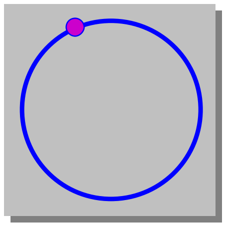
# Differential Constraints

## Beyond Points and Springs

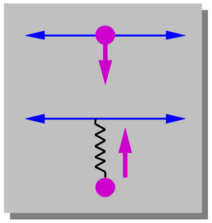
- You can make just about anything out of point masses and springs, *in principle*
- In practice, you can make anything you want as long as it's jello
- Constraints will buy us:
  - Rigid links instead of goopy springs
  - Ways to make interesting contraptions

## A bead on a wire



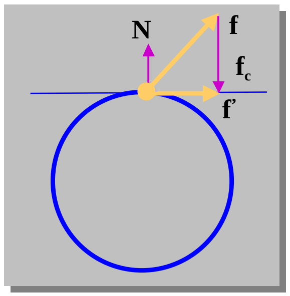
- Desired Behavior:
  - The bead can slide freely *along* the circle
  - It can never come off, however hard we pull
- Question:
  - How does the bead move under applied forces?

## Penalty Constraints



- Why not use a spring to hold the bead on the wire?
- Problem:
  - Weak springs ⇒ goopy constraints
  - Strong springs ⇒ neptune express!
- A classic *stiff system*

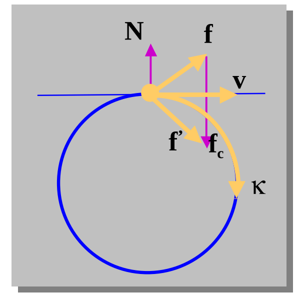
## The basic trick (f = mv version)




- 1st order world.
- *Legal velocity:* tangent to circle ($N \cdot v = 0$)
- *Project* applied force f onto tangent: $f' = f + f_c$
- Added normal-direction force $f_c$: *constraint force*
- No tug-of-war, no stiffness

$$f_c = -\frac{f \cdot N}{N \cdot N} N \qquad f' = f + f_c$$

## f = ma

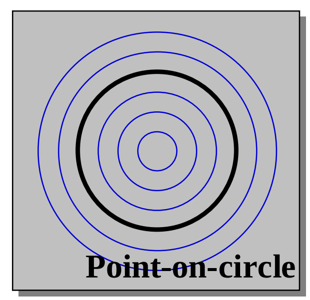



- Same idea, but…
- *Curvature* (κ) has to match.
- κ depends on *both* a and v:
  - the faster you're going, the faster you have to turn
- Calculate $f_c$ to yield a legal *combination* of a and v
- Not as simple!

## Now for the Algebra …

- Fortunately, there's a general recipe for calculating the constraint force
- First, a single constrained particle
- Then, generalize to constrained particle systems

## Representing Constraints



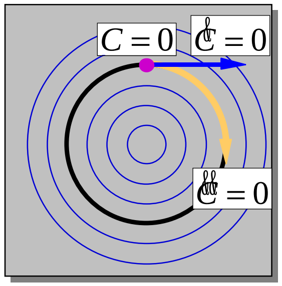

*I. Implicit:*

$$C(\mathbf{x}) = |\mathbf{x}| - r = 0$$

~~*II. Parametric:*~~

$$\cancel{\mathbf{x} = r\,[\cos\theta, \sin\theta]}$$

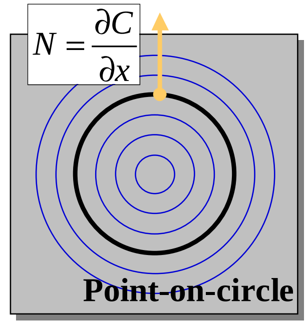
## Maintaining Constraints Differentially




- Start with legal position and velocity.
- Use constraint forces to ensure legal curvature.

| | |
|---|---|
| $C = 0$ | legal position |
| $\dot{C} = 0$ | legal velocity |
| $\ddot{C} = 0$ | legal curvature |

## Constraint Gradient



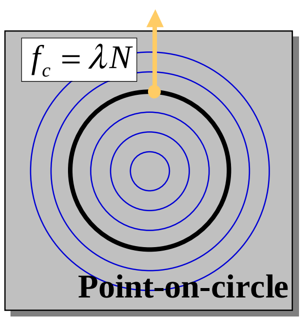

*Implicit:*

$$C(\mathbf{x}) = |\mathbf{x}| - r = 0$$

**Differentiating C gives a normal vector.**

**This is the direction our constraint force will point in.**

## Constraint Forces




**Constraint force: gradient vector times a scalar λ**

**Just one unknown to solve for**

**Assumption: constraint is passive—no energy gain or loss**

## Constraint Force Derivation

$$\boxed{C(x(t))}$$

$$\dot{C} = N \cdot \dot{x}$$

$$\ddot{C} = \frac{\partial}{\partial t}(N \cdot x)$$

$$= \dot{N} \cdot \dot{x} + N \cdot \ddot{x}$$

$$\ddot{x} = \frac{f + f_c}{m}$$

$$f_c = \lambda N$$

**Set $\ddot{C} = 0$, solve for λ:**

$$\lambda = -m\frac{\dot{N} \cdot \dot{x}}{N \cdot N} - \frac{N \cdot f}{N \cdot N}$$

**Constraint force is λN.**

Notation: $N = \frac{\partial C}{\partial x}, \dot{N} = \frac{\partial^2 C}{\partial x \partial t}$

## Example: Point-on-circle

$$C = |\mathbf{x}| - r$$

$$\mathbf{N} = \frac{\partial C}{\partial \mathbf{x}} = \frac{\mathbf{x}}{|\mathbf{x}|}$$

$$\dot{\mathbf{N}} = \frac{\partial^2 C}{\partial \mathbf{x} \partial t} = \frac{1}{|\mathbf{x}|}\left[\dot{\mathbf{x}} - \frac{\mathbf{x}\cdot\dot{\mathbf{x}}}{\mathbf{x}\cdot\mathbf{x}}\mathbf{x}\right]$$

**Write down the constraint equation.**

**Take the derivatives.**

**Substitute into generic template, simplify.**

$$\lambda = -m\frac{\dot{\mathbf{N}}\cdot\dot{\mathbf{x}}}{\mathbf{N}\cdot\mathbf{N}} - \frac{\mathbf{N}\cdot\mathbf{f}}{\mathbf{N}\cdot\mathbf{N}} = \left[m\frac{(\mathbf{x}\cdot\dot{\mathbf{x}})^2}{\mathbf{x}\cdot\mathbf{x}} - m(\dot{\mathbf{x}}\cdot\dot{\mathbf{x}}) - \mathbf{x}\cdot\mathbf{f}\right]\frac{1}{|\mathbf{x}|}$$

## Drift and Feedback

- **In principle, clamping $\ddot{C}$ at zero is enough**
- **Two problems:**
  - **Constraints might not be met initially**
  - **Numerical errors can accumulate**
- **A feedback term handles both problems:**

$$\ddot{C} = -\alpha C - \beta\dot{C}, \text{ instead of}$$
$$\ddot{C} = 0$$

$\boxed{\alpha \text{ and } \beta \text{ are magic constants.}}$

## Tinkertoys

- **Now we know how to simulate a bead on a wire.**
- **Next: a constrained particle *system*.**
  - **E.g. constrain particle/particle distance to make rigid links.**
- **Same idea, but…**

## Constrained particle systems

- **Particle system: a point in state space.**
- **Multiple constraints:**
  - **each is a function $C_i(x_1,x_2,\ldots)$**
  - ***Legal state:* $C_i = 0, \forall i$.**
  - ***Simultaneous* projection.**
  - **Constraint force: *linear combination* of constraint gradients.**
- **Matrix equation.**

## Compact Particle System Notation

$$\ddot{q} = WQ$$

**q:** *3n-long state vector.*

**Q:** *3n-long force vector.*

**M:** *3n x 3n* **diagonal** ***mass matrix.***

**W:** **M-inverse (element- wise reciprocal)**

$$\mathbf{q} = [\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n]$$
$$\mathbf{Q} = [\mathbf{f}_1, \mathbf{f}_2, \cdots, \mathbf{f}_n]$$
$$\mathbf{M} = \begin{bmatrix} m_1 & & & & & & \\ & m_1 & & & & & \\ & & m_1 & & & & \\ & & & \ddots & & & \\ & & & & m_n & & \\ & & & & & m_n & \\ & & & & & & m_n \end{bmatrix}$$
$$\mathbf{W} = \mathbf{M}^{-1}$$

## Particle System Constraint Equations

Matrix equation for $\lambda$

$$\left[\mathbf{JWJ}^T\right]\lambda = -\dot{\mathbf{J}}\dot{\mathbf{q}} - \left[\mathbf{JW}\right]\mathbf{Q}$$

Constrained Acceleration

$$\ddot{\mathbf{q}} = \mathbf{W}\left[\mathbf{Q} + \mathbf{J}^T\lambda\right]$$

More Notation

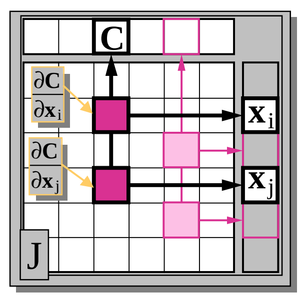
$$\mathbf{C} = [C_1, C_2, \cdots, C_m]$$
$$\lambda = [\lambda_1, \lambda_2, \cdots, \lambda_m]$$
$$\mathbf{J} = \frac{\partial \mathbf{C}}{\partial \mathbf{q}}$$
$$\dot{\mathbf{J}} = \frac{\partial^2 \mathbf{C}}{\partial \mathbf{q} \partial t}$$

Derivation: just like bead-on-wire.

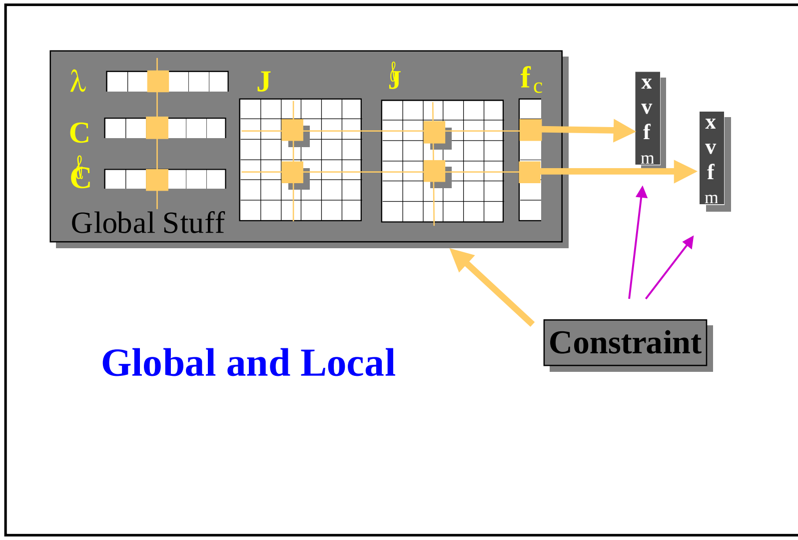
## How do you implement all this?

- **We have a global matrix equation.**
- **We want to build models on the fly, just like masses and springs.**
- **Approach:**
  - **Each constraint adds its own piece to the equation.**

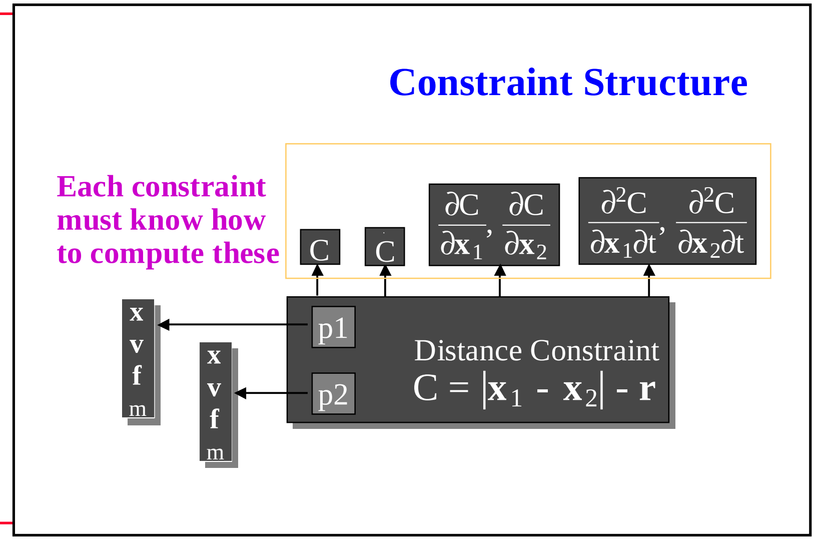
## Matrix Block Structure



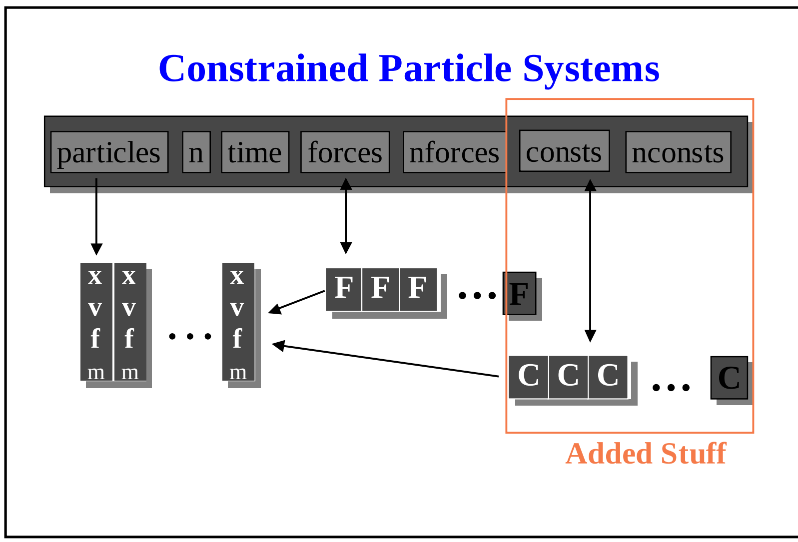

- **Each constraint contributes one or more *blocks* to the matrix.**
- **Sparsity: many empty blocks.**
- **Modularity: let each constraint compute its own blocks.**
- **Constraint and particle indices determine block locations.**



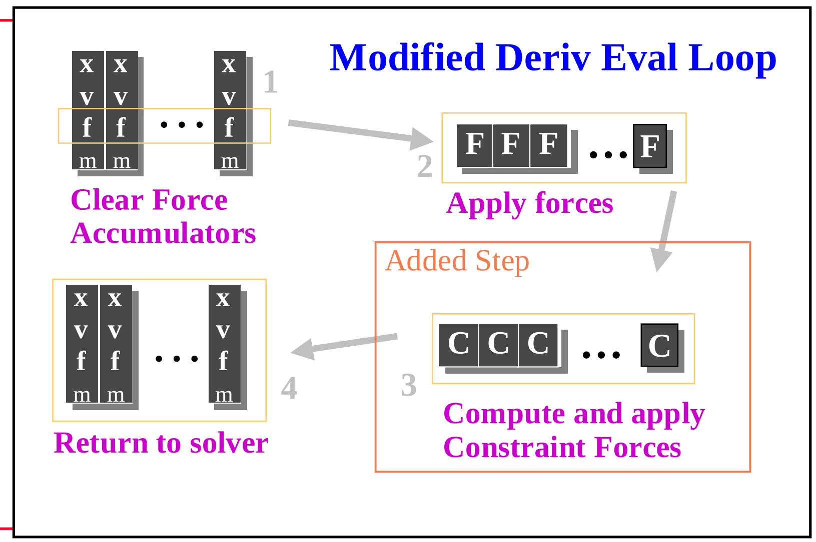
λ
J
C
f c
Global Stuff
x v f m
x v f m
Constraint
Global and Local

Constraint Structure
Each constraint must know how to compute these
C
C
∂C/∂x1, ∂C/∂x2
∂²C/∂x1∂t, ∂²C/∂x2∂t
x v f m
x v f m
p1
p2
Distance Constraint
C = |x1 - x2| - r

Constrained Particle Systems
particles
n
time
forces
nforces
consts
nconsts
x v f m
x v f m
x v f m
F F F
F
C C C
C
Added Stuff

Modified Deriv Eval Loop
1
Clear Force Accumulators
2
F F F
F
Apply forces
Added Step
C C C
C
3
Compute and apply Constraint Forces
4
Return to solver

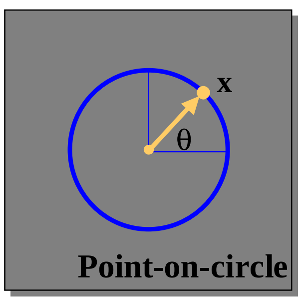
## Constraint Force Eval

- After computing ordinary forces:
  - Loop over constraints, assemble global matrices and vectors.
  - Call matrix solver to get $\lambda$, multiply by $J^T$ to get constraint force.
  - Add constraint force to particle force accumulators.

## Impress your Friends

- The requirement that constraints not add or remove energy is called the *Principle of Virtual Work.*
- The $\lambda$'s are called *Lagrange Multipliers.*
- The derivative matrix, J, is called the *Jacobian Matrix.*

## A whole other way to do it.



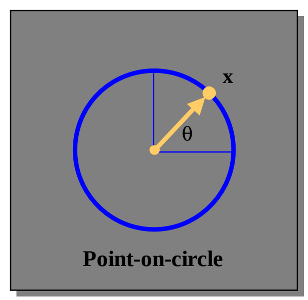

~~*I. Implicit:*~~

$$\cancel{C(\mathbf{x}) = |\mathbf{x}| - r = 0}$$

*II. Parametric:*

$$\mathbf{x} = r\left[\cos\theta, \sin\theta\right]$$

## Parametric Constraints

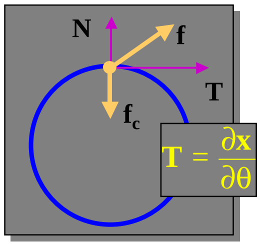



*Parametric:*

$\mathbf{x} = r\left[\cos\theta, \sin\theta\right]$

- Constraint is always met exactly.
- One DOF: $\theta$.
- Solve for $\dot{\theta}$.

## Parametric bead-on-wire (f = mv)




x is not an independent variable.

First step—get rid of it:

$$\dot{x} = \frac{f + f_c}{m} \qquad \text{f = mv } \textit{(constrained)}$$

$$\dot{x} = T\dot{\theta} \qquad \text{chain rule}$$

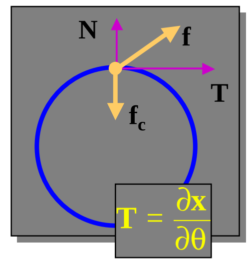
$$T\dot{\theta} = \frac{f + f_c}{m} \qquad \text{combine}$$

## For our next trick…

As before, assume $f_c$ points in the normal direction, so

$\mathbf{T} \cdot \mathbf{f}_c = 0$

We can nuke $f_c$ by dotting T into both sides:




$$T\dot{\theta} = \frac{f + f_c}{m} \qquad \text{from last slide}$$

$$T \cdot T\dot{\theta} = \frac{T \cdot f + \cancel{T \cdot f_c}}{m} \qquad \text{blam!}$$

$$\dot{\theta} = \frac{1}{m}\frac{T \cdot f}{T \cdot T} \qquad \text{rearrange.}$$

## General case

**Lagrange dynamics:**

$$\mathbf{J}^T\mathbf{M}\mathbf{J}\ddot{\mathbf{u}} + \mathbf{J}^T\mathbf{M}\dot{\mathbf{J}}\dot{\mathbf{u}} - \mathbf{J}^T\mathbf{Q} = 0$$

where

$$\mathbf{J} = \frac{\partial \mathbf{q}}{\partial \mathbf{u}}$$

**Not to be confused with:**

$$\left[\mathbf{JWJ}^T\right]\lambda = -\dot{\mathbf{J}}\dot{\mathbf{q}} - [\mathbf{JW}]\mathbf{Q}$$

where

$$\mathbf{J} = \frac{\partial \mathbf{C}}{\partial \mathbf{q}}$$

## Parametric Constraints: Summary

- Generalizations: f = ma, particle systems
  - Like implicit case (see notes.)
- Big advantages:
  - Fewer DOF's.
  - Constraints are always met.
- Big *dis*advantages:
  - Hard to formulate constraints.
  - No easy way to *combine* constraints.
- Offical name: *Lagrangian dynamics.*

## Hybrid systems

$$\left[\mathbf{J}\mathbf{W}\mathbf{J}^T\right]\boldsymbol{\lambda} = -\dot{\mathbf{J}}\dot{\mathbf{u}} - [\mathbf{J}\mathbf{W}]\mathbf{Q}$$

where

$$\mathbf{W} = \mathbf{M}^{-1} = \left[\iiint_i m_i \mathbf{q}_i{}^T \mathbf{q}_i\right]^{-1}$$

$$\mathbf{C}(\mathbf{q}(\mathbf{u}))$$

$$\mathbf{J} = \frac{\partial \mathbf{C}}{\partial \mathbf{q}}\frac{\partial \mathbf{q}}{\partial \mathbf{u}}$$

## Project 1:

- **A bead on a wire (implicit)**
- **A double pendulum**
- **A *triple* pendulum**
- **Simple interactive tinkertoys**