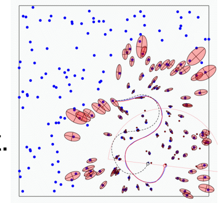
CSE-571
Robotics

SLAM: Simultaneous Localization and Mapping

Many slides courtesy of Ryan Eustice, Cyrill Stachniss, John Leonard

The SLAM Problem



A robot is exploring an unknown, static environment.

Given:

- The robot's controls
- Observations of nearby features

Estimate:

- Map of features
- Path of the robot

SLAM Applications

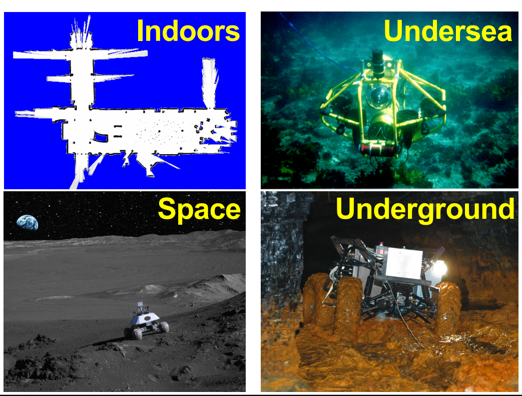
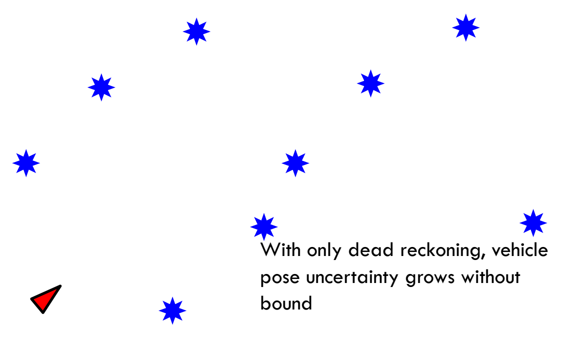


Illustration of SLAM without Landmarks



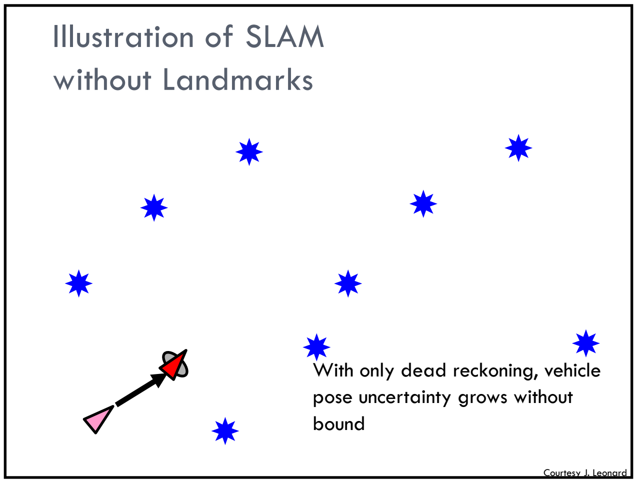
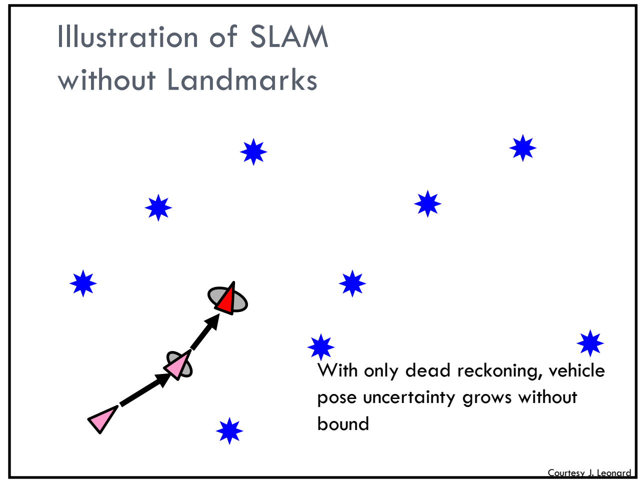


Illustration of SLAM
without Landmarks
With only dead reckoning, vehicle pose uncertainty grows without bound
Courtesy J. Leonard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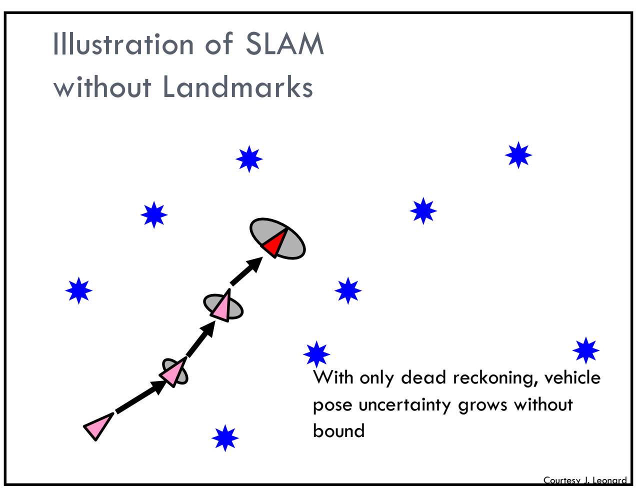
Courtesy J. Leonard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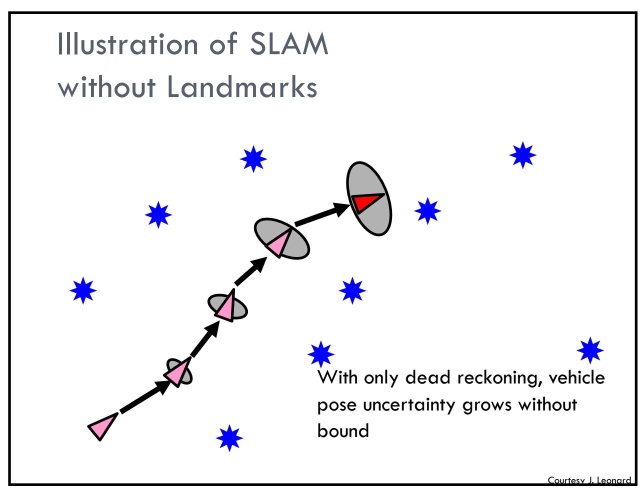
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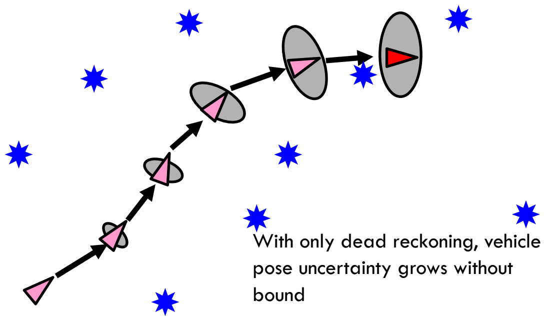
Courtesy J. Leonard

Illustration of SLAM without Landmarks

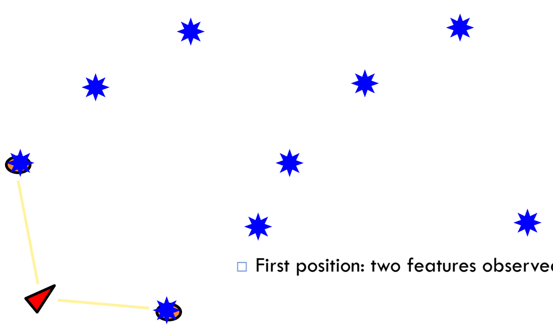


With only dead reckoning, vehicle pose uncertainty grows without bound

Courtesy J. Leonard

Mapping with Raw Odometry

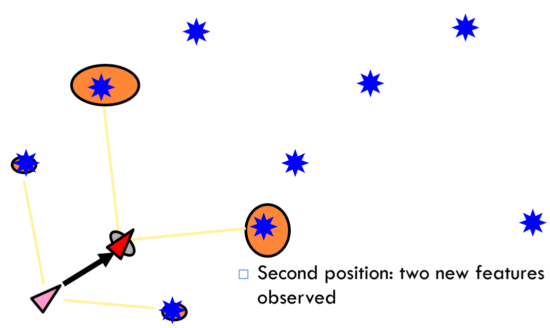
Repeat, with Measurements of Landmarks



- First position: two features observed

Courtesy J. Leonard

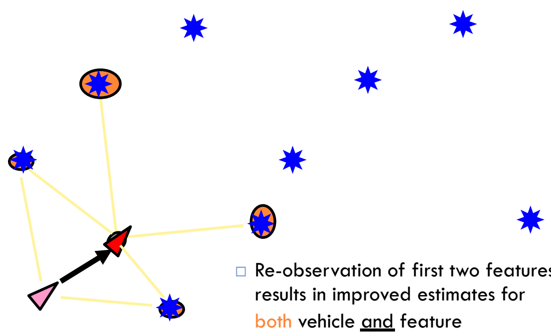
Illustration of SLAM with Landmarks



- Second position: two new features observed

Courtesy J. Leonard

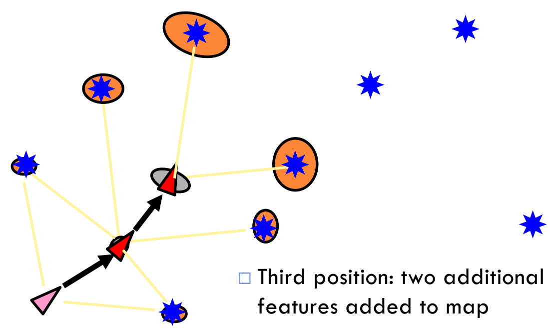
Illustration of SLAM with Landmarks



- Re-observation of first two features results in improved estimates for both vehicle and feature

Courtesy J. Leonard

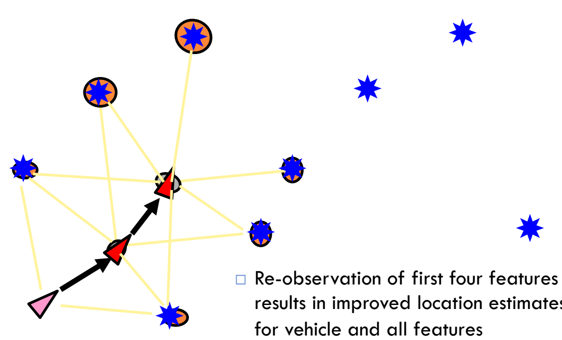
Illustration of SLAM with Landmarks



- Third position: two additional features added to map

Courtesy J. Leonard

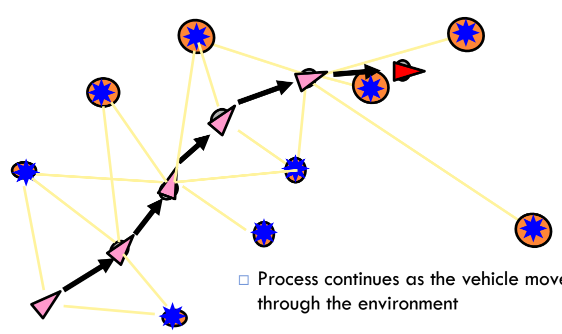
Illustration of SLAM with Landmarks



- Re-observation of first four features results in improved location estimates for vehicle and all features

Courtesy J. Leonard

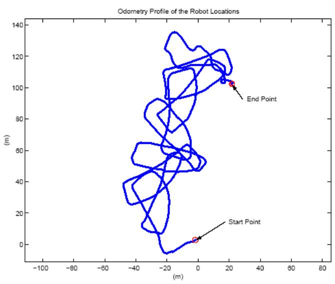
Illustration of SLAM with Landmarks



- Process continues as the vehicle moves through the environment

Courtesy J. Leonard

SLAM Using Landmarks

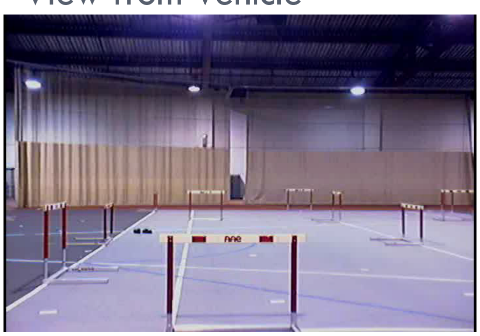


Courtesy J. Leonard

Test Environment (Point Landmarks)

Courtesy J. Leonard

View from Vehicle



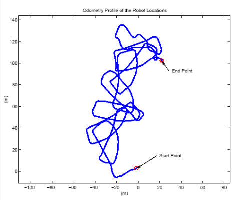
Courtesy J. Leonard

SLAM Using Landmarks

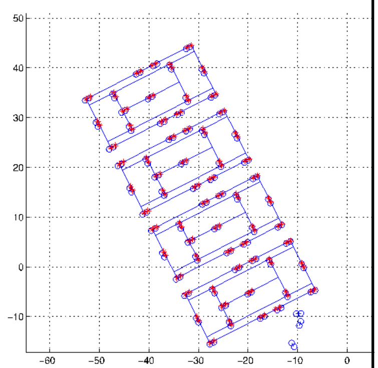
1. Move
2. Sense
3. Associate measurements with known features
4. Update state estimates for robot and previously mapped features
5. Find new features from unassociated measurements
6. Initialize new features
7. Repeat

MIT Indoor Track

Comparison with Ground Truth



odometry

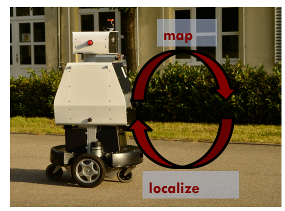


SLAM result

Courtesy J. Leonard

Simultaneous Localization and Mapping (SLAM)

- Building a map and locating the robot in the map at the same time
- Chicken-and-egg problem



Courtesy: Cyrill Stachniss

Definition of the SLAM Problem

Given

- The robot's controls
  $u_{1:T} = \{u_1, u_2, u_3, \ldots, u_T\}$
- Observations
  $z_{1:T} = \{z_1, z_2, z_3, \ldots, z_T\}$

Wanted

- Map of the environment
  m
- Path of the robot
  $x_{0:T} = \{x_0, x_1, x_2, \ldots, x_T\}$

Courtesy: Cyrill Stachniss

Three Main Paradigms

Kalman filter	Particle filter	Graph-based

Courtesy: Cyrill Stachniss

Bayes Filter

- Recursive filter with prediction and correction step
- Prediction

$$\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1})\ bel(x_{t-1})\, dx_{t-1}$$

- Correction

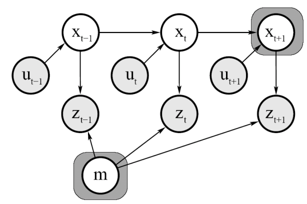
$$bel(x_t) = \eta\ p(z_t \mid x_t)\ \overline{bel}(x_t)$$

Courtesy: Cyrill Stachniss

EKF for Online SLAM

- We consider here the Kalman filter as a solution to the online SLAM problem

$$p(x_t, m \mid z_{1:t}, u_{1:t})$$



Courtesy: Thrun, Burgard, Fox

Extended Kalman Filter Algorithm

1: **Extended_Kalman_filter**$(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t)$**:**

2: $\bar{\mu}_t = g(u_t, \mu_{t-1})$
3: $\bar{\Sigma}_t = G_t\ \Sigma_{t-1}\ G_t^T + R_t$

4: $K_t = \bar{\Sigma}_t\ H_t^T (H_t\ \bar{\Sigma}_t\ H_t^T + Q_t)^{-1}$
5: $\mu_t = \bar{\mu}_t + K_t(z_t - h(\bar{\mu}_t))$
6: $\Sigma_t = (I - K_t\ H_t)\ \bar{\Sigma}_t$
7: return μ_t, Σ_t

Courtesy: Cyrill Stachniss

EKF SLAM

- Application of the EKF to SLAM
- Estimate robot's pose and locations of landmarks in the environment
- Assumption: known correspondences
- State space (for the 2D plane) is

$$x_t = (\underbrace{x, y, \theta}_{\text{robot's pose}}, \underbrace{m_{1,x}, m_{1,y}}_{\text{landmark 1}}, \ldots, \underbrace{m_{n,x}, m_{n,y}}_{\text{landmark n}})^T$$

Courtesy: Cyrill Stachniss

EKF SLAM: State Representation

- Map with n landmarks: (3+2n)-dimensional Gaussian
- Belief is represented by

$$\underbrace{\begin{pmatrix} x \\ y \\ \theta \\ m_{1,x} \\ m_{1,y} \\ \vdots \\ m_{n,x} \\ m_{n,y} \end{pmatrix}}_{\mu} \underbrace{\begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{x\theta} & \sigma_{xm_{1,x}} & \sigma_{xm_{1,y}} & \cdots & \sigma_{xm_{n,x}} & \sigma_{xm_{n,y}} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{y\theta} & \sigma_{ym_{1,x}} & \sigma_{ym_{1,y}} & \cdots & \sigma_{m_{n,x}} & \sigma_{m_{n,y}} \\ \sigma_{\theta x} & \sigma_{\theta y} & \sigma_{\theta\theta} & \sigma_{\theta m_{1,x}} & \sigma_{\theta m_{1,y}} & \cdots & \sigma_{\theta m_{n,x}} & \sigma_{\theta m_{n,y}} \\ \sigma_{m_{1,x}x} & \sigma_{m_{1,x}y} & \sigma_{\theta} & \sigma_{m_{1,x}m_{1,x}} & \sigma_{m_{1,x}m_{1,y}} & \cdots & \sigma_{m_{1,x}m_{n,x}} & \sigma_{m_{1,x}m_{n,y}} \\ \sigma_{m_{1,y}x} & \sigma_{m_{1,y}y} & \sigma_{\theta} & \sigma_{m_{1,y}m_{1,x}} & \sigma_{m_{1,y}m_{1,y}} & \cdots & \sigma_{m_{1,y}m_{n,x}} & \sigma_{m_{1,y}m_{n,y}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \sigma_{m_{n,x}x} & \sigma_{m_{n,x}y} & \sigma_{\theta} & \sigma_{m_{n,x}m_{1,x}} & \sigma_{m_{n,x}m_{1,y}} & \cdots & \sigma_{m_{n,x}m_{n,x}} & \sigma_{m_{n,x}m_{n,y}} \\ \sigma_{m_{n,y}x} & \sigma_{m_{n,y}y} & \sigma_{\theta} & \sigma_{m_{n,y}m_{1,x}} & \sigma_{m_{n,y}m_{1,y}} & \cdots & \sigma_{m_{n,y}m_{n,x}} & \sigma_{m_{n,y}m_{n,y}} \end{pmatrix}}_{\Sigma}$$

Courtesy: Cyrill Stachniss

EKF SLAM: State Representation

- More compactly

$$\underbrace{\begin{pmatrix} x_R \\ m_1 \\ \vdots \\ m_n \end{pmatrix}}_{\mu} \underbrace{\begin{pmatrix} \Sigma_{x_R x_R} & \Sigma_{x_R m_1} & \cdots & \Sigma_{x_R m_n} \\ \Sigma_{m_1 x_R} & \Sigma_{m_1 m_1} & \cdots & \Sigma_{m_1 m_n} \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_{m_n x_R} & \Sigma_{m_n m_1} & \cdots & \Sigma_{m_n m_n} \end{pmatrix}}_{\Sigma}$$

Courtesy: Cyrill Stachniss

EKF SLAM: State Representation

- Even more compactly (note: $x_R \rightarrow x$)

$$\underbrace{\begin{pmatrix} x \\ m \end{pmatrix}}_{\mu} \underbrace{\begin{pmatrix} \Sigma_{xx} & \Sigma_{xm} \\ \Sigma_{mx} & \Sigma_{mm} \end{pmatrix}}_{\Sigma}$$

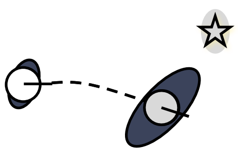
Courtesy: Cyrill Stachniss

EKF SLAM: Filter Cycle

1. State prediction
2. Measurement prediction
3. Measurement
4. Data association
5. Update

Courtesy: Cyrill Stachniss

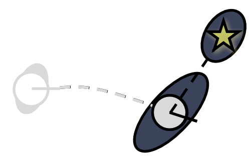
EKF SLAM: State Prediction



$$\underbrace{\begin{pmatrix} x_R \\ m_1 \\ \vdots \\ m_n \end{pmatrix}}_{\mu} \quad \underbrace{\begin{pmatrix} \Sigma_{x_R x_R} & \Sigma_{x_R m_1} & \dots & \Sigma_{x_R m_n} \\ \Sigma_{m_1 x_R} & \Sigma_{m_1 m_1} & \dots & \Sigma_{m_1 m_n} \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_{m_n x_R} & \Sigma_{m_n m_1} & \dots & \Sigma_{m_n m_n} \end{pmatrix}}_{\Sigma}$$

Courtesy: Cyrill Stachniss

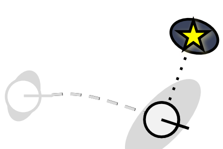
EKF SLAM: Measurement Prediction



$$\underbrace{\begin{pmatrix} x_R \\ m_1 \\ \vdots \\ m_n \end{pmatrix}}_{\mu} \quad \underbrace{\begin{pmatrix} \Sigma_{x_R x_R} & \Sigma_{x_R m_1} & \dots & \Sigma_{x_R m_n} \\ \Sigma_{m_1 x_R} & \Sigma_{m_1 m_1} & \dots & \Sigma_{m_1 m_n} \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_{m_n x_R} & \Sigma_{m_n m_1} & \dots & \Sigma_{m_n m_n} \end{pmatrix}}_{\Sigma}$$

Courtesy: Cyrill Stachniss

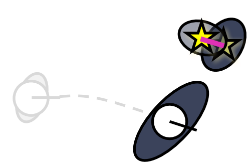
EKF SLAM: Obtained Measurement



$$\underbrace{\begin{pmatrix} x_R \\ m_1 \\ \vdots \\ m_n \end{pmatrix}}_{\mu} \quad \underbrace{\begin{pmatrix} \Sigma_{x_R x_R} & \Sigma_{x_R m_1} & \dots & \Sigma_{x_R m_n} \\ \Sigma_{m_1 x_R} & \Sigma_{m_1 m_1} & \dots & \Sigma_{m_1 m_n} \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_{m_n x_R} & \Sigma_{m_n m_1} & \dots & \Sigma_{m_n m_n} \end{pmatrix}}_{\Sigma}$$

Courtesy: Cyrill Stachniss

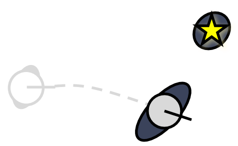
EKF SLAM: Data Association and Difference Between h(x) and z



$$\underbrace{\begin{pmatrix} x_R \\ m_1 \\ \vdots \\ m_n \end{pmatrix}}_{\mu} \quad \underbrace{\begin{pmatrix} \Sigma_{x_R x_R} & \Sigma_{x_R m_1} & \dots & \Sigma_{x_R m_n} \\ \Sigma_{m_1 x_R} & \Sigma_{m_1 m_1} & \dots & \Sigma_{m_1 m_n} \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_{m_n x_R} & \Sigma_{m_n m_1} & \dots & \Sigma_{m_n m_n} \end{pmatrix}}_{\Sigma}$$

Courtesy: Cyrill Stachniss

EKF SLAM: Update Step



$$\underbrace{\begin{pmatrix} x_R \\ m_1 \\ \vdots \\ m_n \end{pmatrix}}_{\mu} \quad \underbrace{\begin{pmatrix} \Sigma_{x_R x_R} & \Sigma_{x_R m_1} & \dots & \Sigma_{x_R m_n} \\ \Sigma_{m_1 x_R} & \Sigma_{m_1 m_1} & \dots & \Sigma_{m_1 m_n} \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_{m_n x_R} & \Sigma_{m_n m_1} & \dots & \Sigma_{m_n m_n} \end{pmatrix}}_{\Sigma}$$

Courtesy: Cyrill Stachniss

EKF SLAM: Concrete Example

Setup

- Robot moves in the 2D plane
- Velocity-based motion model
- Robot observes point landmarks
- Range-bearing sensor
- Known data association
- Known number of landmarks

Courtesy: Cyrill Stachniss

Initialization

- Robot starts in its own reference frame (all landmarks unknown)
- 2N+3 dimensions

$$\mu_0 = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \end{pmatrix}^T$$

$$\Sigma_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \infty & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \infty \end{pmatrix}$$

Courtesy: Cyrill Stachniss

Extended Kalman Filter Algorithm

1: **Extended_Kalman_filter**$(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t)$**:**

2: $\bar{\mu}_t = g(u_t, \mu_{t-1})$
3: $\bar{\Sigma}_t = G_t\, \Sigma_{t-1}\, G_t^T + R_t$

4: $K_t = \bar{\Sigma}_t\, H_t^T (H_t\, \bar{\Sigma}_t\, H_t^T + Q_t)^{-1}$
5: $\mu_t = \bar{\mu}_t + K_t(z_t - h(\bar{\mu}_t))$
6: $\Sigma_t = (I - K_t\, H_t)\, \bar{\Sigma}_t$
7: return μ_t, Σ_t

Courtesy: Cyrill Stachniss

Prediction Step (Motion)

- Goal: Update state space based on the robot's motion
- Robot motion in the plane

$$\begin{pmatrix} x' \\ y' \\ \theta' \end{pmatrix} = \underbrace{\begin{pmatrix} x \\ y \\ \theta \end{pmatrix} + \begin{pmatrix} -\frac{v_t}{\omega_t}\sin\theta + \frac{v_t}{\omega_t}\sin(\theta + \omega_t\Delta t) \\ \frac{v_t}{\omega_t}\cos\theta - \frac{v_t}{\omega_t}\cos(\theta + \omega_t\Delta t) \\ \omega_t\Delta t \end{pmatrix}}_{g_{x,y,\theta}(u_t,(x,y,\theta)^T)}$$

- How to map that to the 2N+3 dim space?

Courtesy: Cyrill Stachniss

Update the State Space

- From the motion in the plane

$$\begin{pmatrix} x' \\ y' \\ \theta' \end{pmatrix} = \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} + \begin{pmatrix} -\frac{v_t}{\omega_t}\sin\theta + \frac{v_t}{\omega_t}\sin(\theta + \omega_t\Delta t) \\ \frac{v_t}{\omega_t}\cos\theta - \frac{v_t}{\omega_t}\cos(\theta + \omega_t\Delta t) \\ \omega_t\Delta t \end{pmatrix}$$

- to the 2N+3 dimensional space

$$\begin{pmatrix} x' \\ y' \\ \theta' \\ \vdots \end{pmatrix} = \underbrace{\begin{pmatrix} x \\ y \\ \theta \\ \vdots \end{pmatrix} + \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0\ldots0 \\ 0 & 1 & 0 & 0\ldots0 \\ 0 & 0 & 1 & \underbrace{0\ldots0}_{2N cols} \end{pmatrix}^T}_{F_x^T} \begin{pmatrix} -\frac{v_t}{\omega_t}\sin\theta + \frac{v_t}{\omega_t}\sin(\theta + \omega_t\Delta t) \\ \frac{v_t}{\omega_t}\cos\theta - \frac{v_t}{\omega_t}\cos(\theta + \omega_t\Delta t) \\ \omega_t\Delta t \end{pmatrix}}_{g(u_t,x_t)}$$

Courtesy: Cyrill Stachniss

Extended Kalman Filter Algorithm

1: $\textbf{Extended\_Kalman\_filter}(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t)$:

2: ~~$\bar{\mu}_t = g(u_t, \mu_{t-1})$~~ DONE

3: $\bar{\Sigma}_t = G_t\ \Sigma_{t-1}\ G_t^T + R_t$

4: $K_t = \bar{\Sigma}_t\ H_t^T(H_t\ \bar{\Sigma}_t\ H_t^T + Q_t)^{-1}$

5: $\mu_t = \bar{\mu}_t + K_t(z_t - h(\bar{\mu}_t))$

6: $\Sigma_t = (I - K_t\ H_t)\ \bar{\Sigma}_t$

7: return μ_t, Σ_t

Courtesy: Cyrill Stachniss

Update Covariance

- The function g only affects the robot's motion and not the landmarks

Jacobian of the motion (3x3)

$$G_t = \begin{pmatrix} G_t^x & 0 \\ 0 & I \end{pmatrix}$$

Identity (2N x 2N)

Courtesy: Cyrill Stachniss

This Leads to the Time Propagation

1: **Extended_Kalman_filter**$(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t)$**:**

2: ~~$\bar{\mu}_t = g(u_t, \mu_{t-1})$~~ Apply & DONE
3: ➡ $\bar{\Sigma}_t = G_t \, \Sigma_{t-1} \, G_t^T + R_t$

$$\begin{aligned}\bar{\Sigma}_t &= G_t \Sigma_{t-1} G_t^T + R_t \\ &= \begin{pmatrix} G_t^x & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \Sigma_{xx} & \Sigma_{xm} \\ \Sigma_{mx} & \Sigma_{mm} \end{pmatrix} \begin{pmatrix} (G_t^x)^T & 0 \\ 0 & I \end{pmatrix} + R_t \\ &= \begin{pmatrix} G_t^x \Sigma_{xx} (G_t^x)^T & G_t^x \Sigma_{xm} \\ (G_t^x \Sigma_{xm})^T & \Sigma_{mm} \end{pmatrix} + R_t\end{aligned}$$

Courtesy: Cyrill Stachniss

Extended Kalman Filter Algorithm

1: **Extended_Kalman_filter**$(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t)$**:**

2: ~~$\bar{\mu}_t = g(u_t, \mu_{t-1})$~~ DONE
3: ~~$\bar{\Sigma}_t = G_t \, \Sigma_{t-1} \, G_t^T + R_t$~~ DONE

4: $K_t = \bar{\Sigma}_t \, H_t^T (H_t \, \bar{\Sigma}_t \, H_t^T + Q_t)^{-1}$
5: $\mu_t = \bar{\mu}_t + K_t(z_t - h(\bar{\mu}_t))$
6: $\Sigma_t = (I - K_t \, H_t) \, \bar{\Sigma}_t$
7: *return* μ_t, Σ_t

Courtesy: Cyrill Stachniss

EKF SLAM: Correction Step

- Known data association
- $c_t^i = j$: *i*-th measurement at time *t* observes the landmark with index *j*
- Initialize landmark if unobserved
- Compute the expected observation
- Compute the Jacobian of h
- Proceed with computing the Kalman gain

Courtesy: Cyrill Stachniss

Range-Bearing Observation

- Range-Bearing observation $z_t^i = (r_t^i, \phi_t^i)^T$
- If landmark has not been observed

$$\begin{pmatrix} \bar{\mu}_{j,x} \\ \bar{\mu}_{j,y} \end{pmatrix} = \begin{pmatrix} \bar{\mu}_{t,x} \\ \bar{\mu}_{t,y} \end{pmatrix} + \begin{pmatrix} r_t^i \cos(\phi_t^i + \bar{\mu}_{t,\theta}) \\ r_t^i \sin(\phi_t^i + \bar{\mu}_{t,\theta}) \end{pmatrix}$$

observed location of landmark j — estimated robot's location — relative measurement

Courtesy: Cyrill Stachniss

Jacobian for the Observation

- Based on

$$\delta = \begin{pmatrix} \delta_x \\ \delta_y \end{pmatrix} = \begin{pmatrix} \bar{\mu}_{j,x} - \bar{\mu}_{t,x} \\ \bar{\mu}_{j,y} - \bar{\mu}_{t,y} \end{pmatrix}$$

$$q = \delta^T \delta$$

$$\hat{z}_t^i = \begin{pmatrix} \sqrt{q} \\ \operatorname{atan2}(\delta_y, \delta_x) - \bar{\mu}_{t,\theta} \end{pmatrix}$$

- Compute the Jacobian

$$ {}^{\text{low}}H_t^i = \frac{\partial h(\bar{\mu}_t)}{\partial \bar{\mu}_t} = \frac{1}{q} \begin{pmatrix} -\sqrt{q}\delta_x & -\sqrt{q}\delta_y & 0 & +\sqrt{q}\delta_x & \sqrt{q}\delta_y \\ \delta_y & -\delta_x & -q & -\delta_y & \delta_x \end{pmatrix}$$

Courtesy: Cyrill Stachniss

Jacobian for the Observation

- Use the computed Jacobian

$$ {}^{\text{low}}H_t^i = \frac{1}{q} \begin{pmatrix} -\sqrt{q}\delta_x & -\sqrt{q}\delta_y & 0 & +\sqrt{q}\delta_x & \sqrt{q}\delta_y \\ \delta_y & -\delta_x & -q & -\delta_y & \delta_x \end{pmatrix}$$

- map it to the high dimensional space

$$H_t^i = {}^{\text{low}}H_t^i \, F_{x,j}$$

$$F_{x,j} = \begin{pmatrix} 1 & 0 & 0 & 0\cdots0 & 0 & 0 & 0\cdots0 \\ 0 & 1 & 0 & 0\cdots0 & 0 & 0 & 0\cdots0 \\ 0 & 0 & 1 & 0\cdots0 & 0 & 0 & 0\cdots0 \\ 0 & 0 & 0 & 0\cdots0 & 1 & 0 & 0\cdots0 \\ 0 & 0 & 0 & \underbrace{0\cdots0}_{2j-2} & 0 & 1 & \underbrace{0\cdots0}_{2N-2j} \end{pmatrix}$$

Courtesy: Cyrill Stachniss

Next Steps as Specified…

1: **Extended_Kalman_filter**$(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t)$:

2: ~~$\bar{\mu}_t = g(u_t, \mu_{t-1})$~~ DONE

3: ~~$\bar{\Sigma}_t = G_t \, \Sigma_{t-1} \, G_t^T + R_t$~~ DONE

4: ➡ $K_t = \bar{\Sigma}_t \, H_t^T (H_t \, \bar{\Sigma}_t \, H_t^T + Q_t)^{-1}$

5: $\mu_t = \bar{\mu}_t + K_t(z_t - h(\bar{\mu}_t))$

6: $\Sigma_t = (I - K_t \, H_t) \, \bar{\Sigma}_t$

7: return μ_t, Σ_t

Courtesy: Cyrill Stachniss

Extended Kalman Filter Algorithm

1: **Extended_Kalman_filter**$(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t)$:

2: ~~$\bar{\mu}_t = g(u_t, \mu_{t-1})$~~ DONE

3: ~~$\bar{\Sigma}_t = G_t \, \Sigma_{t-1} \, G_t^T + R_t$~~ DONE

4: ~~$K_t = \bar{\Sigma}_t \, H_t^T (H_t \, \bar{\Sigma}_t \, H_t^T + Q_t)^{-1}$~~ Apply & DONE

5: ~~$\mu_t = \bar{\mu}_t + K_t(z_t - h(\bar{\mu}_t))$~~ Apply & DONE

6: ~~$\Sigma_t = (I - K_t \, H_t) \, \bar{\Sigma}_t$~~ Apply & DONE

7: ➡ return μ_t, Σ_t

Courtesy: Cyrill Stachniss

EKF SLAM – Correction (1/2)

EKF_SLAM_Correction

$Q_t = \begin{pmatrix} {\sigma_r}^2 & 0 \\ 0 & {\sigma_\phi}^2 \end{pmatrix}$

for all observed features $z_t^i = (r_t^i, \phi_t^i)^T$ *do*

$\quad j = c_t^i$

$\quad$ *if landmark* j *never seen before*

$$\begin{pmatrix}\bar\mu_{j,x}\\ \bar\mu_{j,y}\end{pmatrix} = \begin{pmatrix}\bar\mu_{t,x}\\ \bar\mu_{t,y}\end{pmatrix} + \begin{pmatrix} r_t^i \cos(\phi_t^i+\bar\mu_{t,\theta}) \\ r_t^i \sin(\phi_t^i + \bar\mu_{t,\theta})\end{pmatrix}$$

$\quad$ *endif*

$$\delta = \begin{pmatrix}\delta_x\\ \delta_y\end{pmatrix} = \begin{pmatrix}\bar\mu_{j,x}-\bar\mu_{t,x}\\ \bar\mu_{j,y}-\bar\mu_{t,y}\end{pmatrix}$$

$q = \delta^T\delta$

$$\hat z_t^i = \begin{pmatrix}\sqrt{q} \\ \text{atan2}(\delta_y,\delta_x) - \bar\mu_{t,\theta}\end{pmatrix}$$

Courtesy: Cyrill Stachniss

EKF SLAM – Correction (2/2)

$$F_{x,j} = \begin{pmatrix} 1 & 0 & 0 & 0\cdots0 & 0 & 0 & 0\cdots0 \\ 0 & 1 & 0 & 0\cdots0 & 0 & 0 & 0\cdots0 \\ 0 & 0 & 1 & 0\cdots0 & 0 & 0 & 0\cdots0 \\ 0 & 0 & 0 & 0\cdots0 & 1 & 0 & 0\cdots0 \\ 0 & 0 & 0 & \underbrace{0\cdots0}_{2j-2} & 0 & 1 & \underbrace{0\cdots0}_{2N-2j} \end{pmatrix}$$

$$H_t^i = \frac{1}{q}\begin{pmatrix} -\sqrt{q}\delta_x & -\sqrt{q}\delta_y & 0 & +\sqrt{q}\delta_x & \sqrt{q}\delta_y \\ \delta_y & -\delta_x & -q & -\delta_y & +\delta_x\end{pmatrix} F_{x,j}$$

$K_t^i = \bar\Sigma_t\, H_t^{iT}(H_t^i\, \bar\Sigma_t\, H_t^{iT} + Q_t)^{-1}$

$\bar\mu_t = \bar\mu_t + K_t^i(z_t^i - \hat z_t^i)$

$\bar\Sigma_t = (I - K_t^i\, H_t^i)\, \bar\Sigma_t$

endfor

$\mu_t = \bar\mu_t$

$\Sigma_t = \bar\Sigma_t$

return μ_t, Σ_t

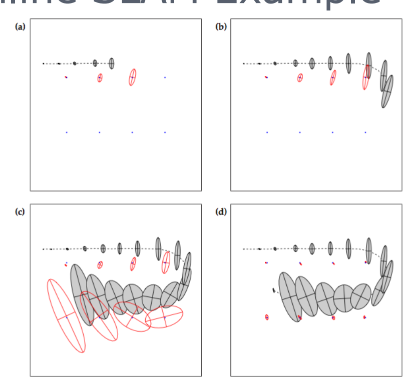
Courtesy: Cyrill Stachniss

EKF SLAM Complexity

- Cubic complexity depends only on the measurement dimensionality
- Cost per step: dominated by the number of landmarks: $O(n^2)$
- Memory consumption: $O(n^2)$
- The EKF becomes computationally intractable for large maps!

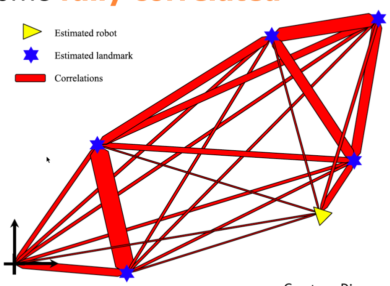
Courtesy: Cyrill Stachniss

Online SLAM Example



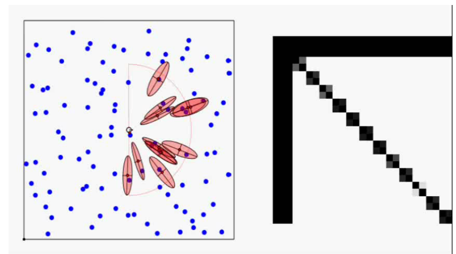
EKF SLAM Correlations

- In the limit, the landmark estimates become **fully correlated**



Courtesy: Dissanayake

EKF SLAM Correlations

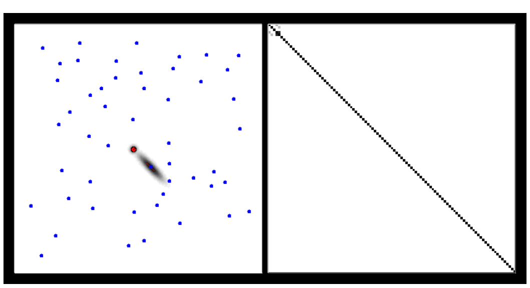


Blue path = true path **Red path** = estimated path **Black path** = odometry

- Approximate the SLAM posterior with a high-dimensional Gaussian [Smith & Cheesman, 1986] …
- Single hypothesis data association

Courtesy: M. Montemerlo

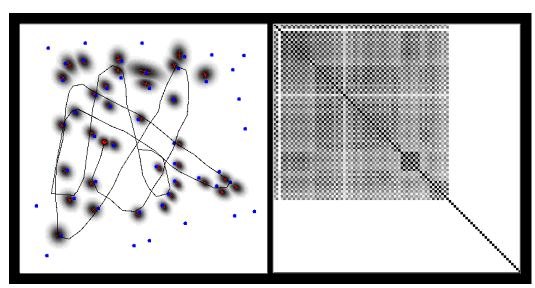
EKF SLAM Correlations



Map Correlation matrix

Courtesy: M. Montemerlo

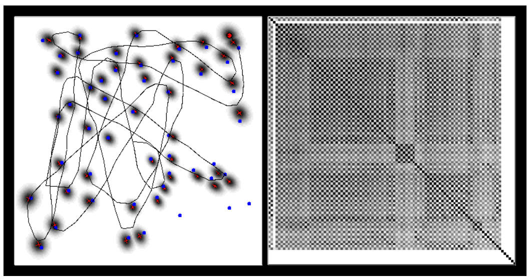
EKF SLAM Correlations



Map Correlation matrix

Courtesy: M. Montemerlo

EKF SLAM Correlations

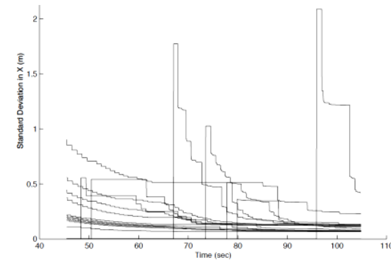


Map Correlation matrix

Courtesy: M. Montemerlo

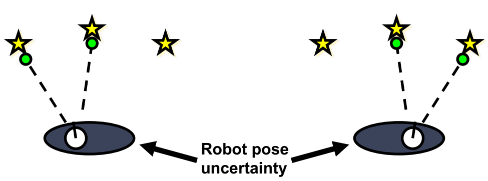
EKF SLAM Uncertainties

- The **determinant** of any sub-matrix of the map covariance matrix **decreases monotonically**
- New landmarks are initialized with **maximum uncertainty**



Courtesy: Dissanayake

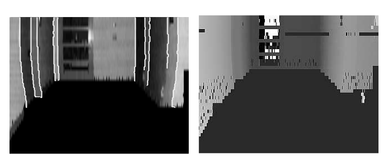
Data Association in SLAM



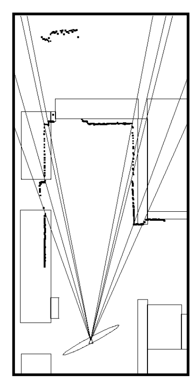
- In the real world, the mapping between observations and landmarks is unknown
- Picking wrong data associations can have catastrophic consequences
  - EKF SLAM is brittle in this regard
- Pose error correlates data associations

Data Association

- Given an environment map
- And a set of sensor observations
- Associate observations with map elements



Vision Laser

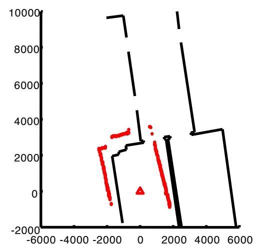


Difficulties: clutter

- Influence of the type, density, precision and robustness of features considered:

Laser scanner:

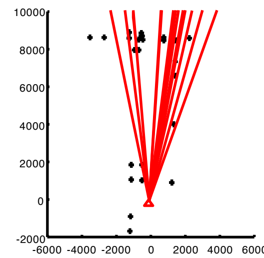
- *Small amount of features (n)*
- *Small amount of measurements (m)*
- Low spuriousness



Low clutter

Difficulties: clutter

- Vertical Edge Monocular vision:

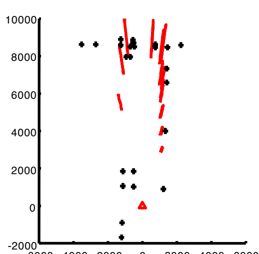


High clutter

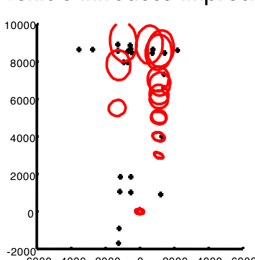
- Many features (n large)
- Many measurements (m large)
- no depth information
- higher spuriousness

Difficulties: imprecision

- Both the sensor and the vehicle introduce imprecision



Vertical Edge Trinocular vision:
variable depth precision
good angular precision



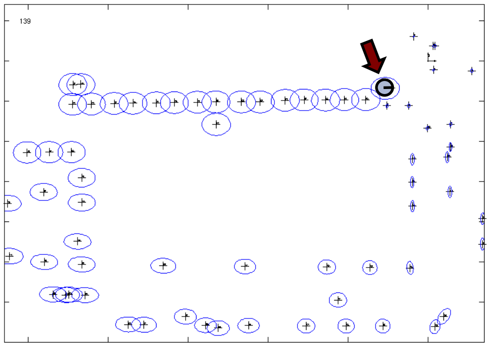
Robot imprecision:
introduces CORRELATED error

Loop-Closing

- Loop-closing means recognizing an already mapped area
- Data association under
  - high ambiguity
  - possible environment symmetries
- Uncertainties **collapse** after a loop-closure (whether the closure was correct or not)

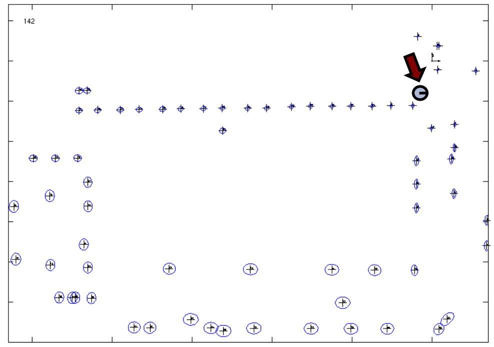
Courtesy: Cyrill Stachniss

Before the Loop-Closure



Courtesy: K. Arras

After the Loop-Closure



Courtesy: K. Arras

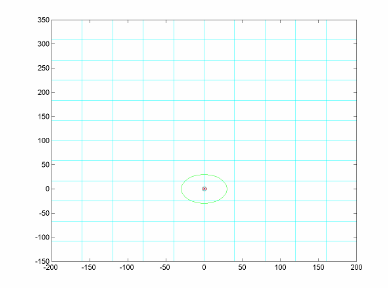
Example: Victoria Park Dataset

Courtesy: E. Nebot

Victoria Park: Data Acquisition

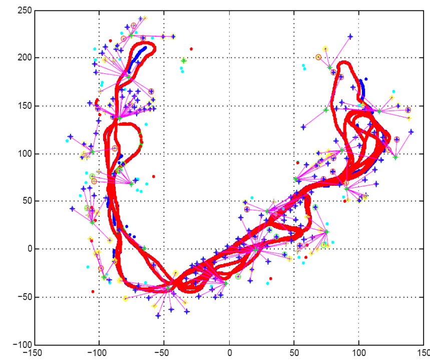
Courtesy: E. Nebot

Victoria Park: EKF Estimate



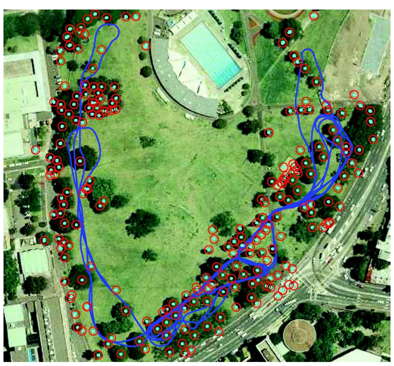
Courtesy: E. Nebot

Victoria Park: EKF Estimate



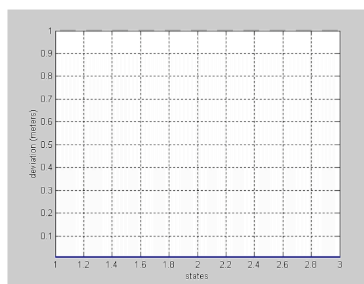
Courtesy: E. Nebot

Victoria Park: Landmarks



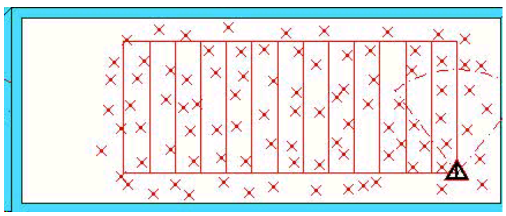
Courtesy: E. Nebot

Victoria Park: Landmark Covariance



Courtesy: E. Nebot

Sub-maps for EKF SLAM



[Leonard et al 1998]

Andrew Davison: MonoSLAM

EKF SLAM Summary

- Quadratic in the number of landmarks: $O(n^2)$
- Convergence results for the linear case.
- Can diverge if nonlinearities are large!
- Have been applied successfully in large-scale environments.
- Approximations reduce the computational complexity.

Literature

EKF SLAM

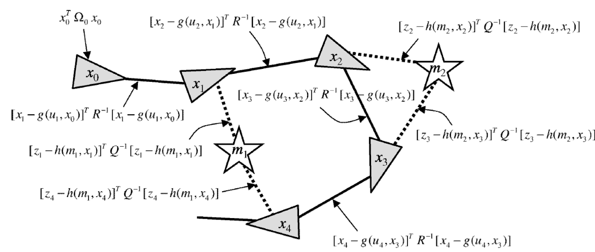
- Thrun et al.: "Probabilistic Robotics", Chapter 10
- Smith, Self, & Cheeseman: "Estimating Uncertain Spatial Relationships in Robotics"
- Dissanayake et al.: "A Solution to the Simultaneous Localization and Map Building (SLAM) Problem"
- Durrant-Whyte & Bailey: "SLAM Part 1" and "SLAM Part 2" tutorials

Courtesy: Cyrill Stachniss

Graph-SLAM

- Full SLAM technique
- Generates probabilistic links
- Computes map only occasionally
- Based on Information Filter form

Graph-SLAM Idea



Sum of all constraints:

$$J_{\text{GraphSLAM}} = x_0^T \Omega_0 x_0 + \sum_t [x_t - g(u_t, x_{t-1})]^T R^{-1} [x_t - g(u_t, x_{t-1})] + \sum_t [z_t - h(m_{c_t}, x_t)]^T Q^{-1} [z_t - h(m_{c_t}, x_t)]$$

Information Form

- Represent posterior in canonical form

  $\Omega = \Sigma^{-1}$ Information matrix

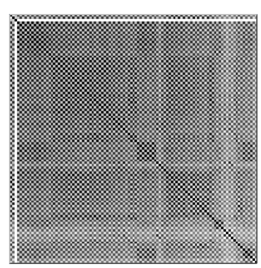
  $\xi = \Sigma^{-1}\mu$ Information vector

- One-to-one transform between canonical and moment representation

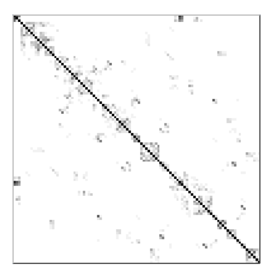
  $\Sigma = \Omega^{-1}$

  $\mu = \Omega^{-1}\xi$

Information vs. Moment Form

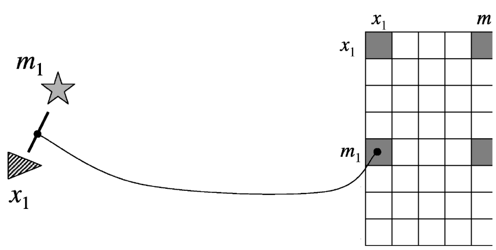


Correlation matrix

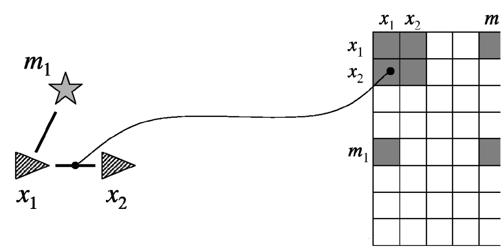


Information matrix

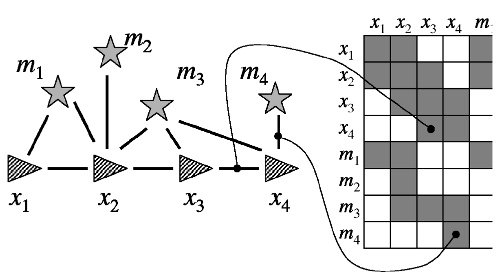
Graph-SLAM Idea (1)



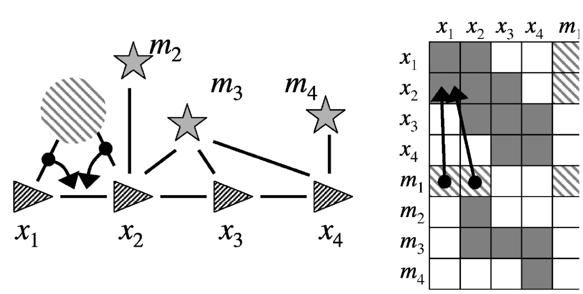
Graph-SLAM Idea (2)



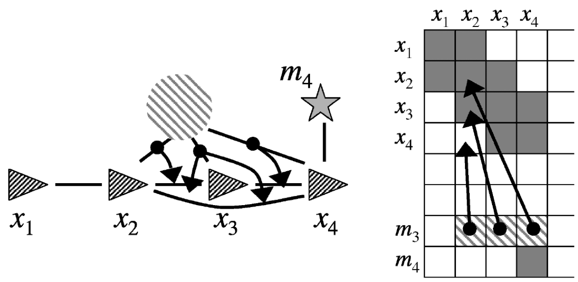
Graph-SLAM Idea (3)



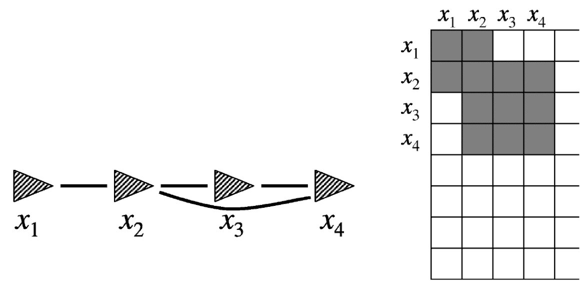
Graph-SLAM Inference (1)



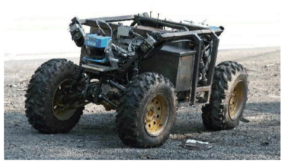
Graph-SLAM Inference (2)

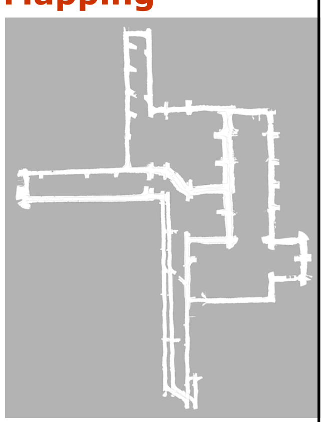


Graph-SLAM Inference (3)

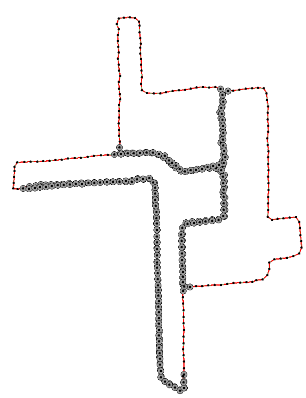


Mine Mapping





Mine Mapping: Data Associations



Efficient Map Recovery

- Information matrix inversion can be avoided if only best map estimate is required
- Minimize constraint function $J_{GraphSLAM}$ using standard optimization techniques (gradient descent, Levenberg Marquardt, conjugate gradient)

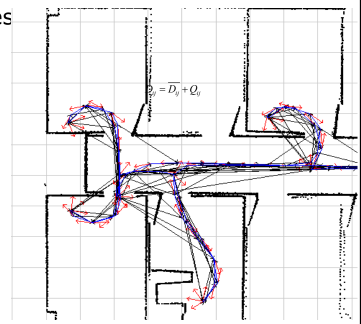
Robot Poses and Scans [Lu and Milios 1997]

- Successive robot poses connected by odometry
- Sensor readings yield constraints between poses
- Constraints represented by Gaussians

$$D_{ij} = \overline{D_{ij}} + Q_{ij}$$

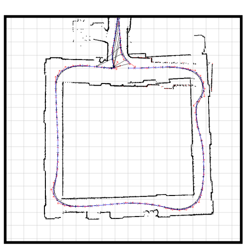
- Globally optimal estimate

$$\arg\max_{X_i} \left[P(D_{ij} \mid \overline{D_{ij}}) \right]$$

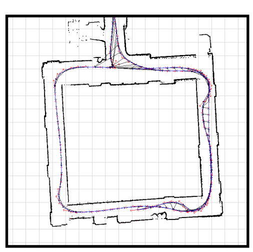


Loop Closure

- Use scan patches to detect loop closure
- Add new position constraints
- Deform the network based on covariances of matches



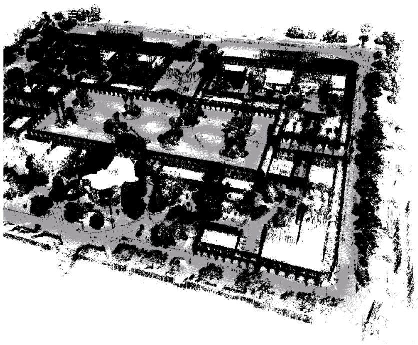
Before loop closure

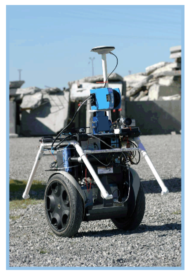


After loop closure

Mapping the Allen Center

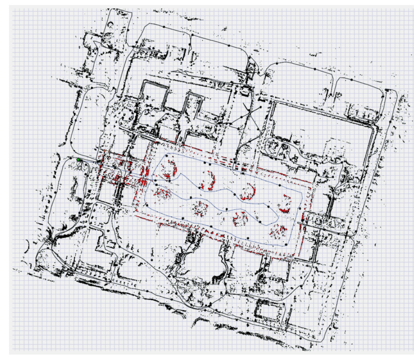
3D Outdoor Mapping





10^8 features, 10^5 poses, only few secs using cg.

Map Before Optimization



Map After Optimization

Graph-SLAM Summary

- Adresses full SLAM problem
- Constructs link graph between poses and poses/landmarks
- Graph is sparse: number of edges linear in number of nodes
- Inference performed by building information matrix and vector (linearized form)
- Map recovered by reduction to robot poses, followed by conversion to moment representation, followed by estimation of landmark positions
- ML estimate by minimization of $J_{GraphSLAM}$
- Data association by iterative greedy search