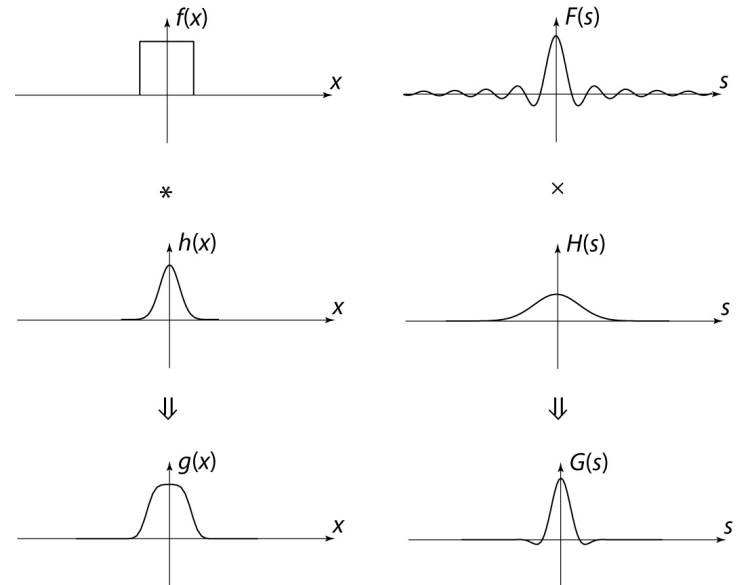
$$f \cdot h \longleftrightarrow F * H$$

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1D convolution theorem example

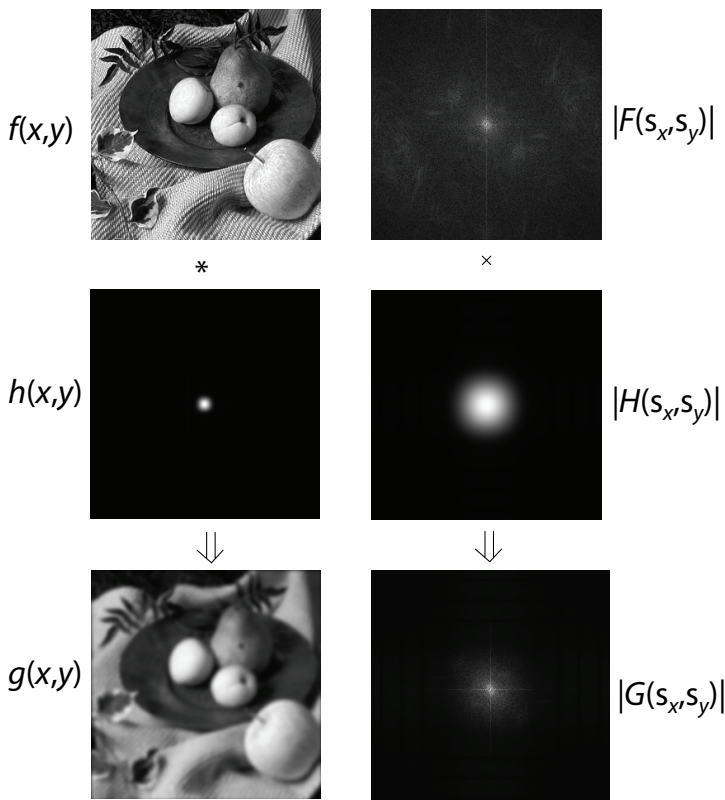
Spatial domain

Frequency domain



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2D convolution theorem example



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Convolution properties

Convolution exhibits a number of basic, but important properties...easily proved in the Fourier domain.

Commutativity:

$$a(x) * b(x) = b(x) * a(x)$$

Associativity:

$$[a(x) * b(x)] * c(x) = a(x) * [b(x) * c(x)]$$

Linearity:

$$a(x) * [k \cdot b(x)] = k \cdot [a(x) * b(x)]$$

$$a(x) * (b(x) + c(x)) = a(x) * b(x) + a(x) * c(x)$$

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The delta function

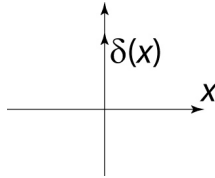
The **Dirac delta function** (or **impulse function**), $\delta(x)$, is a handy tool for sampling theory.

It has zero width, infinite height, and unit area.

Can be computed as a limit of various functions, e.g.:

$$\delta(x) = \lim_{\sigma \rightarrow 0} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-x^2}{2\sigma^2}\right) = \lim_{w \rightarrow 0} \frac{1}{w} \Pi\left(\frac{x}{w}\right)$$

It is usually drawn as:



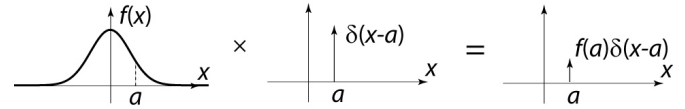
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Sifting and shifting

For sampling, the delta function has two important properties.

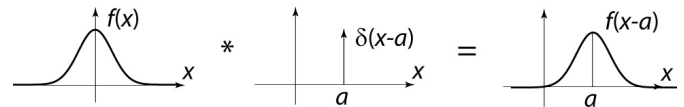
Sifting:

$$f(x)\delta(x-a) = f(a)\delta(x-a)$$



Shifting:

$$f(x) * \delta(x-a) = f(x-a)$$



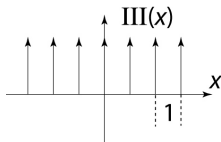
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The shah/comb function

A string of delta functions is the key to sampling. The resulting function is called the **shah** or **comb** function or **impulse train**:

$$\text{III}(x) = \sum_{n=-\infty}^{\infty} \delta(x-n)$$

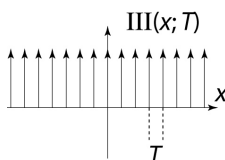
which looks like:



We can also define an impulse train in terms of a desired delta function spacing, T :

$$\text{III}(x; T) = \sum_{n=-\infty}^{\infty} \delta(x-nT)$$

which looks like:



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The shah/comb function, cont'd

If we multiply an input function by the impulse train, we get:

$$f(x)\text{III}(x; T) = f(x) \sum_{n=-\infty}^{\infty} \delta(x-nT)$$

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The shah/comb function, cont'd

Amazingly, the Fourier transform of the shah function is also the shah function:

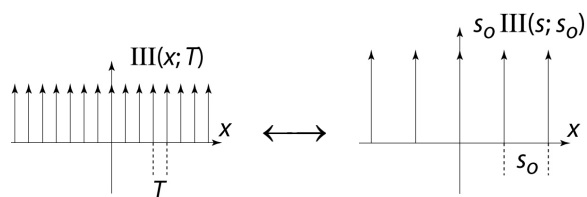
$$\text{III}(x) \longleftrightarrow \text{III}(s)$$

One can also show that:

$$\text{III}(x; T) \longleftrightarrow \frac{1}{T} \text{III}(s; 1/T) = s_0 \text{III}(s; s_0)$$

where $s_0 = 1/T$.

We can visualize this as:

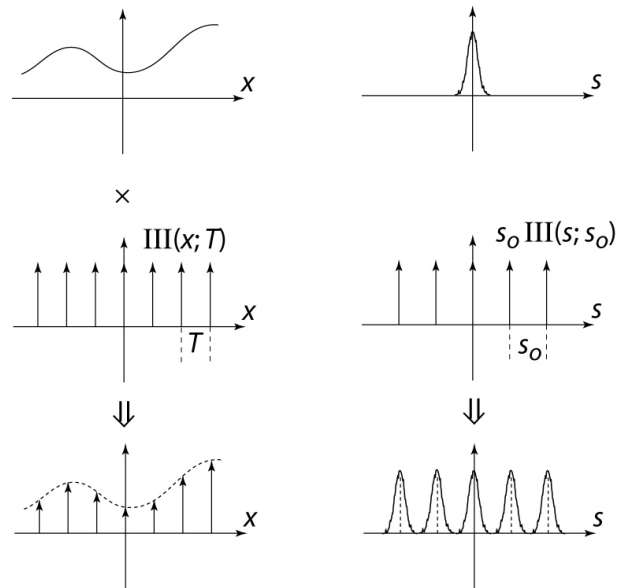


For convenience, I won't draw the delta functions as scaled vertically, though mathematically, one must keep track of these scale factors.

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Sampling

Now, we can talk about sampling.

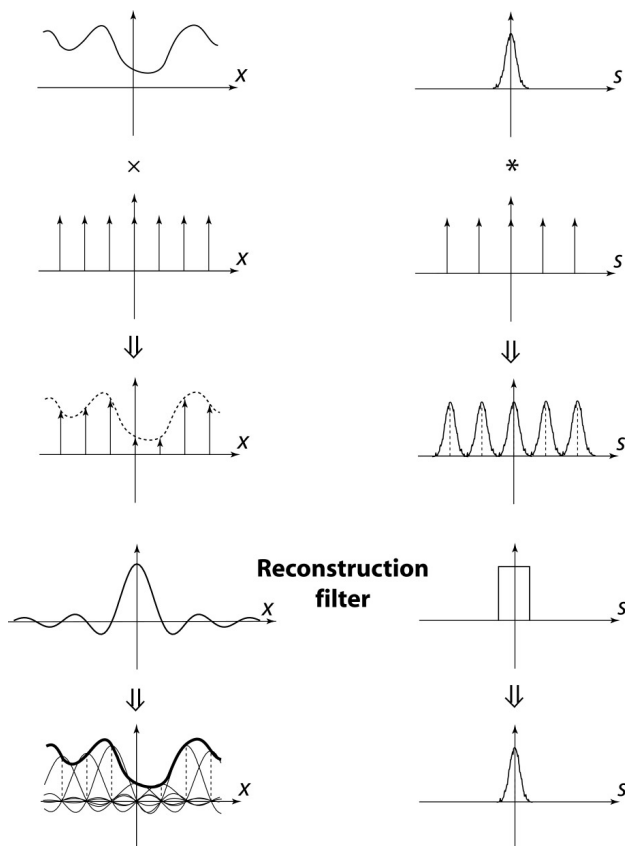


The Fourier spectrum gets *replicated* by spatial sampling!

How do we recover the signal?

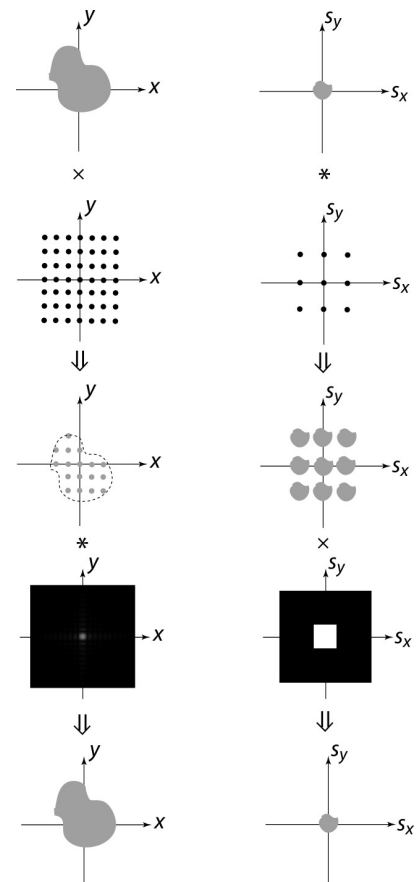
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Sampling and reconstruction



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Sampling and reconstruction in 2D



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Sampling theorem

This result is known as the **Sampling Theorem** and is due to Claude Shannon who first discovered it in 1949:

A signal can be reconstructed from its samples without loss of information, if the original signal has no frequencies above $\frac{1}{2}$ the sampling frequency.

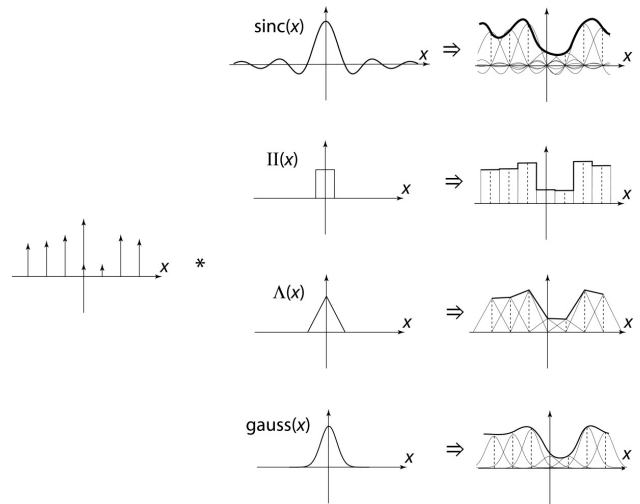
For a given **bandlimited** function, the minimum rate at which it must be sampled is the **Nyquist frequency**.

Reconstruction filters

The sinc filter, while "ideal", has two drawbacks:

- ♦ It has large support (slow to compute)
- ♦ It introduces ringing in practice

We can choose from many other filters...



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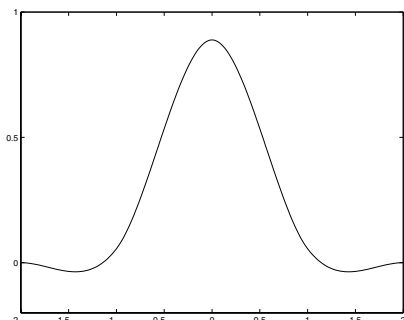
Cubic filters

Mitchell and Netravali (1988) experimented with cubic filters, reducing them all to the following form:

$$r(x) = \begin{cases} \frac{1}{6} \left\{ \begin{aligned} (12 - 9B - 6C)|x|^3 + (-18 + 12B + 6C)|x|^2 + (6 - 2B) & |x| < 1 \\ ((-B - 6C)|x|^3 + (6B + 30C)|x|^2 + (-12B - 48C)|x| + (8B + 24C)) & 1 \leq |x| < 2 \\ 0 & \text{otherwise} \end{aligned} \right. \end{cases}$$

The choice of B or C trades off between being too blurry or having too much ringing. B=C=1/3 was their "visually best" choice.

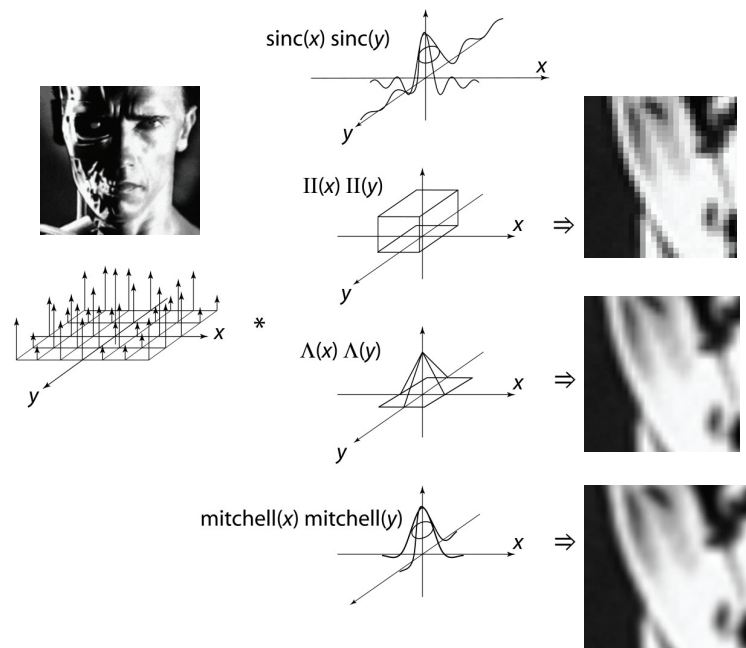
The resulting reconstruction filter is often called the "Mitchell filter."



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Reconstruction filters in 2D

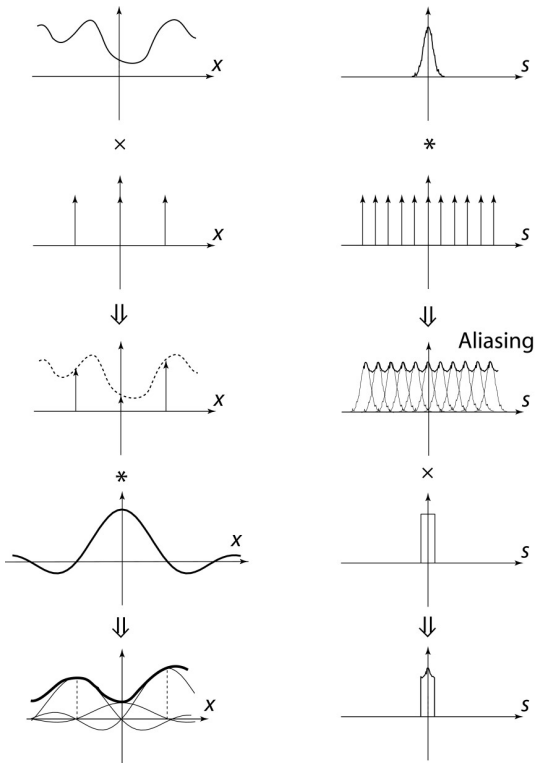
We can also perform reconstruction in 2D...



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Aliasing

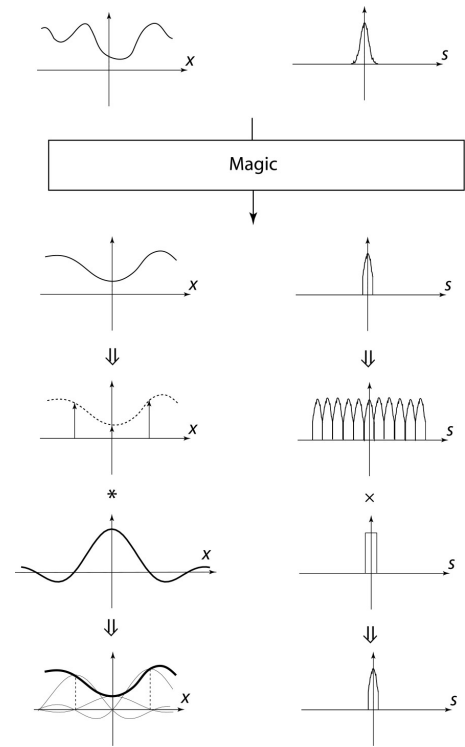
What if we go below the Nyquist frequency?



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Anti-aliasing

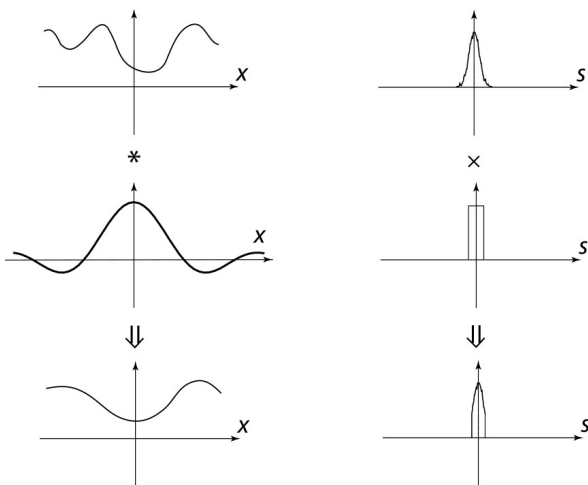
Anti-aliasing is the process of *removing* the frequencies before they alias.



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Anti-aliasing by analytic prefiltering

We can fill the "magic" box with analytic pre-filtering of the signal:

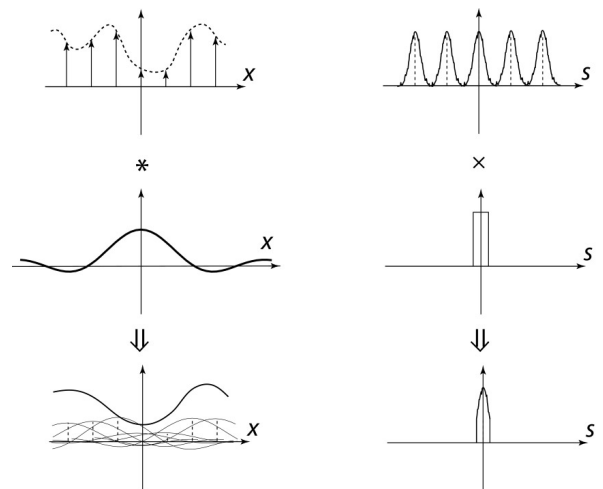


Why may this not generally be possible?

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Filtered downsampling

Alternatively, we can sample the image at a higher rate, and then filter that signal:



We can now sample the signal at a lower rate. The whole process is called **filtered downsampling** or **supersampling and averaging down**.

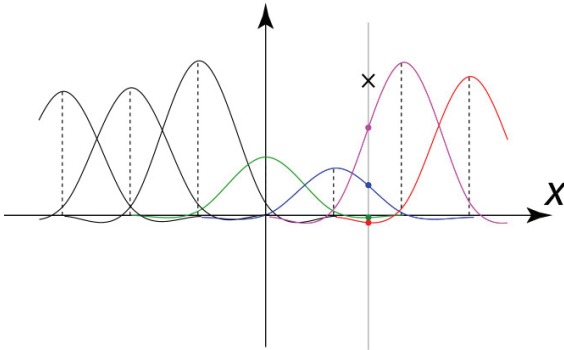
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Practical upsampling

When resampling a function (e.g., when resizing an image), you do not need to reconstruct the complete continuous function.

For zooming in on a function, you need only use a reconstruction filter and evaluate as needed for each new sample.

Here's an example using a cubic filter:

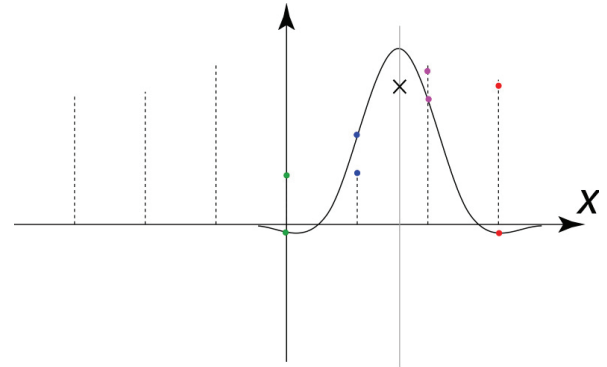


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Practical upsampling

This can also be viewed as:

1. putting the reconstruction filter at the desired location
2. evaluating at the original sample positions
3. taking products with the sample values themselves
4. summing it up



Important: filter should always be normalized!

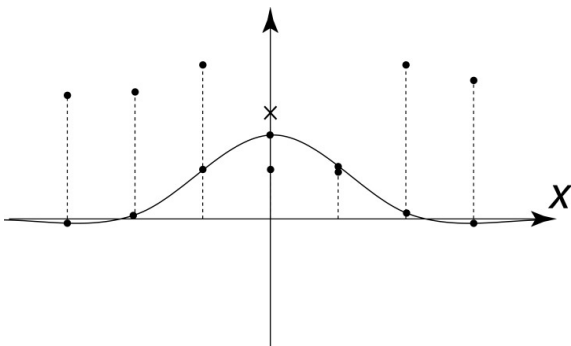
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Practical downsampling

Downsampling is similar, but filter has larger support and smaller amplitude.

Operationally:

1. Choose reconstruction filter in downsampled space.
2. Compute the downsampling rate, d , ratio of new sampling rate to old sampling rate
3. Stretch the filter by $1/d$ and scale it down by d
4. Follow upsampling procedure (previous slides) to compute new values



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2D resampling

We've been looking at **separable** filters:

$$r_{2D}(x, y) = r_{1D}(x)r_{1D}(y)$$

How might you use this fact for efficient resampling in 2D?

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