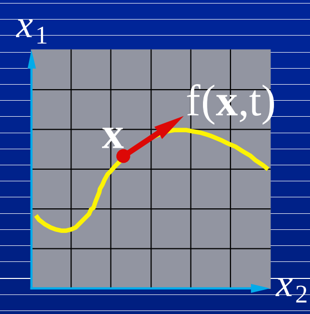
Differential Equations

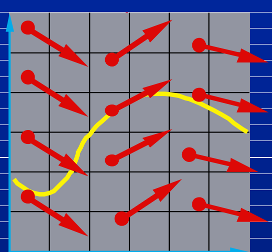
A Canonical Differential Equation



$$\dot{\mathbf{x}} = f(\mathbf{x},t)$$

- **x(t): a moving point.**
- **f(x,t): x's velocity.**

Vector Field

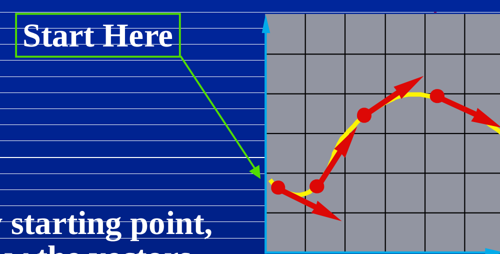


The differential equation

$$\dot{x} = \mathbf{f}(x,t)$$

defines a vector field over x.

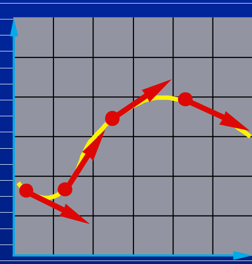
Integral Curves



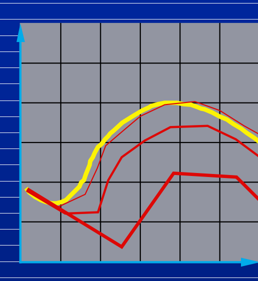
Pick any starting point, and follow the vectors.

Initial Value Problems

Given the starting point, follow the integral curve.



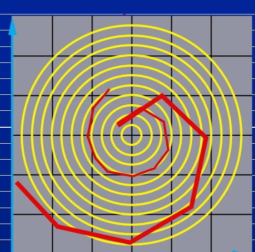
Euler's Method



- Simplest numerical solution method
- Discrete time steps
- Bigger steps, bigger errors.

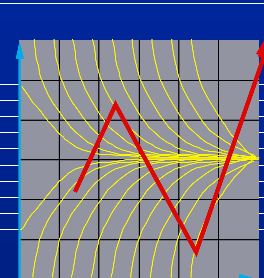
$$\mathbf{x}(t + \Delta t) = \mathbf{x}(t) + \Delta t\, f(\mathbf{x},t)$$

Problem I: Inaccuracy



Error turns x(t) from a circle into the spiral of your choice.

Problem II: Instability



Accuracy of the Euler method

$$x(t+h) = x(t) + hf(x,t) + O(h^2)$$

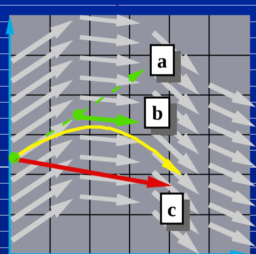
What do we need to do to get accuracy of O(h^3)

The midpoint method

Also knows as second-order Runge-Kutta

$$k_1 = hf(x_0, t_0)$$
$$k_2 = hf(x_0 + \frac{k_1}{2}, t_0 + \frac{h}{2})$$
$$x(t_0 + h) = x_0 + k_2 + O(h^3)$$

The Midpoint Method



a. Compute an Euler step

$$\Delta \mathbf{x} = \Delta t\, \mathbf{f}(\mathbf{x}, t)$$

b. Evaluate f at the midpoint

$$\mathbf{f}_{mid} = \mathbf{f}\left(\frac{\mathbf{x} + \Delta \mathbf{x}}{2}, \frac{t + \Delta t}{2}\right)$$

c. Take a step using the midpoint value

$$\mathbf{x}(t + \Delta t) = \mathbf{x}(t) + \Delta t\, \mathbf{f}_{mid}$$

q-stage Runge-Kutta method

General form

$$x(t_0 + h) = x(t_0) + h \sum_{i=1}^{q} w_i k_i$$

$$k_i = f\left(x_0 + h \sum_{j=1}^{q} \beta_{ij} k_j\right)$$

Find the constants which ensure a given accuracy O(h^n).
What if β matrix is full?

p-order RK method

A method has p-order of p is the largest integer for which

$$x(t_0+h)-x(t_0)-h\sum_{i=1}^{q} w_i k_i = O\left(h^{p+1}\right)$$

Fourth order RK

$$k_1 = hf(x_0, t_0)$$

$$k_2 = hf(x_0+\frac{k_1}{2}, t_0+\frac{h}{2})$$

$$k_3 = hf(x_0+\frac{k_2}{2}, t_0+\frac{h}{2})$$

$$k_4 = hf(x_0+k_3, t_0+h)$$

$$x(t_0+h) = x_0+\frac{1}{6}k_1+\frac{1}{6}k_1+\frac{1}{3}k_2+\frac{1}{3}k_3+\frac{1}{6}k_4$$

Why is this method often used?

Order vs. Stages

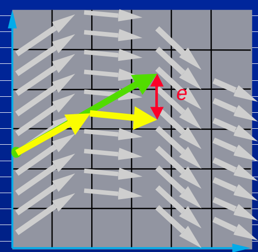
order (p)	1	2	3	4	5	6	7	8	9	10
min stage (q)	1	2	3	4	6	7	9	11	$12 \le q \le 17$	$13 \le q \le 17$

Nothing is known for orders > 10

Step control

1. **Step doubling (Richardson extrapolation)**
2. **Embedding estimate**
3. **Variable step, variable order**

Step doubling



Compute 2 estimates:

1. x_a – with step size h
2. x_b – with 2 solver steps of size h/2

$$e = |x_a - x_b|$$

Given error tollerance ε adjust the step size by a factor

$$\left(\frac{\varepsilon}{e}\right)^{1/p}$$

Embedding estimate

aka Runge-Kutta-Fehlberg

Compare:

1. Fifth order RK method with 6 stages
2. Fourth order RK method with 6 stages

Variable step, variable order

Change between methods of different order as well as step based on obtained error estimates.

These methods are currently the last word in numerical integration.

Scaling error estimates

In some set of differential equations dependent variables differ enormously in magnitude

$$\left|\frac{\Delta_0}{\Delta_1}\right|^{1/p}$$

$$\Delta_0 = \varepsilon \mathbf{s}$$

More methods…

- Bulrisch-Stoer Method
  - Best choice for getting accuracy in smooth ODEs
- Predictor-Corrector methods

Implicit methods

Useful for stiff equations

$$x(t_0+h) = x(t_0) + h\sum_{i=1}^{q} w_i k_i$$

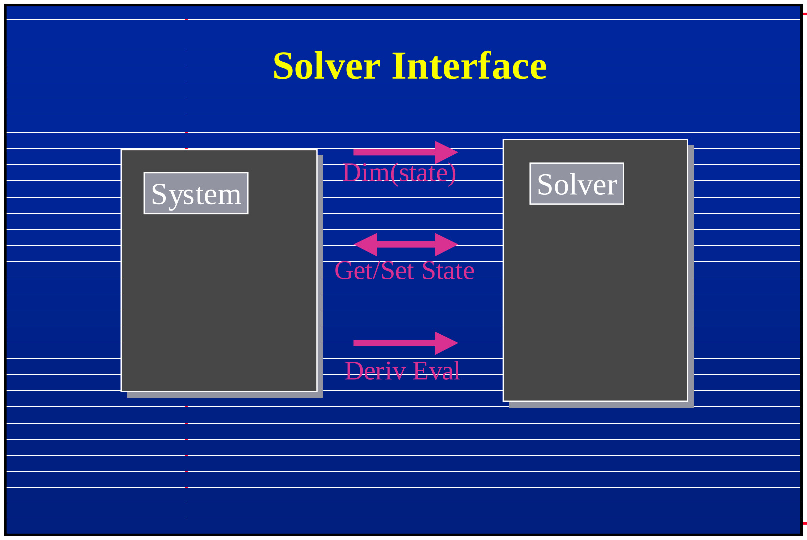
$$k_i = f\left(x_0 + h\sum_{j=1}^{q} \beta_{ij} k_j \right)$$

Helpful hints

- *Don't* use Euler's method
- *Do* Use adaptive step size
- For stiff equations use implicit methods
  - more on that later in the course

Modular Implementation

- Generic operations:
  - Get dim(x)
  - Get/set x and t
  - Deriv Eval at current (x,t)
- Write solvers in terms of these.
  - Re-usable solver code.
  - Simplifies model implementation.


Solver Interface
System
Dim(state)
Get/Set State
Deriv Eval
Solver