Topics in Articulated Animation

Reading

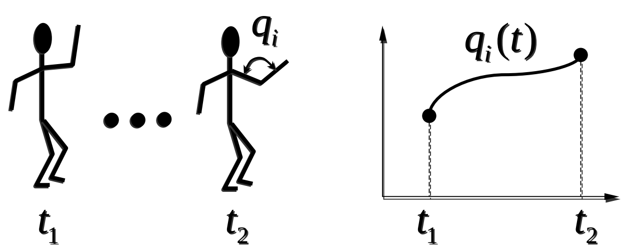
Shoemake, "Quaternions Tutorial"

Animation

Articulated models:

- rigid parts
- connected by joints

They can be animated by specifying the joint angles (or other display parameters) as functions of time.



Character Representation

Character Models are rich, complex

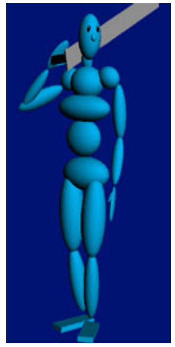
- hair, clothes (particle systems)
- muscles, skin (FFD's *etc.*)

Focus is rigid-body Degrees of Freedom (DOFs)

- joint angles

Simple Rigid Body → Skeleton

vs.



Kinematics and dynamics

Kinematics: how the positions of the parts vary as a function of the joint angles.

Dynamics: how the positions of the parts vary as a function of applied forces.

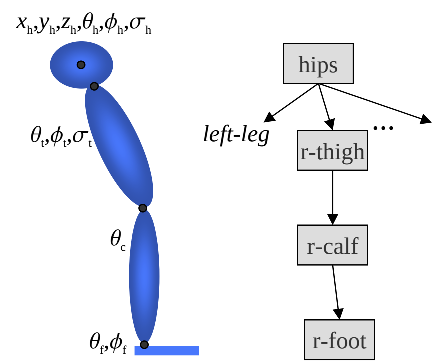
Key-frame animation

- Each joint specified at various **key frames** (not necessarily the same as other joints)
- System does interpolation or **in-betweening**

Doing this well requires:

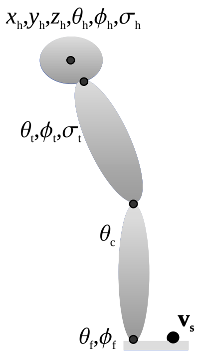
- A way of smoothly interpolating key frames: **splines**
- A good interactive system
- A lot of skill on the part of the animator

Efficient Skeleton: Hierarchy



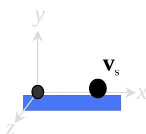
- each bone relative to parent
- easy to limit joint angles

Computing a Sensor Position



Forward kinematics

- uses vector-matrix multiplication
- transformation matrix is composition of all joint transforms between sensor/effector and root



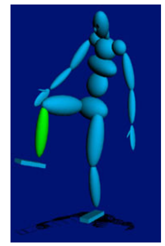
$$\mathbf{v}_w = \mathbf{T}(x_h, y_h, z_h)\mathbf{R}(\theta_h, \phi_h, \sigma_h)\ \mathbf{TR}(\theta_t, \phi_t, \sigma_t)\ \mathbf{TR}(\theta_c)\ \mathbf{TR}(\theta_f, \phi_f)\ \mathbf{v}_s$$

Joints = Rotations

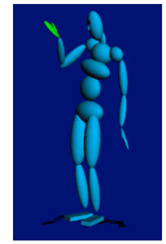
To specify a pose, we specify the joint-angle rotations

Each joint can have up to three rotational DOFs

1 DOF: knee



2 DOF: wrist



3 DOF: arm

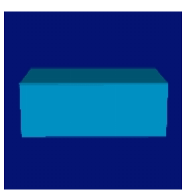
Euler angles

An Euler angle is a rotation about a single Cartesian axis

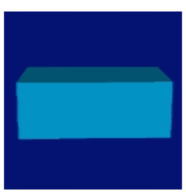
Create multi-DOF rotations by concatenating Eulers

Can get three DOF by concatenating:

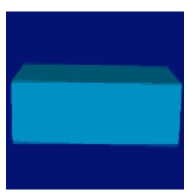
Euler-X



Euler-Y



Euler-Z



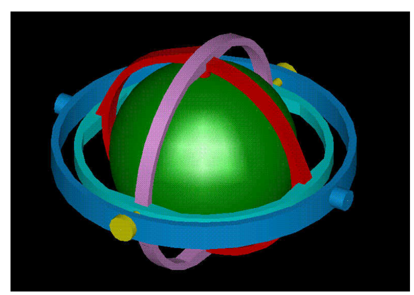
Singularities

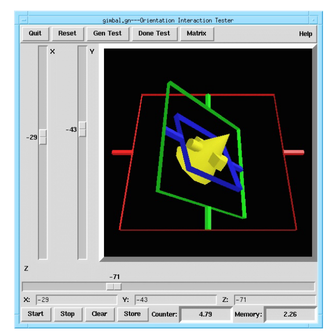
What *is* a singularity?

- continuous subspace of parameter space all of whose elements map to same rotation

Why is this bad?

- induces **gimbal lock** - two or more axes align, results in loss of rotational DOFs (*i.e.* derivatives)



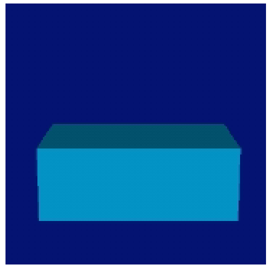


Singularities in Action

An object whose orientation is controlled by Euler rotation XYZ(θ,ϕ,σ)

(0,0,0) : Okay

(0, ± 90°, 0) : X and Z axes align

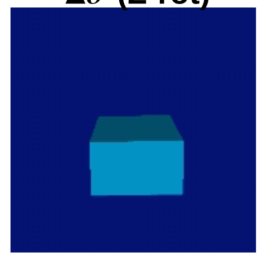


Eliminates a DOF

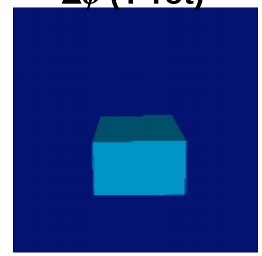
In this configuration, changing θ (X Euler angle) and σ (Z Euler angle) produce the same result.

No way to rotate around world X axis!

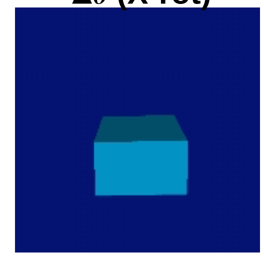
$\Delta\sigma$ (Z-rot)



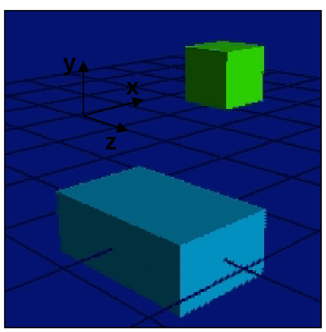
$\Delta\phi$ (Y-rot)



$\Delta\theta$ (X-rot)



Resulting Behavior



No applied force or other stimuli can induce rotation about world X-axis

The object locks up!!

Singularities in Euler Angles

Cannot be avoided (occur at 0° or 90°)

Difficult to work around

But, only affects three DOF rotations

Other Properties of Euler Angles

Several important tasks are easy:

- interactive specification (sliders, *etc.*)
- joint limits
- Euclidean interpolation (Hermites, Beziers, *etc.*)
  - May be funky for tumbling bodies
  - fine for most joints

Quaternions

But… singularities are unacceptable for IK, optimization

Traditional solution: Use unit quaternions to represent rotations

- S^3 has same topology as rotation space (a sphere), so no singularities

History of Quaternions

Invented by Sir William Rowan Hamilton in 1843

$$H = w + \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$$
$$\text{where } \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$$

I still must assert that this discovery appears to me to be as important for the middle of the nineteenth century as the discovery of fluxions [the calculus] was for the close of the seventeenth.

Hamilton

[quaternions] … although beautifully ingenious, have been an unmixed evil to those who have touched them in any way.

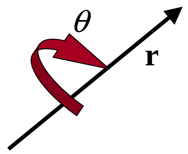
Thompson

Quaternion as a 4 vector

$$\mathbf{q} = \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} w \\ \mathbf{v} \end{pmatrix}$$

Axis-angle rotation as a quaternion

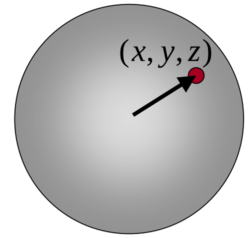
$$\mathbf{q} = \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} w \\ \mathbf{v} \end{pmatrix}$$



$$\mathbf{q} = \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2)\mathbf{r} \end{pmatrix}$$

Unit Quaternions

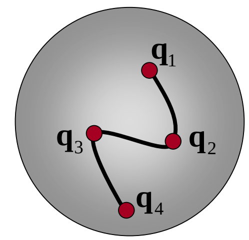
$$\mathbf{q} = \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}$$



$$w = \sqrt{1-(x^2+y^2+z^2)}$$

$$|\mathbf{q}| = 1$$

$$x^2 + y^2 + z^2 + w^2 = 1$$



Quaternion Product

$$\begin{pmatrix} w_1 \\ \mathbf{v}_1 \end{pmatrix}\begin{pmatrix} w_2 \\ \mathbf{v}_2 \end{pmatrix} = \begin{pmatrix} w_1 w_2 - \mathbf{v}_1 \cdot \mathbf{v}_2 \\ w_1\mathbf{v}_2 + w_2\mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2 \end{pmatrix}$$

$$\begin{pmatrix} w_1 \\ \mathbf{v}_1 \end{pmatrix}\begin{pmatrix} w_2 \\ \mathbf{v}_2 \end{pmatrix} \neq \begin{pmatrix} w_2 \\ \mathbf{v}_2 \end{pmatrix}\begin{pmatrix} w_1 \\ \mathbf{v}_1 \end{pmatrix}$$

Quaternion Conjugate

$$\mathbf{q}^* = \begin{pmatrix} w_1 \\ \mathbf{v}_1 \end{pmatrix}^* = \begin{pmatrix} w_1 \\ -\mathbf{v}_1 \end{pmatrix}$$

$$(\mathbf{p}^*)^* = \mathbf{p}$$

$$(\mathbf{pq})^* = \mathbf{q}^*\mathbf{p}^*$$

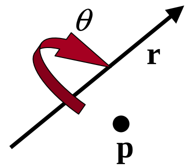
$$(\mathbf{p}+\mathbf{q})^* = \mathbf{p}^* + \mathbf{q}^*$$

Quaternion Inverse

$$\mathbf{q}^{-1}\mathbf{q} = 1$$

$$\mathbf{q}^{-1} = \mathbf{q}^{*} / |\mathbf{q}| = \begin{pmatrix} w \\ -\mathbf{v} \end{pmatrix} / |\mathbf{q}| = \begin{pmatrix} w \\ -\mathbf{v} \end{pmatrix} / (w^2 + \mathbf{v}\cdot\mathbf{v})$$

Quaternion Rotation

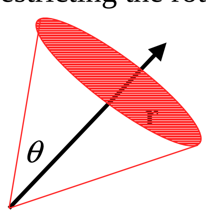


$$\mathbf{q}\mathbf{p}\mathbf{q}^{-1} = \begin{pmatrix} w \\ \mathbf{v} \end{pmatrix}\begin{pmatrix} 0 \\ \mathbf{p} \end{pmatrix}\begin{pmatrix} w \\ -\mathbf{v} \end{pmatrix}$$
$$= \begin{pmatrix} w \\ \mathbf{v} \end{pmatrix}\begin{pmatrix} \mathbf{p}\cdot\mathbf{v} \\ w\mathbf{p} - \mathbf{p}\times\mathbf{v} \end{pmatrix}$$
$$= \begin{pmatrix} w\mathbf{p}\cdot\mathbf{v} - w\mathbf{p}\cdot\mathbf{v} = 0 \\ w(w\mathbf{p} - \mathbf{p}\mathbf{v}) + (\mathbf{p}\cdot\mathbf{v})\mathbf{v} + \mathbf{v}(w\mathbf{p} - \mathbf{p}\times\mathbf{v}) \end{pmatrix}$$

What about a quaternion product $\mathbf{q}_1\mathbf{q}_2$?

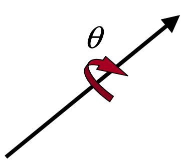
Quaternion constraints

Restricting the rotation cone



$$\frac{1-\cos(\theta_x)}{2} = q_y^{\ 2} + q_z^{\ 2}$$

Restricting the rotation twist around an axis



$$\tan(\theta/2) = \frac{q_{axis}}{q_w}$$

Matrix Form

$$\mathbf{q} = \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}$$

$$\mathbf{M} = \begin{pmatrix} 1-2y^2-2z^2 & 2xy+2wz & 2xz-2wy \\ 2xy-2wz & 1-2x^2-2z^2 & 2yz+2wx \\ 2xz+2wy & 2yz-2wx & 1-2x^2-2y^2 \end{pmatrix}$$

Normalized Quaternion Matrix Form

$$\mathbf{q}_n = \alpha \mathbf{q}_n$$

$$\mathbf{q}_n = \frac{1}{\|\mathbf{q}\|}\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}$$

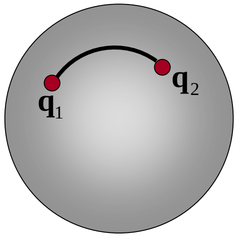
$$\mathbf{M}_n = \frac{1}{\|\mathbf{q}\|^2}\begin{pmatrix} \|\mathbf{q}\|^2 - 2y^2 - 2z^2 & 2xy + 2wz & 2xz - 2wy \\ 2xy - 2wz & \|\mathbf{q}\|^2 - 2x^2 - 2z^2 & 2yz + 2wx \\ 2xz + 2wy & 2yz - 2wx & \|\mathbf{q}\|^2 - 2x^2 - 2y^2 \end{pmatrix}$$

Quaternion interpolation

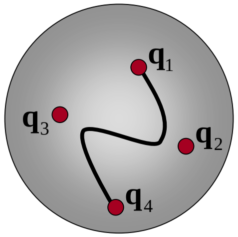
Spherical linear interpolation (SLERP)

$$slerp(t; q_1, q_2) = q_1 \frac{\sin(\theta)(1-t)}{\sin(\theta)} + q_2 \frac{\sin(\theta)t}{\sin(\theta)}$$

$$\cos(\theta) = q_1 \cdot q_2$$



Spherical cubic interpolation (SQUAD)



$$squad(t; q_1, q_2, q_3, q_4) = slerp\left(2t(1-t); slerp(t; q_1, q_4), slerp(t; q_2, q_3)\right)$$

Quaternions: What Works

Simple formulae for converting to rotation matrix

Continuous derivatives - no singularities

“Optimal” interpolation - geodesics map to shortest paths in rotation space

Nice calculus (corresponds to rotations)