CSE 573: Artificial Intelligence
Autumn 2010

Lecture 7: MDPs/RL

10/21/2010

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Many slides over the course adapted from either Dan Klein,
Stuart Russell or Andrew Moore

Outline

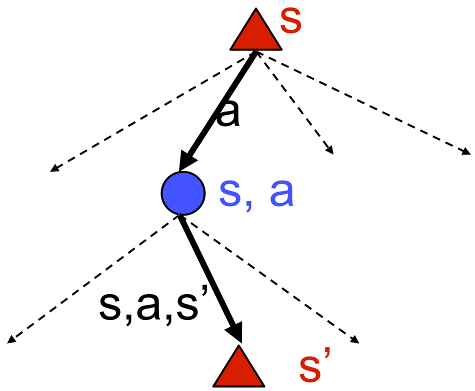
- Markov decision processes
  - Review Optimality / Value Iteration
  - Value Iteration convergence / complexity
  - Policy Iteration
- Reinforcement Learning
  - Passive Learning
  - TD Updates
  - Q-learning
- 3:30: Tom Mitchell's Distinguished Lecture
  - EEB-105

Homework Rant

- PS2 Due Tuesday!
- PS1 will be handed back this afternoon
  - Admissibility was hard, but overall everyone did well!
  - Next time: Follow the instructions!!!
    - Only hand in the one/two requested files (and don’t zip/tar them)
    - Don’t change any other files
    - Turn off your debug printouts
    - Comment your code (if you want partial credit)

Recap: MDPs

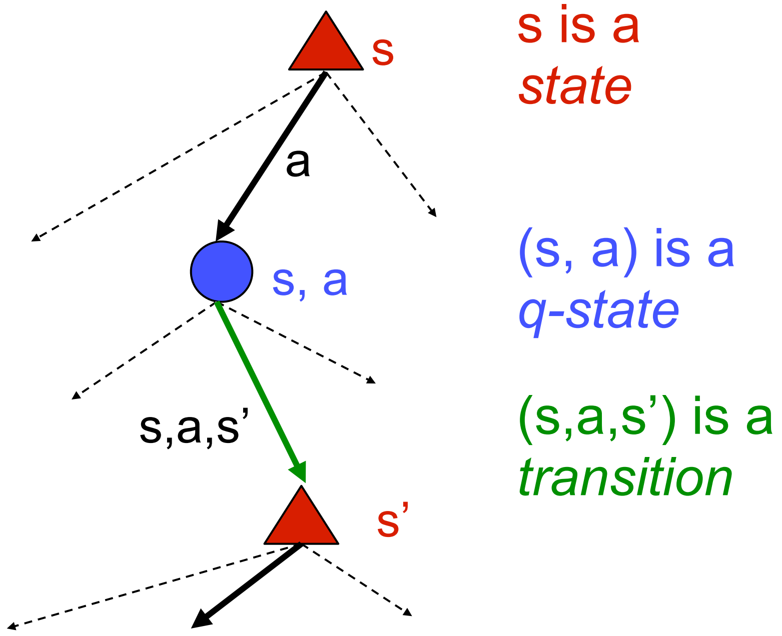
- Markov decision processes:
  - States S
  - Actions A
  - Transitions P(s'|s,a) (or T(s,a,s'))
  - Rewards R(s,a,s') (and discount γ)
  - Start state s_0



- Quantities:
  - Policy = map of states to actions
  - Utility = sum of discounted rewards
  - Values = expected future utility from a state
  - Q-Values = expected future utility from a q-state

Recap: Optimal Utilities

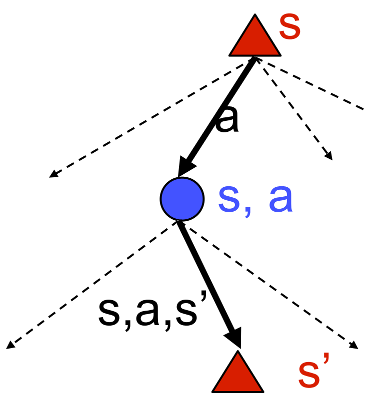
- The utility of a state s:
  $V^*(s)$ = expected utility starting in s and acting optimally
- The utility of a q-state (s,a):
  $Q^*(s,a)$ = expected utility starting in s, taking action a and thereafter acting optimally
- The optimal policy:
  $\pi^*(s)$ = optimal action from state s



Recap: Bellman Equations

- Definition of utility leads to a simple one-step lookahead relationship amongst optimal utility values:

Total optimal rewards = maximize over choice of (first action plus optimal future)



- Formally:

$$V^*(s) = \max_a Q^*(s,a)$$

$$Q^*(s,a) = \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V^*(s')\right]$$

$$V^*(s) = \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V^*(s')\right]$$

Practice: Computing Actions

- Which action should we chose from state s:
  - Given optimal values V?

$$\arg\max_a \sum_{s'} T(s,a,s')[R(s,a,s') + \gamma V^*(s')]$$

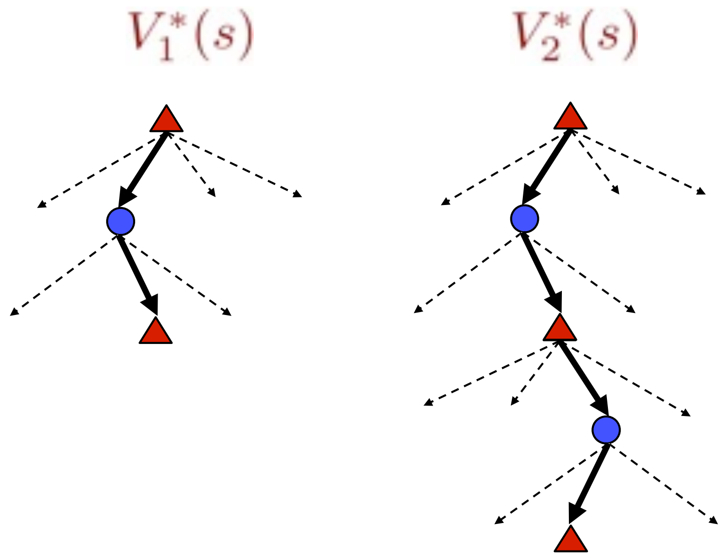
  - Given optimal q-values Q?

$$\arg\max_a Q^*(s,a)$$

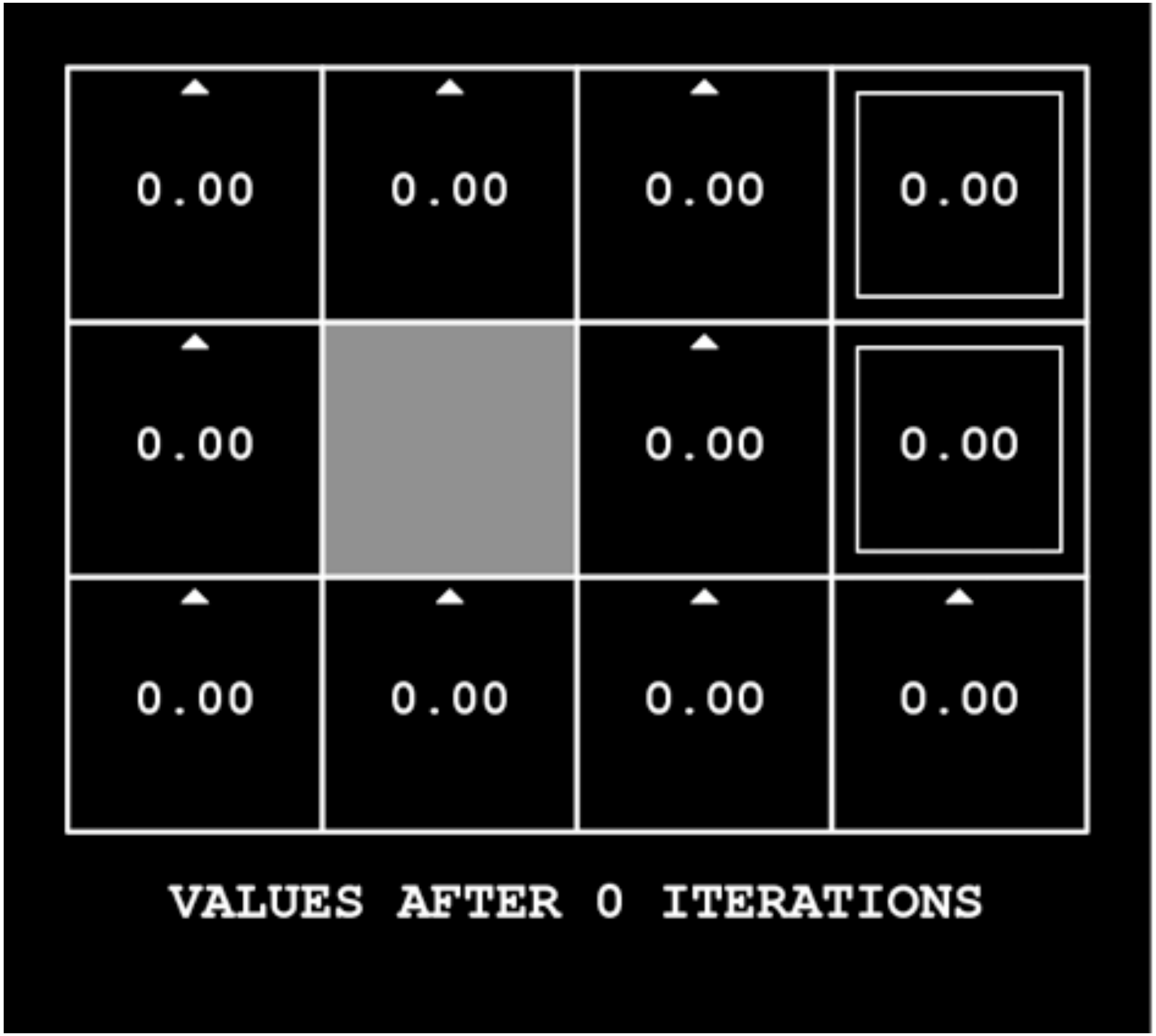
  - Lesson: actions are easier to select from Q's!

Value Estimates

- Calculate estimates $V_k^*(s)$
  - Not the optimal value of s!
  - The optimal value considering only next k time steps (k rewards)
  - As $k \rightarrow \infty$, it approaches the optimal value
- Value Iteration: dynamic programming



Example: Value Iteration



Value Iteration

- Idea:
  - Start with $V_0^*(s) = 0$, which we know is right (why?)
  - Given V_i^*, calculate the values for all states for depth i+1:

$$V_{i+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V_i(s')\right]$$

  - Throw out old vector V_i^*
  - Repeat until convergence
  - This is called a value update or Bellman update
- Theorem: will converge to unique optimal values
  - Basic idea: approximations get refined towards optimal values
  - Policy may converge long before values do

Convergence

- Define the max-norm: $||U|| = \max_s |U(s)|$

- Theorem: For any two value vectors U and V

$$||U^{t+1} - V^{t+1}|| \leq \gamma \, ||U^t - V^t||$$

  - I.e. any distinct approximations must get closer to each other, so, in particular, any approximation must get closer to the true U and value iteration converges to a unique, stable, optimal solution

- Theorem:

$$||U^{t+1} - U^t|| < \epsilon, \Rightarrow ||U^{t+1} - U|| < 2\epsilon\gamma/(1-\gamma)$$

  - I.e. once the change in our approximation is small, it must also be close to correct

Value Iteration Complexity

- Problem size:
  - |A| actions and |S| states
- Each Iteration
  - Computation: $O(|A| \cdot |S|^2)$
  - Space: $O(|S|)$
- Num of iterations
  - Can be exponential in the discount factor γ

Asynchronous Value Iteration*

- In value iteration, we update every state in each iteration

- Actually, *any* sequences of Bellman updates will converge if every state is visited infinitely often

- In fact, we can update the policy as seldom or often as we like, and we will still converge

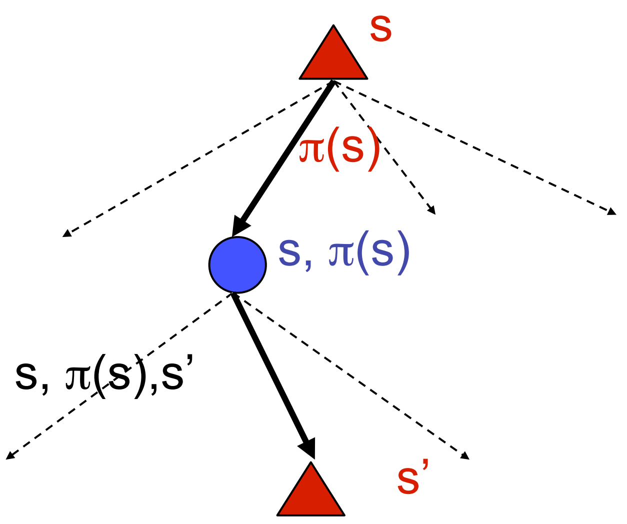
- Idea: Update states whose value we expect to change:
  If $|V_{i+1}(s) - V_i(s)|$ is large then update predecessors of s

Utilities for Fixed Policies

- Another basic operation: compute the utility of a state s under a fix (general non-optimal) policy
- Define the utility of a state s, under a fixed policy π:

  $V^\pi(s)$ = expected total discounted rewards (return) starting in s and following π

- Recursive relation (one-step look-ahead / Bellman equation):



$$V^\pi(s) = \sum_{s'} T(s, \pi(s), s')[R(s, \pi(s), s') + \gamma V^\pi(s')]$$

Policy Evaluation

- How do we calculate the V's for a fixed policy?
- Idea one: modify Bellman updates

$$V_0^{\pi}(s) = 0$$

$$V_{i+1}^{\pi}(s) \leftarrow \sum_{s'} T(s, \pi(s), s')[R(s, \pi(s), s') + \gamma V_i^{\pi}(s')]$$

- Idea two: it's just a linear system, solve with Matlab (or whatever)

Policy Iteration

- Problem with value iteration:
  - Considering all actions each iteration is slow: takes |A| times longer than policy evaluation
  - But policy doesn't change each iteration, time wasted

- Alternative to value iteration:
  - Step 1: Policy evaluation: calculate utilities for a fixed policy (not optimal utilities!) until convergence (fast)
  - Step 2: Policy improvement: update policy using one-step lookahead with resulting converged (but not optimal!) utilities (slow but infrequent)
  - Repeat steps until policy converges

Policy Iteration

- Policy evaluation: with fixed current policy π, find values with simplified Bellman updates:
  - Iterate until values converge

$$V_{i+1}^{\pi_k}(s) \leftarrow \sum_{s'} T(s, \pi_k(s), s') \left[R(s, \pi_k(s), s') + \gamma V_i^{\pi_k}(s')\right]$$

- Policy improvement: with fixed utilities, find the best action according to one-step look-ahead

$$\pi_{k+1}(s) = \arg\max_a \sum_{s'} T(s, a, s') \left[R(s, a, s') + \gamma V^{\pi_k}(s')\right]$$

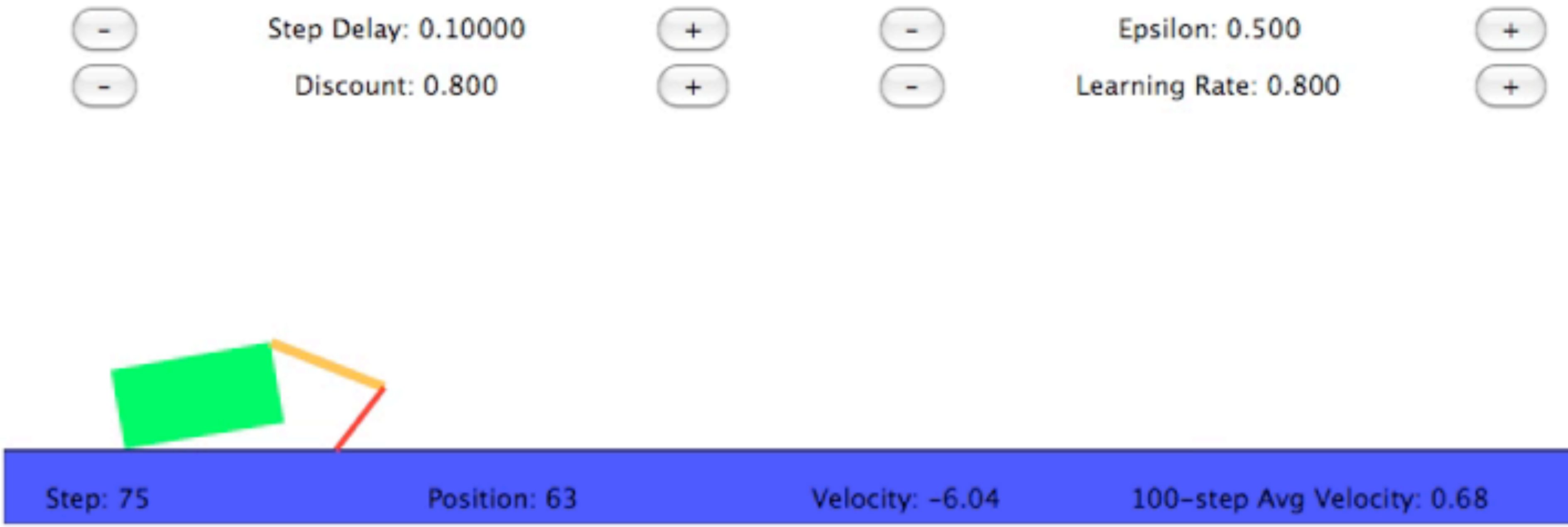
Policy Iteration Complexity

- Problem size:
  - |A| actions and |S| states
- Each Iteration
  - Computation: $O(|S|^3 + |A| \cdot |S|^2)$
  - Space: $O(|S|)$
- Num of iterations
  - Unknown, but can be faster in practice
  - Convergence is guaranteed

Comparison

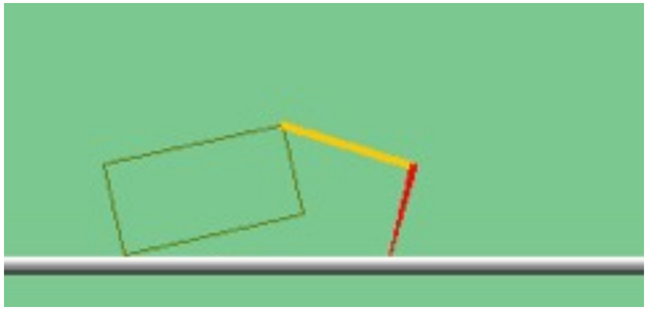
- In value iteration:
  - Every pass (or “backup”) updates both utilities (explicitly, based on current utilities) and policy (possibly implicitly, based on current policy)
- In policy iteration:
  - Several passes to update utilities with frozen policy
  - Occasional passes to update policies
- Hybrid approaches (asynchronous policy iteration):
  - Any sequences of partial updates to either policy entries or utilities will converge if every state is visited infinitely often

What is it doing?



Reinforcement Learning

- Reinforcement learning:
  - Still have an MDP:
    - A set of states s ∈ S
    - A set of actions (per state) A
    - A model T(s,a,s')
    - A reward function R(s,a,s')
  - Still looking for a policy $\pi(s)$
  - New twist: don't know T or R
    - I.e. don't know which states are good or what the actions do
    - Must actually try actions and states out to learn



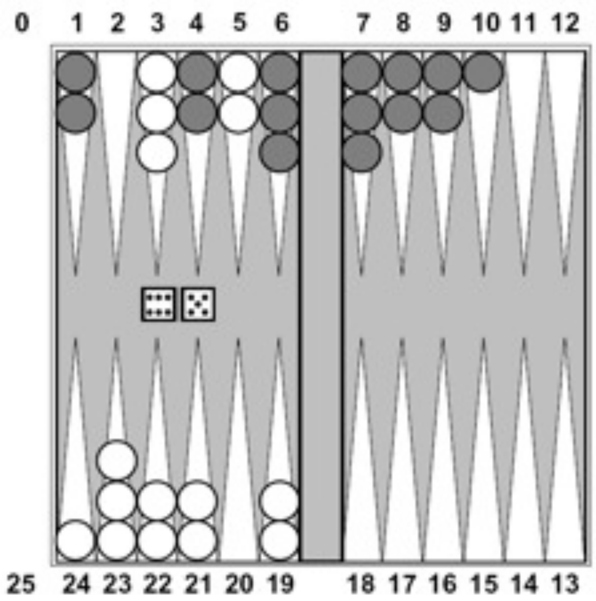
Example: Animal Learning

- RL studied experimentally for more than 60 years in psychology
  - Rewards: food, pain, hunger, drugs, etc.
  - Mechanisms and sophistication debated

- Example: foraging
  - Bees learn near-optimal foraging plan in field of artificial flowers with controlled nectar supplies
  - Bees have a direct neural connection from nectar intake measurement to motor planning area

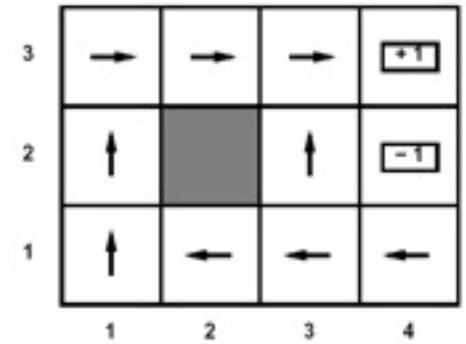
Example: Backgammon

- Reward only for win / loss in terminal states, zero otherwise
- TD-Gammon learns a function approximation to V(s) using a neural network
- Combined with depth 3 search, one of the top 3 players in the world



- You could imagine training Pacman this way…
- … but it's tricky! (It's also P3)

Passive Learning



- Simplified task
  - You don't know the transitions T(s,a,s')
  - You don't know the rewards R(s,a,s')
  - You are given a policy $\pi(s)$
  - Goal: learn the state values (and maybe the model)
  - I.e., policy evaluation
- In this case:
  - Learner "along for the ride"
  - No choice about what actions to take
  - Just execute the policy and learn from experience
  - We'll get to the active case soon
  - This is NOT offline planning!

Detour: Sampling Expectations

- Want to compute an expectation weighted by P(x):

$$E[f(x)] = \sum_x P(x) f(x)$$

- Model-based: estimate P(x) from samples, compute expectation

$$x_i \sim P(x)$$

$$\hat{P}(x) = \text{count}(x)/k$$

$$E[f(x)] \approx \sum_x \hat{P}(x) f(x)$$

- Model-free: estimate expectation directly from samples

$$x_i \sim P(x)$$

$$E[f(x)] \approx \frac{1}{k} \sum_i f(x_i)$$

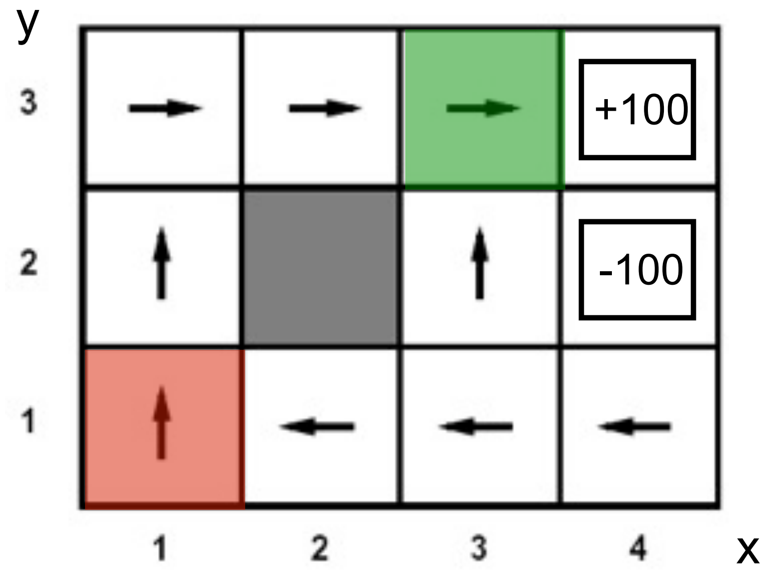
- Why does this work? Because samples appear with the right frequencies!

Example: Direct Estimation

- Episodes:

(1,1) up -1
(1,2) up -1
(1,2) up -1
(1,3) right -1
(2,3) right -1
(3,3) right -1
(3,2) up -1
(3,3) right -1
(4,3) exit +100
(done)

(1,1) up -1
(1,2) up -1
(1,3) right -1
(2,3) right -1
(3,3) right -1
(3,2) up -1
(4,2) exit -100
(done)



$\gamma = 1$, $R = -1$

$V(1,1) \sim (92 + -106) / 2 = -7$

$V(3,3) \sim (99 + 97 + -102) / 3 = 31.3$

Model-Based Learning

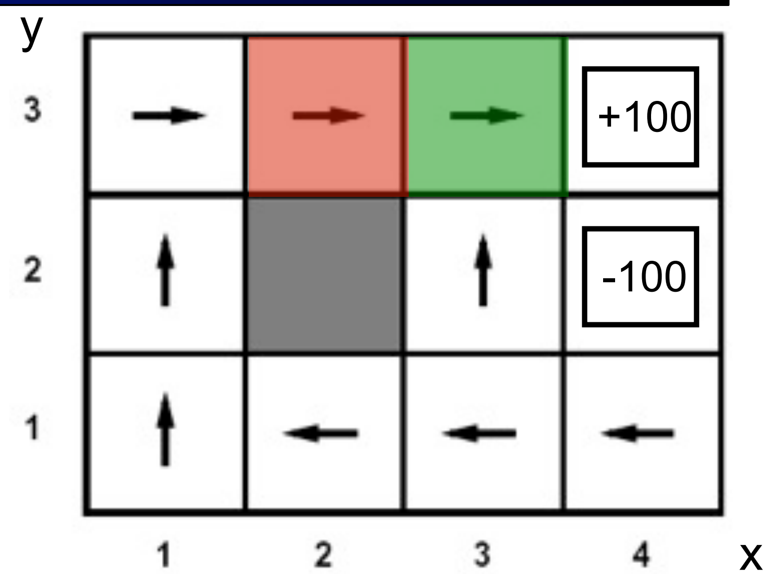
- Idea:
  - Learn the model empirically (rather than values)
  - Solve the MDP as if the learned model were correct
  - Better than direct estimation?
- Empirical model learning
  - Simplest case:
    - Count outcomes for each s,a
    - Normalize to give estimate of T(s,a,s')
    - Discover R(s,a,s') the first time we experience (s,a,s')
  - More complex learners are possible (e.g. if we know that all squares have related action outcomes, e.g. "stationary noise")

Example: Model-Based Learning

- Episodes:

(1,1) up -1
(1,2) up -1
(1,2) up -1
(1,3) right -1
(2,3) right -1
(3,3) right -1
(3,2) up -1
(3,3) right -1
(4,3) exit +100
(done)

(1,1) up -1
(1,2) up -1
(1,3) right -1
(2,3) right -1
(3,3) right -1
(3,2) up -1
(4,2) exit -100
(done)



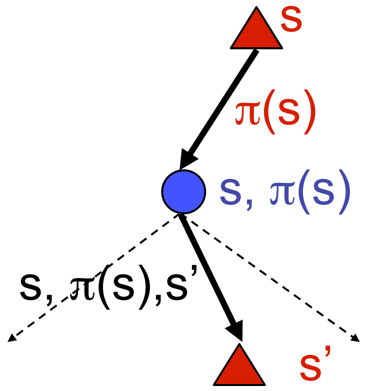
$\gamma = 1$

T(<3,3>, right, <4,3>) = 1 / 3

T(<2,3>, right, <3,3>) = 2 / 2

Recap: Model-Based Policy Evaluation

- Simplified Bellman updates to calculate V for a fixed policy:
  - New V is expected one-step-look-ahead using current V
  - Unfortunately, need T and R



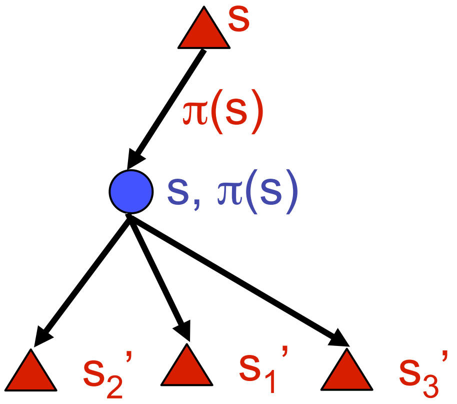
$$V_0^{\pi}(s) = 0$$

$$V_{i+1}^{\pi}(s) \leftarrow \sum_{s'} T(s, \pi(s), s')[R(s, \pi(s), s') + \gamma V_i^{\pi}(s')]$$

Sample Avg to Replace Expectation?

$$V_{i+1}^{\pi}(s) \leftarrow \sum_{s'} T(s, \pi(s), s')[R(s, \pi(s), s') + \gamma V_i^{\pi}(s')]$$

- Who needs T and R? Approximate the expectation with samples (drawn from T!)



$$sample_1 = R(s, \pi(s), s_1') + \gamma V_i^{\pi}(s_1')$$

$$sample_2 = R(s, \pi(s), s_2') + \gamma V_i^{\pi}(s_2')$$

$$\ldots$$

$$sample_k = R(s, \pi(s), s_k') + \gamma V_i^{\pi}(s_k')$$

$$V_{i+1}^{\pi}(s) \leftarrow \frac{1}{k} \sum_i sample_i$$

Exponential Moving Average

- Exponential moving average
  - Makes recent samples more important

$$\bar{x}_n = \frac{x_n + (1-\alpha)\cdot x_{n-1} + (1-\alpha)^2 \cdot x_{n-2} + \ldots}{1 + (1-\alpha) + (1-\alpha)^2 + \ldots}$$

  - Forgets about the past (distant past values were wrong anyway)
  - Easy to compute from the running average

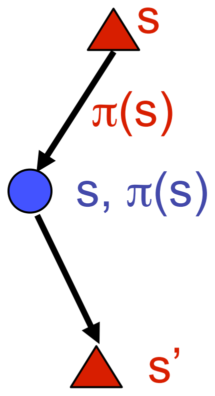
$$\bar{x}_n = (1-\alpha)\cdot \bar{x}_{n-1} + \alpha \cdot x_n$$

- Decreasing learning rate can give converging averages

Model-Free Learning

$$V^\pi(s) \leftarrow \sum_{s'} T(s,\pi(s),s')[R(s,\pi(s),s') + \gamma V^\pi(s')]$$

- Big idea: why bother learning T?
  - Update V each time we experience a transition
- Temporal difference learning (TD)
  - Policy still fixed!
  - Move values toward value of whatever successor occurs: running average!



$$sample = R(s,\pi(s),s') + \gamma V^\pi(s')$$

$$V^\pi(s) \leftarrow (1-\alpha)V^\pi(s) + (\alpha)sample$$

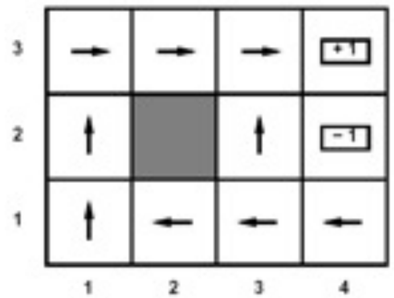
$$V^\pi(s) \leftarrow V^\pi(s) + \alpha(sample - V^\pi(s))$$

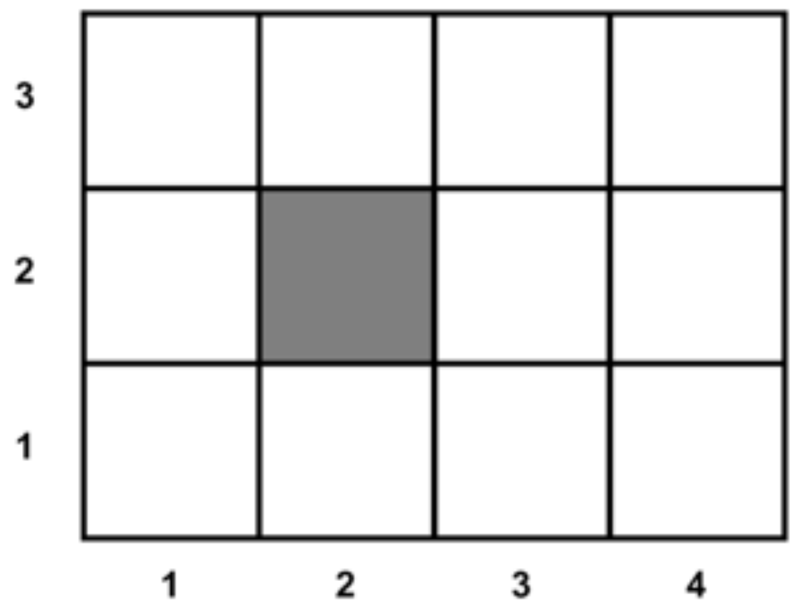
Example: TD Policy Evaluation

$$V^{\pi}(s) \leftarrow (1-\alpha)V^{\pi}(s) + \alpha\left[R(s,\pi(s),s') + \gamma V^{\pi}(s')\right]$$

(1,1) up -1
(1,2) up -1
(1,2) up -1
(1,3) right -1
(2,3) right -1
(3,3) right -1
(3,2) up -1
(3,3) right -1
(4,3) exit +100
(done)

(1,1) up -1
(1,2) up -1
(1,3) right -1
(2,3) right -1
(3,3) right -1
(3,2) up -1
(4,2) exit -100
(done)





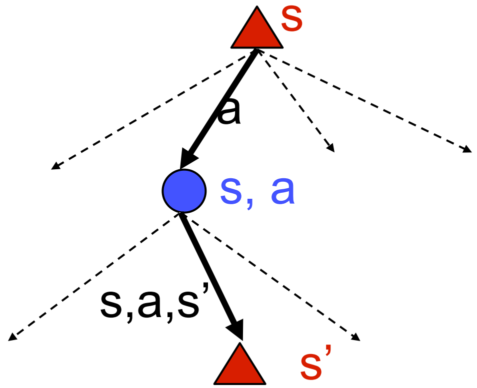
Take $\gamma = 1$, $\alpha = 0.5$

Problems with TD Value Learning

- TD value leaning is model-free for policy evaluation (passive learning)
- However, if we want to turn our value estimates into a policy, we're sunk:

$$\pi(s) = \arg\max_a Q^*(s,a)$$

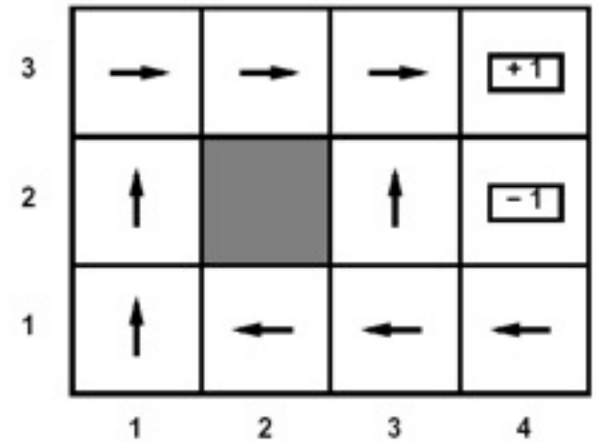
$$Q^*(s,a) = \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V^*(s')\right]$$



- Idea: learn Q-values directly
- Makes action selection model-free too!

Active Learning

- Full reinforcement learning
  - You don't know the transitions T(s,a,s')
  - You don't know the rewards R(s,a,s')
  - You can choose any actions you like
  - Goal: learn the optimal policy
  - … what value iteration did!



- In this case:
  - Learner makes choices!
  - Fundamental tradeoff: exploration vs. exploitation
  - This is NOT offline planning! You actually take actions in the world and find out what happens…

Detour: Q-Value Iteration

- Value iteration: find successive approx optimal values
  - Start with $V_0^*(s) = 0$
  - Given V_i^*, calculate the values for all states for depth i+1:

$$V_{i+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V_i(s')\right]$$

- But Q-values are more useful!
  - Start with $Q_0^*(s,a) = 0$
  - Given Q_i^*, calculate the q-values for all q-states for depth i+1:

$$Q_{i+1}(s,a) \leftarrow \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma \max_{a'} Q_i(s',a')\right]$$

Q-Learning Update

- Q-Learning: sample-based Q-value iteration

$$Q^*(s,a) = \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma \max_{a'} Q^*(s',a') \right]$$

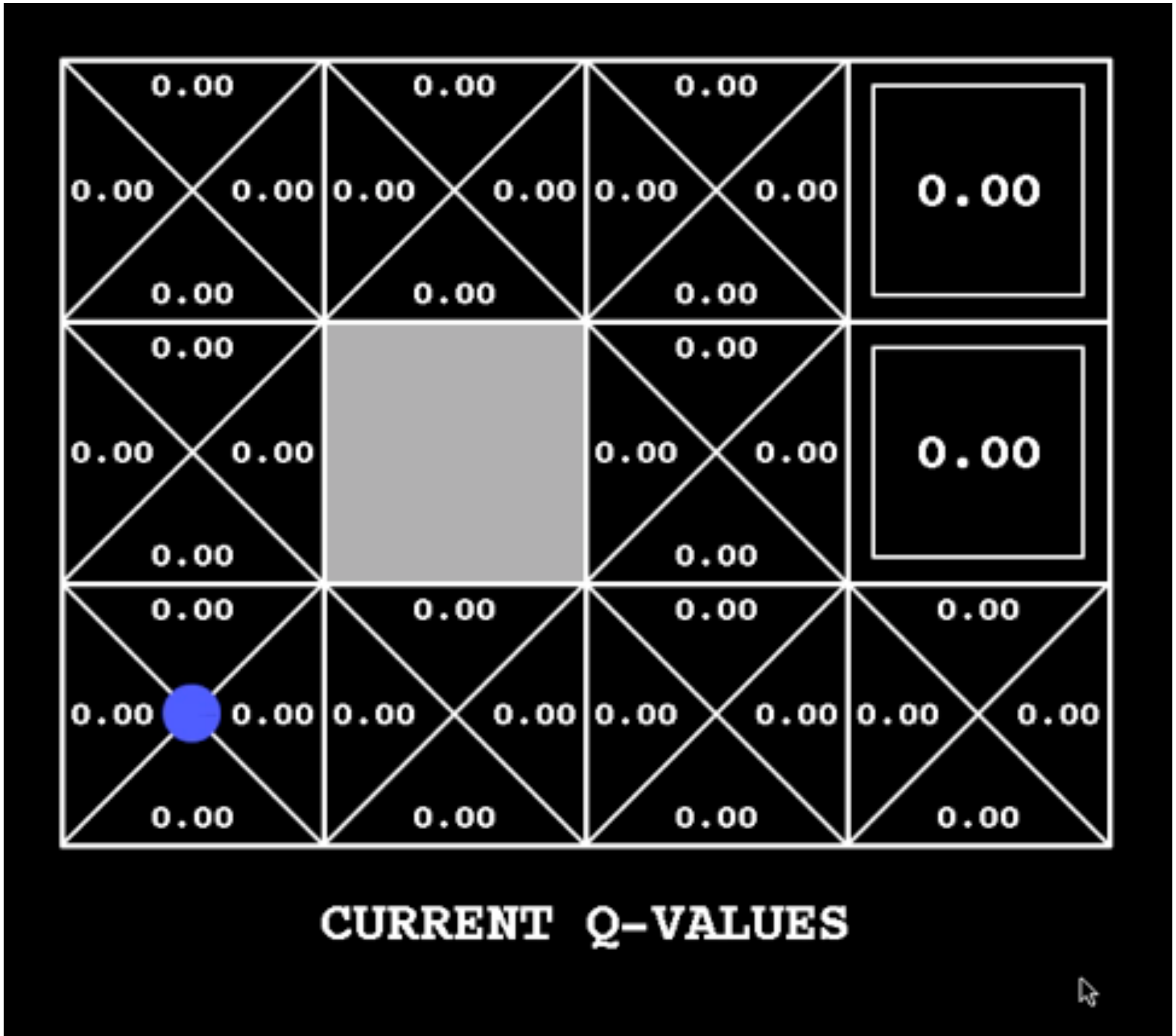
- Learn Q*(s,a) values
  - Receive a sample (s,a,s',r)
  - Consider your old estimate: $Q(s,a)$
  - Consider your new sample estimate:

$$sample = R(s,a,s') + \gamma \max_{a'} Q(s',a')$$

  - Incorporate the new estimate into a running average:

$$Q(s,a) \leftarrow (1-\alpha)Q(s,a) + (\alpha)\,[sample]$$

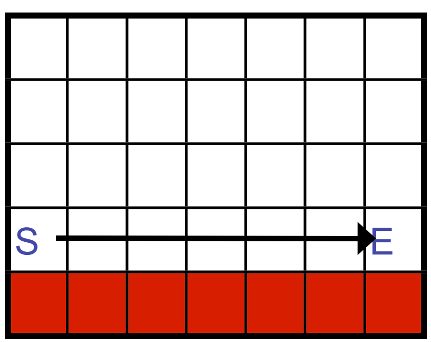
Q-Learning: Fixed Policy

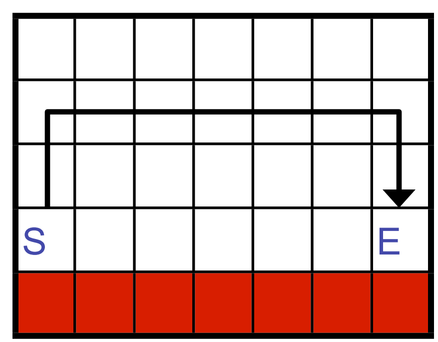


Q-Learning Properties

- Amazing result: Q-learning converges to optimal policy
  - If you explore enough
  - If you make the learning rate small enough
  - … but not decrease it too quickly!
  - Not too sensitive to how you select actions (!)

- Neat property: off-policy learning
  - learn optimal policy without following it (some caveats)

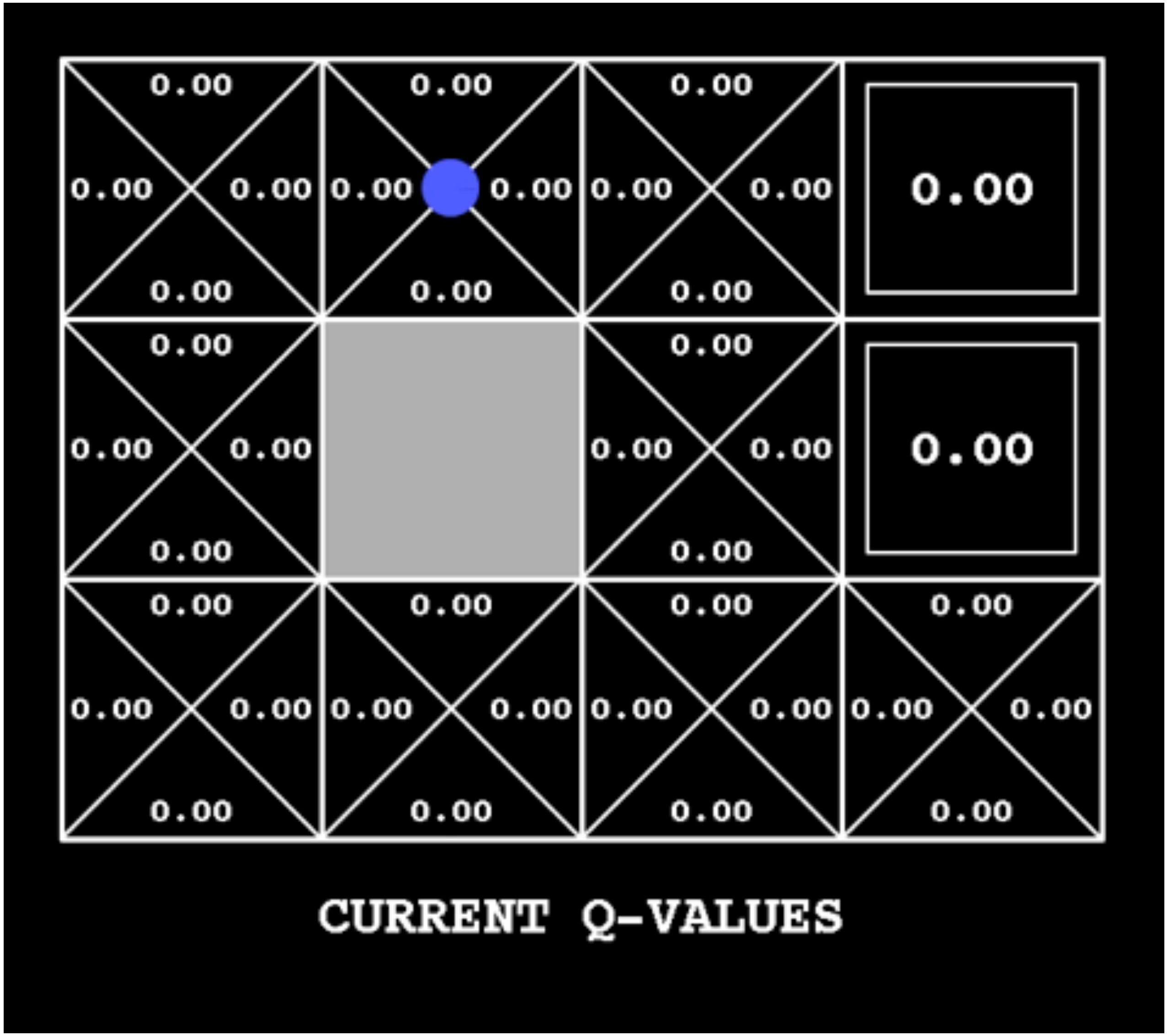




Exploration / Exploitation

- Several schemes for action selection
  - Simplest: random actions (ε greedy)
    - Every time step, flip a coin
    - With probability ε, act randomly
    - With probability $1-\varepsilon$, act according to current policy
  - Problems with random actions?
    - You do explore the space, but keep thrashing around once learning is done
    - One solution: lower ε over time
    - Another solution: exploration functions

Q-Learning: ε Greedy



Exploration Functions

- When to explore
  - Random actions: explore a fixed amount
  - Better idea: explore areas whose badness is not (yet) established
- Exploration function
  - Takes a value estimate and a count, and returns an optimistic utility, e.g. $f(u,n) = u + k/n$ (exact form not important)
  - Exploration policy $\pi(s)$=

$$\max_{a'} Q_i(s',a') \quad \text{vs.} \quad \max_{a'} f(Q_i(s',a'), N(s',a'))$$

Q-Learning Final Solution

- Q-learning produces tables of q-values: