Dynamics of Transformation Hierarchies

Kinetic Energy

$$
\begin{aligned}
T_j &= \frac{1}{2}\int_i \dot{\mathbf{p}}_i^T\,\dot{\mathbf{p}}_i\;\tau_i\,dx\,dy\,dz \\
&= \frac{1}{2}\int_i \mathbf{x}_i^T\,\dot{\mathbf{W}}_j{}^T\,\dot{\mathbf{W}}_j\mathbf{x}_i\;\tau_i\,dx\,dy\,dz \\
&= \frac{1}{2}\int_i tr\left(\dot{\mathbf{W}}_j\mathbf{x}_i\mathbf{x}_i^T\,\dot{\mathbf{W}}_j{}^T\right)\;\tau_i\,dx\,dy\,dz \\
&= \frac{1}{2}\,tr\left(\dot{\mathbf{W}}_j\left[\int_i \mathbf{x}_i\mathbf{x}_i^T\;\tau_i\,dx\,dy\,dz\right]\dot{\mathbf{W}}_j{}^T\right) \\
&= \frac{1}{2}\,tr\left(\dot{\mathbf{W}}_j\mathbf{M}_j\dot{\mathbf{W}}_j{}^T\right)
\end{aligned}
$$

where $\mathbf{M}_i$ is the primitive mass tensor

Kinetic Energy Lagrangian Contribution

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial T_i}{\partial \dot{q}_j} - \frac{\partial T_i}{\partial q_j} &= \frac{1}{2}\;tr\left(\frac{\partial \mathbf{W}_i}{\partial q_j}\mathbf{M}_i\ddot{\mathbf{W}}_i{}^T + \ddot{\mathbf{W}}_i\mathbf{M}_i\frac{\partial \mathbf{W}_i{}^T}{\partial q_j}\right) \\
&= tr\left(\frac{\partial \mathbf{W}_i}{\partial q_j}\mathbf{M}_i\ddot{\mathbf{W}}_i{}^T\right)
\end{aligned}
$$

Gereralized Force Equation

$$
\begin{aligned}
C_j = &\sum_i\left[tr\left(\frac{\partial \mathbf{W}_i}{\partial q_j}\mathbf{M}_i\ddot{\mathbf{W}}_i{}^T\right) + m_i\mathbf{g}\frac{\partial \mathbf{W}_i}{\partial q_j}\mathbf{c}_i\right] \\
&+\sum_k\left[\frac{\partial(\mathbf{F}_k\mathbf{p}_k)}{\partial q_j}\right] \\
&+\sum_l\left[\lambda_l\frac{\partial \mathbf{C}_l}{\partial q_j}\right]
\end{aligned}
$$

where i ranges over all primitives, k over all point forces, and l over all mechanical constraints.

Recursive Formulation

$S(i)$ is the set of all node indices in the subtree rooted at node i.

$$\begin{aligned}
&\sum_i tr\left(\frac{\partial \mathbf{W}_i}{\partial q_j}\mathbf{M}_i\ddot{\mathbf{W}}_i^T\right) + m_i\mathbf{g}\frac{\partial \mathbf{W}_i}{\partial q_j}\mathbf{c}_i \\
&= \sum_{i\in S(j)} tr\left(\frac{\partial \mathbf{W}_i}{\partial q_j}\mathbf{M}_i\ddot{\mathbf{W}}_i^T\right) + m_i\mathbf{g}\frac{\partial \mathbf{W}_i}{\partial q_j}\mathbf{c}_i \\
&= tr\left(\frac{\partial \mathbf{W}_j}{\partial q_j}\sum_{i\in S(j)}\mathbf{W}_i{}^j\mathbf{M}_i\ddot{\mathbf{W}}_i^T\right) \\
&\quad + \mathbf{g}\frac{\partial \mathbf{W}_j}{\partial q_j}\sum_{i\in S(j)} m_i\mathbf{W}_i{}^j\mathbf{c}_i.
\end{aligned}$$

Recursive Formulation

We introduce two new recursively defined variables

$$\hat{\mathbf{c}}_i = m_i\mathbf{c}_i + \sum_{j\in S(i)}\mathbf{R}_j\hat{\mathbf{c}}_j$$

$$\hat{\ddot{\mathbf{M}}}_i = \mathbf{M}_i\ddot{\mathbf{W}}_i^T + \sum_{j\in S(i)}\mathbf{R}_j\hat{\ddot{\mathbf{M}}}_j$$

. Finally, recursive definition of the Newtonian constraint

$$\begin{aligned}
C_j =& tr\left(\frac{\partial \mathbf{W}_j}{\partial q_j}\hat{\ddot{\mathbf{M}}}_j\right) + \mathbf{g}\frac{\partial \mathbf{W}_j}{\partial q_j}\hat{\mathbf{c}}_j \\
&+ \sum_k\left[\frac{\partial \mathbf{F}_k\mathbf{p}_{F_k}}{\partial q_j}\right] + \sum_l\left[\mathbf{q}_l^\lambda\frac{\partial \mathbf{C}_{m_l}}{\partial q_j}\right].
\end{aligned}$$