CSE 573: Artificial Intelligence

Constraint Satisfaction

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Slides adapted from Dan Klein, Stuart Russell, Andrew Moore & Luke Zettlemoyer

Space of Search Strategies

- Blind Search
  - DFS, BFS, IDS
- Informed Search
  - Systematic: Uniform cost, greedy, A*, IDA*
  - Stochastic: Hill climbing w/ random walk & restarts
- **Constraint Satisfaction**
- Adversary Search
  - Min-max, alpha-beta, expectimax, MDPS…

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Recap: Search Problem

- States
  - configurations of the world
- Successor function:
  - function from states to lists of triples (state, action, cost)
- Start state
- Goal test

Constraint Satisfaction

- Kind of ***search*** in which
  - States are ***factored*** into sets of variables
  - Search = assigning values to these variables
  - Goal test is encoded with constraints
    - → Gives ***structure*** to search space
    - Exploration of one part informs others
- Special techniques add speed
  - Propagation
  - Variable ordering
  - Preprocessing

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Constraint Satisfaction Problems

- Subset of search problems
- State is ***factored*** - defined by
  - Variables X_i with values from a
  - Domain D (often D depends on i)
- Goal test is a set of constraints

WHY STUDY?
- Simple example of a ***formal representation language***
- Allows more ***powerful search algorithms***

Example: Map-Coloring

- Variables: $WA,\ NT,\ Q,\ NSW,\ V,\ SA,\ T$
- Domain: $D = \{red, green, blue\}$
- Constraints: adjacent regions must have different colors

$$WA \neq NT$$

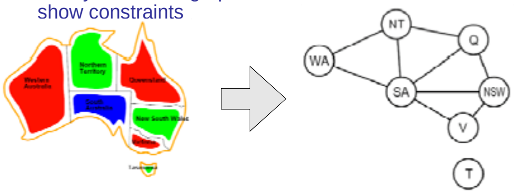
$$(WA, NT) \in \{(red, green), (red, blue), (green, red), \ldots\}$$

- Solutions are assignments satisfying all constraints, e.g.:

$$\{WA = red, NT = green, Q = red,$$
$$NSW = green, V = red, SA = blue, T = green\}$$

Constraint Graphs

- Binary CSP: each constraint relates (at most) two variables
- Binary constraint graph: nodes are variables, arcs show constraints



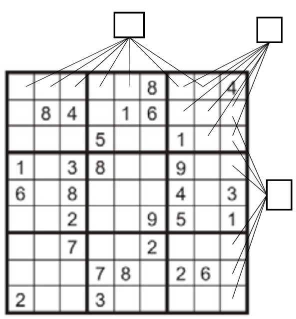
- General-purpose CSP algorithms use the graph structure to speed up search. E.g., Tasmania is an independent subproblem!

Real-World CSPs

- Assignment problems: e.g., who teaches what class
- Timetabling problems: e.g., which class is offered when and where?
- Hardware configuration
- Gate assignment in airports
- Transportation scheduling
- Factory scheduling
- Fault diagnosis
- … lots more!

- Many real-world problems involve real-valued variables…

Example: Sudoku



- Variables:
  - Each (open) square
- Domains:
  - {1,2,…,9}
- Constraints:

9-way alldiff for each column

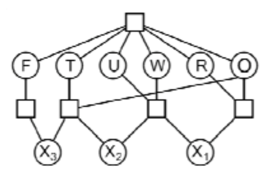
9-way alldiff for each row

9-way alldiff for each region

Example: Cryptarithmetic

- Variables (circles):
  $F\ T\ U\ W\ R\ O\ X_1\ X_2\ X_3$
- Domains:
  $\{0,1,2,3,4,5,6,7,8,9\}$
- Constraints (boxes):
  $\text{alldiff}(F,T,U,W,R,O)$
  $O + O = R + 10 \cdot X_1$
  $\cdots$

```
  T W O
+ T W O
F O U R
```



Crossword Puzzle

- Variables & domains?
- Constraints?

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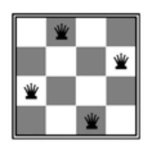
Example: N-Queens

- CSP Formulation 1:
  - Variables: X_{ij}
  - Domains: $\{0,1\}$
  - Constraints



$$\forall i,j,k \quad X_{ij} + X_{ik} \leq 1$$
$$\forall i,j,k \quad X_{ij} + X_{kj} \leq 1$$
$$\forall i,j,k \quad X_{ij} + X_{i+k,j+k} \leq 1$$
$$\forall i,j,k \quad X_{ij} + X_{i+k,j-k} \leq 1$$

$$\sum_{i,j} X_{ij} = N$$

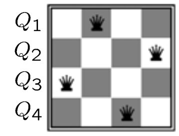
Example: N-Queens

- CSP Formulation 1:
  - Variables: X_{ij}
  - Domains: $\{0, 1\}$
  - Constraints

$$\forall i, j, k \;\; (X_{ij}, X_{ik}) \in \{(0,0),(0,1),(1,0)\}$$
$$\forall i, j, k \;\; (X_{ij}, X_{kj}) \in \{(0,0),(0,1),(1,0)\}$$
$$\forall i, j, k \;\; (X_{ij}, X_{i+k,j+k}) \in \{(0,0),(0,1),(1,0)\}$$
$$\forall i, j, k \;\; (X_{ij}, X_{i+k,j-k}) \in \{(0,0),(0,1),(1,0)\}$$
$$\sum_{i,j} X_{ij} = N$$

Example: N-Queens

- Formulation 2:
  - Variables: Q_k
  - Domains: $\{1, 2, 3, \ldots N\}$



  - Constraints:

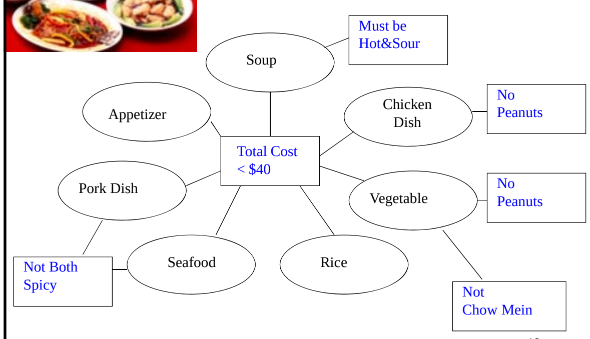
Implicit: $\forall i, j$ non-threatening(Q_i, Q_j)

-or-

Explicit: $(Q_1, Q_2) \in \{(1,3),(1,4),\ldots\}$

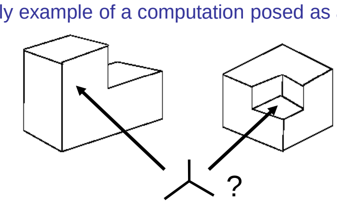
$\cdots$

Chinese Constraint Network



Example: The Waltz Algorithm

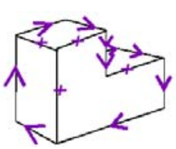
- The Waltz algorithm is for interpreting line drawings of solid polyhedra
- An early example of a computation posed as a CSP



- Look at all intersections
- Adjacent intersections impose constraints on each other

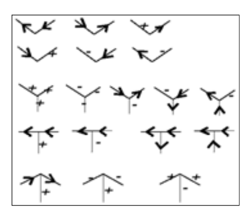
Waltz on Simple Scenes

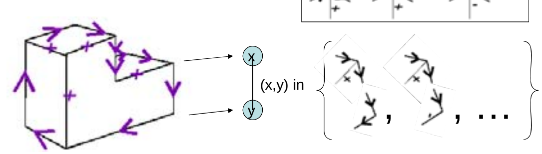
- Assume all objects:
  - Have no shadows or cracks
  - Three-faced vertices
  - "General position": no junctions change with small movements of the eye.
- Then each line on image is one of the following:
  - Boundary line (edge of an object) (>) with right hand of arrow denoting "solid" and left hand denoting "space"
  - Interior convex edge (+)
  - Interior concave edge (-)



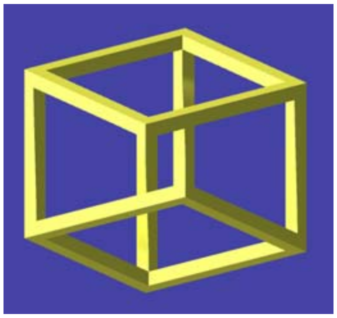
Legal Junctions

- Only certain junctions are physically possible
- How can we formulate a CSP to label an image?
- Variables: vertices
- Domains: junction labels
- Constraints: both ends of a line should have the same label





Local *vs* Global Consistency



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Varieties of CSPs

- Discrete Variables
  - Finite domains
    - Size d means $O(d^n)$ complete assignments
    - E.g., Boolean CSPs, including Boolean satisfiability (NP-complete)
  - Infinite domains (integers, strings, etc.)
    - E.g., job scheduling, variables are start/end times for each job
    - Linear constraints solvable, nonlinear undecidable
- Continuous variables
  - E.g., start/end times for Hubble Telescope observations
  - Linear constraints solvable in polynomial time by LP methods

Varieties of Constraints

- Varieties of Constraints
  - Unary constraints involve a single variable (equiv. to shrinking domains):

    $$SA \neq green$$

  - Binary constraints involve pairs of variables:

    $$SA \neq WA$$

  - Higher-order constraints involve 3 or more variables:
    e.g., cryptarithmetic column constraints
- Preferences (soft constraints):
  - E.g., red is better than green
  - Often representable by a cost for each variable assignment
  - Gives constrained optimization problems
  - (We'll ignore these until we get to Bayes' nets)

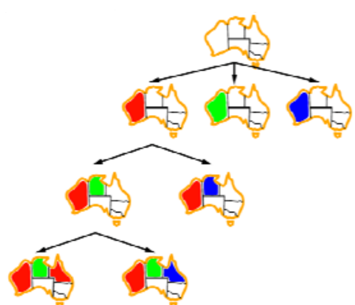
CSPs as Search?

- States?
- Successor function?
- Start state?
- Goal test?

Standard Search Formulation

- States are defined by the values assigned so far
- Initial state: the empty assignment, {}
- Successor function:
  - assign value to an unassigned variable
- Goal test:
  - the current assignment is complete &
  - satisfies all constraints

Backtracking Example



Backtracking Search

- Note 1: Only consider a single variable at each point
  - Variable assignments are commutative, so ***fix ordering of variables***
    - I.e., [WA = red then NT = blue] same as
      - [NT = blue then WA = red]
  - What is ***branching factor*** of this search?

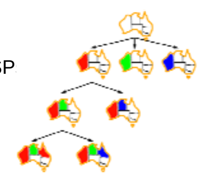
Backtracking Search

Note 2: Only allow legal assignments at each point
- I.e. Ignore values which conflict previous assignments
- Might need some computation to eliminate such conflicts
- "Incremental goal test"

"Backtracking Search"

Depth-first search for CSPs with these two ideas
- One variable at a time, fixed order
- Only trying consistent assignments

Is called "Backtracking Search"
- Basic uninformed algorithm for CSP
- Can solve n-queens for n = 25



Backtracking Search

```
function BACKTRACKING-SEARCH(csp) returns solution/failure
   return RECURSIVE-BACKTRACKING({ }, csp)
function RECURSIVE-BACKTRACKING(assignment, csp) returns soln/failure
   if assignment is complete then return assignment
   var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[csp], assignment, csp)
   for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
      if value is consistent with assignment given CONSTRAINTS[csp] then
         add {var = value} to assignment
         result ← RECURSIVE-BACKTRACKING(assignment, csp)
         if result ≠ failure then return result
         remove {var = value} from assignment
   return failure
```

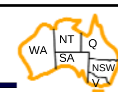
- What are the choice points?

Improving Backtracking

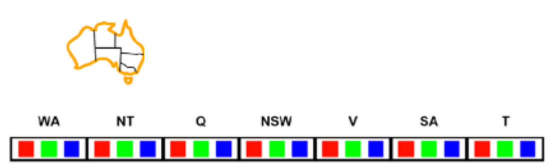
General-purpose ideas give huge gains in speed

- Ordering:
  - Which variable should be assigned next?
  - In what order should its values be tried?
- Filtering: Can we detect inevitable failure early?
- Structure: Can we exploit the problem structure?

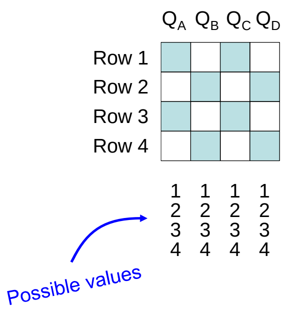
Forward Checking



- Idea: Keep track of remaining legal values for unassigned variables (using immediate constraints)
- Idea: Terminate when any variable has no legal values



Forward Checking



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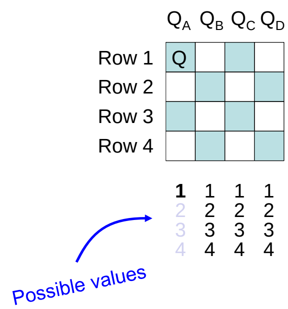
Forward Checking



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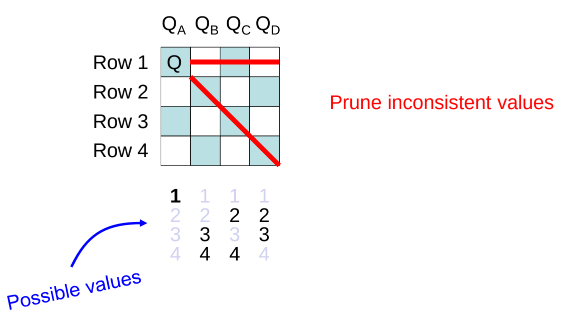
Forward Checking



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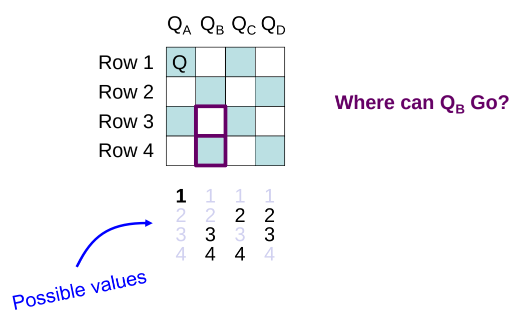
Forward Checking



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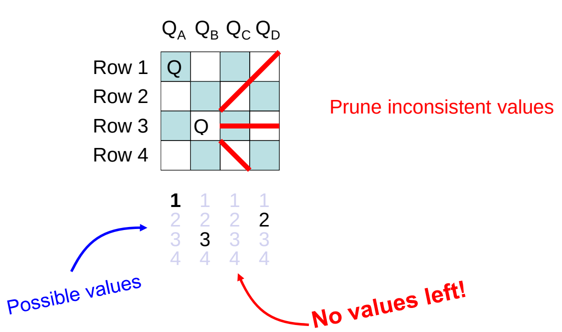
Forward Checking



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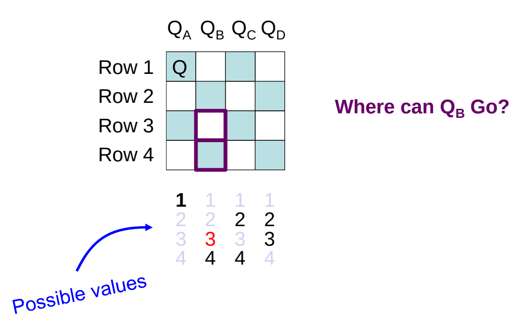
Forward Checking



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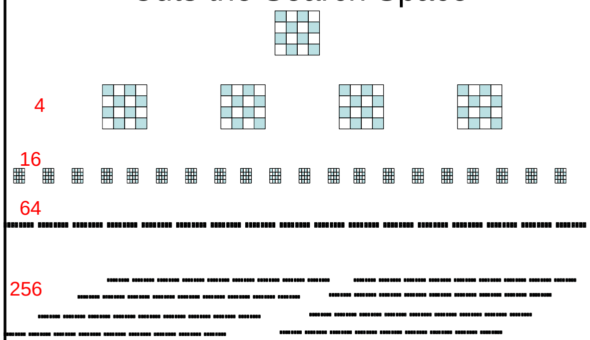
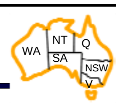
Forward Checking Cuts the Search Space



Are We Done?

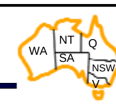
Constraint Propagation



- Forward checking propagates information from assigned to adjacent unassigned variables, but doesn't detect more distant failures:

- NT and SA cannot both be blue!
- Why didn't we detect this yet?
- *Constraint propagation* repeatedly enforces constraints (locally)

Arc Consistency



- Simplest form of propagation makes each arc *consistent*
  - X ~ Y is consistent iff for *every* value x there is *some* allowed y

- If X loses a value, neighbors of X need to be rechecked!
- Arc consistency detects failure earlier than forward checking
- What's the downside of arc consistency?
- Can be run as a preprocessor or after each assignment

Arc Consistency

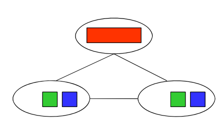
```
function AC-3( csp) returns the CSP, possibly with reduced domains
   inputs: csp, a binary CSP with variables {X1, X2, ..., Xn}
   local variables: queue, a queue of arcs, initially all the arcs in csp

   while queue is not empty do
      (Xi, Xj) ← REMOVE-FIRST(queue)
      if REMOVE-INCONSISTENT-VALUES(Xi, Xj) then
         for each Xk in NEIGHBORS[Xi] do
            add (Xk, Xi) to queue
─────────────────────────────────────────────
function REMOVE-INCONSISTENT-VALUES( Xi, Xj) returns true iff succeeds
   removed ← false
   for each x in DOMAIN[Xi] do
      if no value y in DOMAIN[Xj] allows (x,y) to satisfy the constraint Xi ↔ Xj
         then delete x from DOMAIN[Xi]; removed ← true
   return removed
```

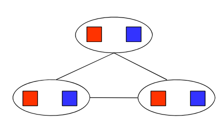
- Runtime: $O(n^2d^3)$, can be reduced to $O(n^2d^2)$
- … but detecting all possible future problems is NP-hard – why?

[demo: arc consistency animation]

Limitations of Arc Consistency

After running arc consistency:
- Can have one solution left
- Can have multiple solutions left
- Can have no solutions left
  (and not know it)

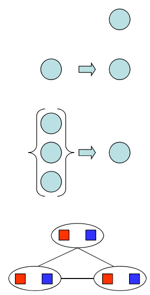




What went wrong here?

K-Consistency*

- Increasing degrees of consistency
  - 1-Consistency (Node Consistency): Each single node's domain has a value which meets that node's unary constraints
  - 2-Consistency (Arc Consistency): For each pair of nodes, any consistent assignment to one can be extended to the other
  - K-Consistency: For each k nodes, any consistent assignment to k-1 can be extended to the k^{th} node.
- Higher k more expensive to compute



Variable Ordering Heuristics

- Minimum remaining values (MRV):
  - Choose the variable with the fewest legal values

- Why min rather than max?
- Also called "most constrained variable"
- "Fail-fast" ordering

Ordering: Degree Heuristic

- Tie-breaker among MRV variables
- Degree heuristic:
  - Choose the variable participating in the most constraints on remaining variables

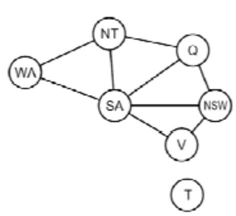
- Why most rather than fewest constraints?

Ordering: Least Constraining Value

- Given a choice of variable:
  - Choose the *least constraining value*
  - The one that rules out the fewest values in the remaining variables
  - Note that it may take some computation to determine this!
- Why least rather than most?
- Combining these heuristics makes 1000 queens feasible

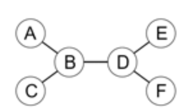
Problem Structure

- Tasmania and mainland are independent subproblems
- Identifiable as connected components of constraint graph
- Suppose each subproblem has c variables out of n total
- Worst-case solution cost is $O((n/c)(d^c))$, linear in n
  - E.g., n = 80, d = 2, c =20
  - 2^{80} = 4 billion years at 10 million nodes/sec
  - $(4)(2^{20})$ = 0.4 seconds at 10 million nodes/sec



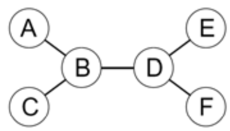
Tree-Structured CSPs

- Choose a variable as root, order variables from root to leaves such that every node's parent precedes it in the ordering



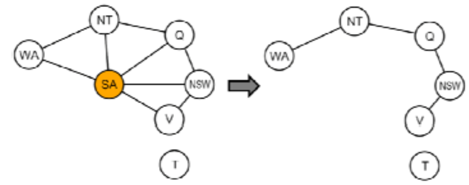
- For i = n : 2, apply RemoveInconsistent(Parent(X_i),X_i)
- For i = 1 : n, assign X_i consistently with Parent(X_i)
- Runtime: $O(n\ d^2)$

Tree-Structured CSPs



- Theorem: if the constraint graph has no loops, the CSP can be solved in O(n d^2) time!
  - Compare to general CSPs, where worst-case time is O(d^n)
- This property also applies to logical and probabilistic reasoning: an important example of the relation between syntactic restrictions and the complexity of reasoning.

Nearly Tree-Structured CSPs

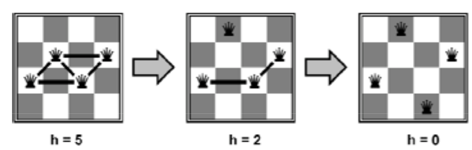


- Conditioning: instantiate a variable, prune its neighbors' domains
- Cutset conditioning: instantiate (in all ways) a set of variables such that the remaining constraint graph is a tree
- Cutset size c gives runtime O((d^c) (n-c) d^2), very fast for small c

Local Search for CSPs

- Greedy and stochastic methods typically search over “complete” states, i.e., all variables assigned
- To apply to CSPs:
  - Allow states with unsatisfied constraints
  - Operators *reassign* variable values
- Variable selection: randomly select any conflicted variable
- Value selection heuristic:
  - Min-conflicts
  - Choose value that violates the fewest constraints
  - I.e., hill climb with h(n) = total number of violated constraints

Example: 4-Queens

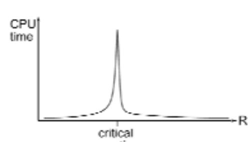


- States: 4 queens in 4 columns (4^4 = 256 states)
- Operators: move queen in column
- Goal test: no attacks
- Evaluation: h(n) = number of attacks

Performance of Min-Conflicts

- Given random initial state, can solve n-queens in almost constant time for large n (e.g., 10,000,000) with high probability
- The same appears to be true for any randomly-generated CSP *except* in a narrow range of the ratio

$$R = \frac{\text{number of constraints}}{\text{number of variables}}$$



CSP Summary

- CSPs are a special (factored) kind of search problem:
  - States defined by values (domains) of a fixed set of variables
  - Goal test defined by constraints on variable values
- Backtracking = DFS - one legal variable assigned per node
- Variable ordering and value selection heuristics help
- Forward checking prevents assignments that fail later
- Constraint propagation (e.g., arc consistency)
  - does additional work to constrain values and detect inconsistencies
- Constraint graph representation
  - Allows analysis of problem structure
- Tree-structured CSPs can be solved in linear time
- Local (stochastic) search often effective in practice
  - Iterative min-conflicts