Algebraic Geometry
A Personal View

CSE 590B

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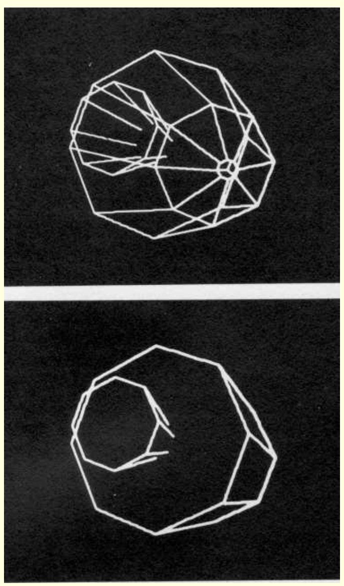
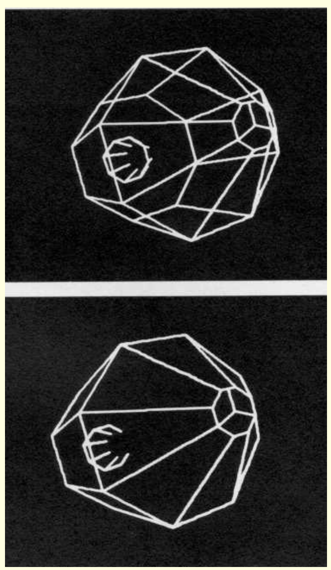
University of Michigan

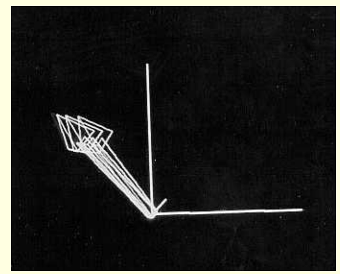
1967-1974

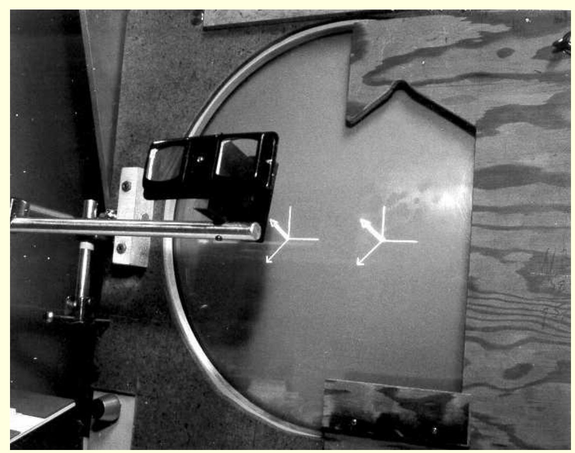
University of Michigan

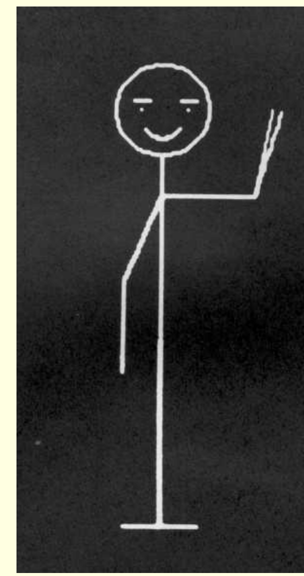
1967-1974





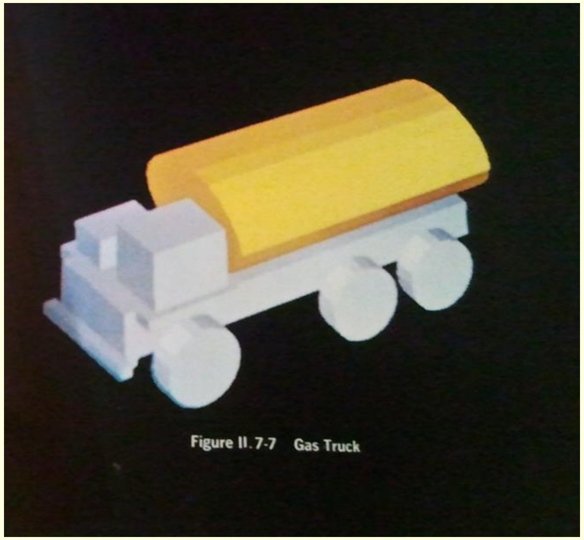


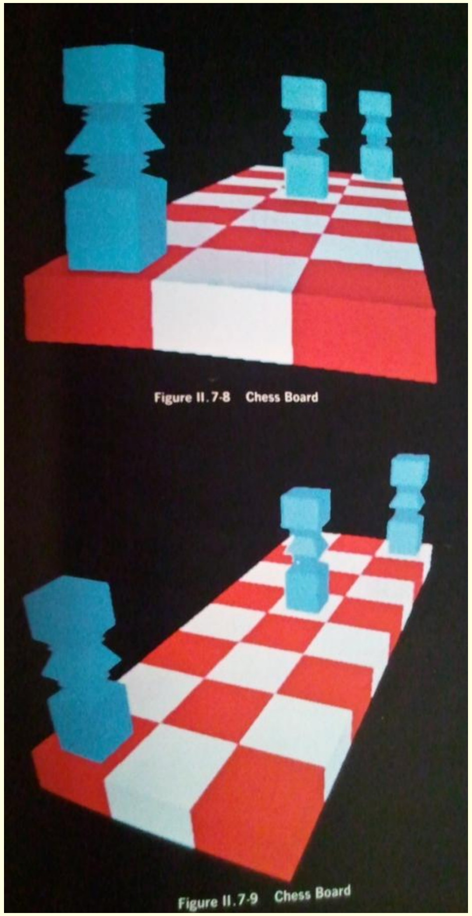




Gordon Romney (U Utah)

1969

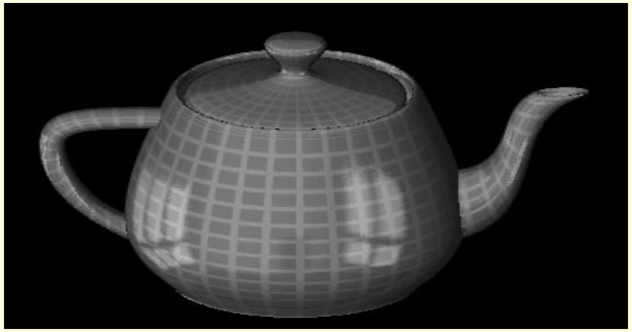




+ Appendix

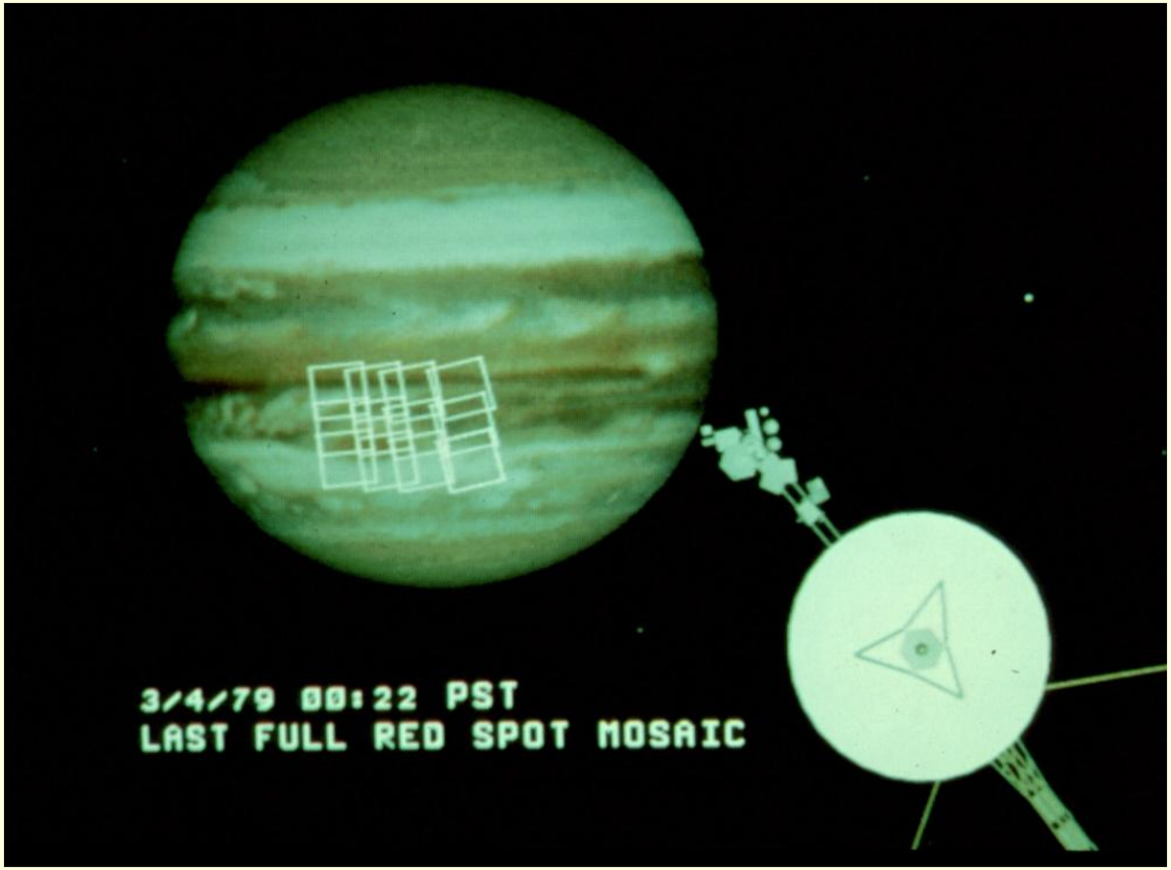
University of Utah

1974-1977



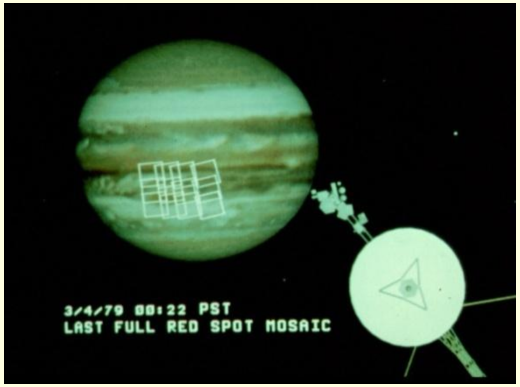
JPL/Caltech

1977-1995

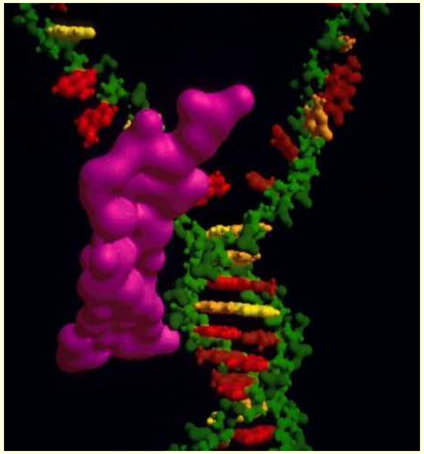


JPL/Caltech

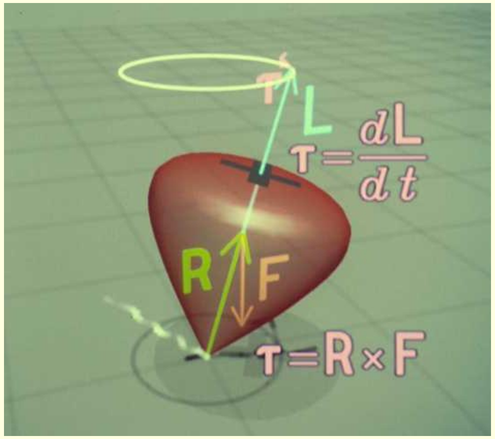
1977-1995



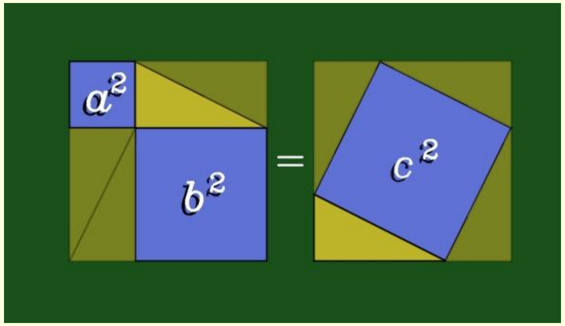
Voyager



Cosmos



The Mechanical Universe

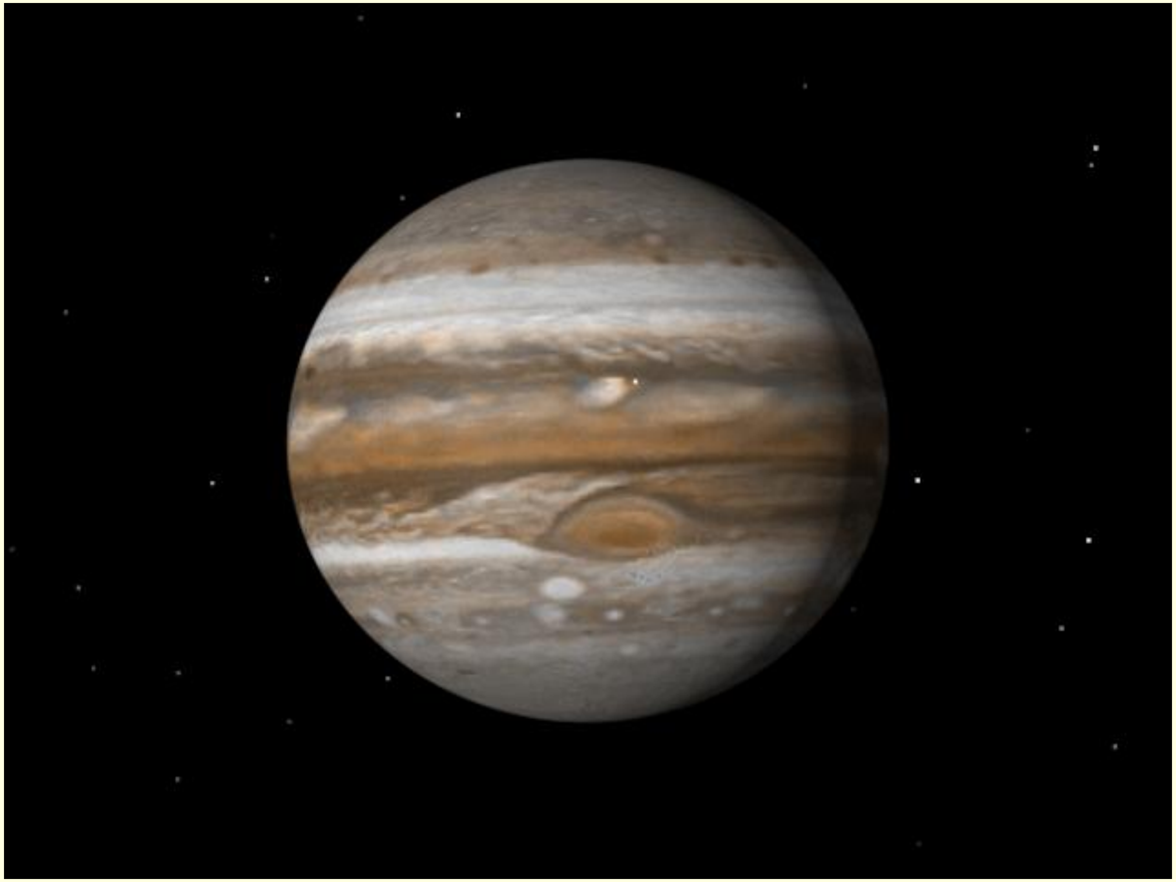


Mathematics!

Render 3D Objects

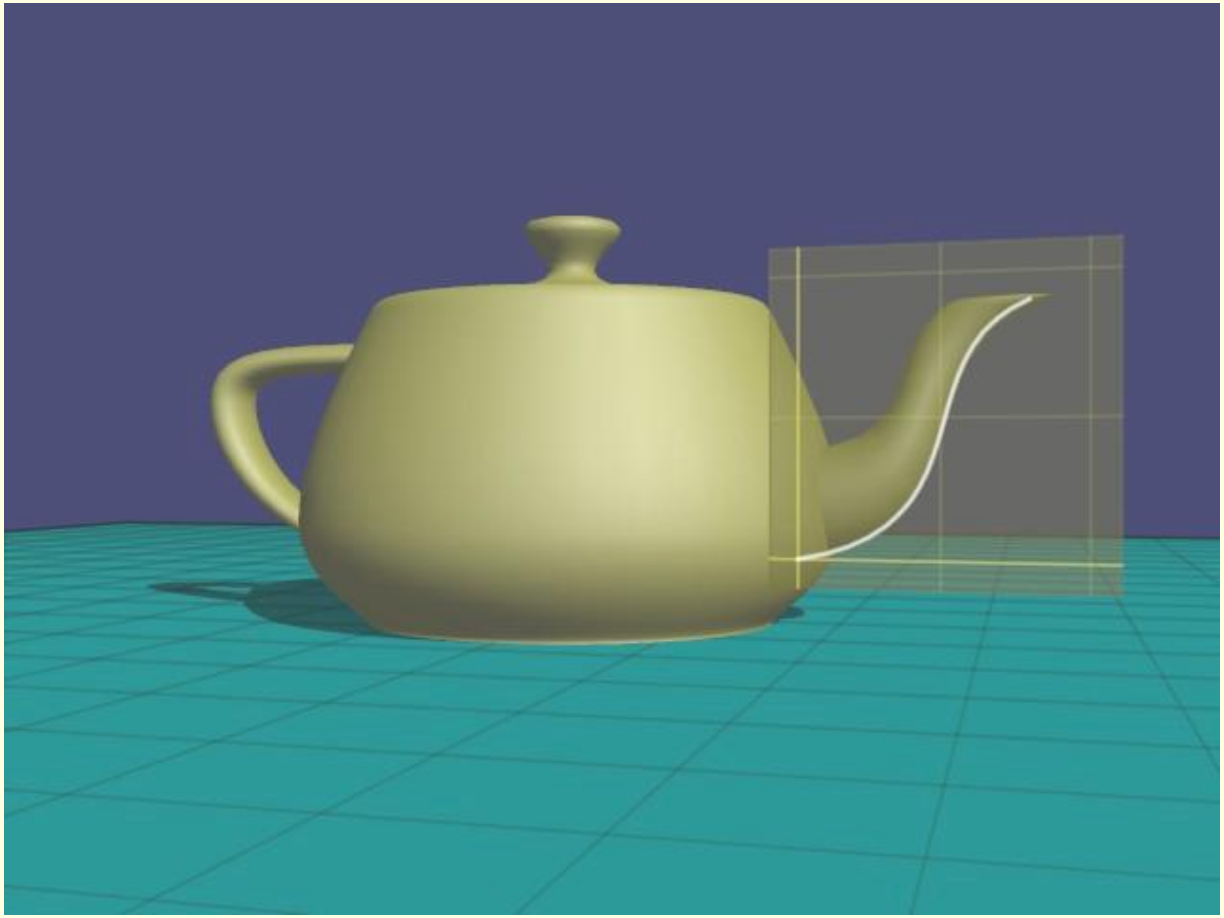
Planar Polygons = First Order Surfaces

Render 3D Objects



Second Order Surfaces

Render 3D Objects



Third (and higher) Order Surfaces

UM, UU, JPL, Microsoft and Now

1962-present

Studying Algebraic Geometry

Algebraic Equations

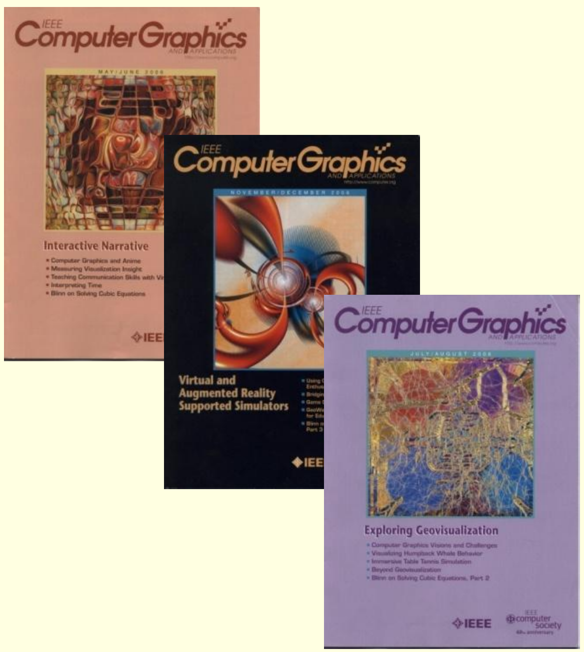
↕

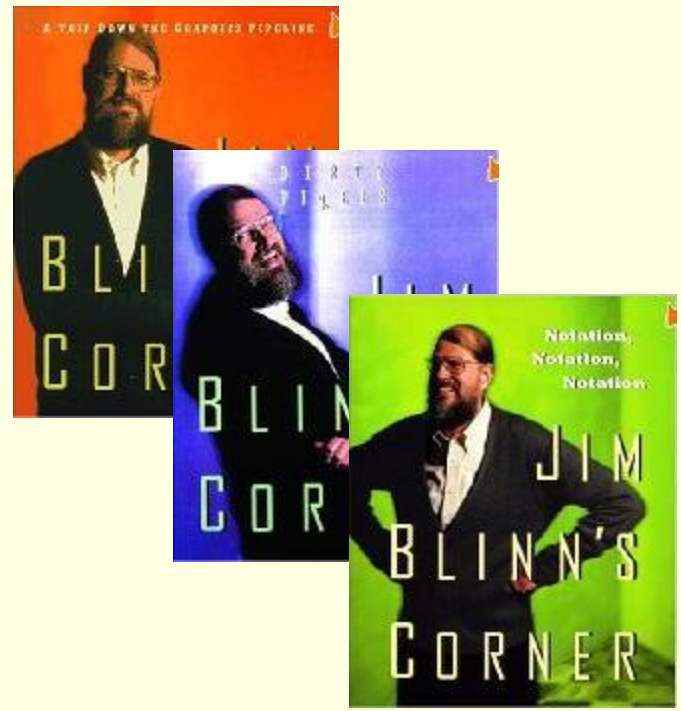
Geometric Shapes

Making Algebraic Geometry More Understandable

Jim Blinn's Corner Articles

1987 - 2007





Many of them on Algebraic Geometry

Why Am I Here

- Share my enthusiasms
- Help me organize my ideas
  - I work better if I have an audience (M.B.)
  - Updates to old articles
  - Unpublished articles
  - Keep me from repeating myself
  - Publish on web site
- One Session every 2 weeks
- Later meetings may get more sketchy
- Discuss open questions

Why Are You Here

- Varied Audience
  - Go slowly at first
  - Prerequisites:
    - vectors and matrices
    - homogeneous coords
- See old stuff in new ways
- See new stuff

What I will talk about

- Real Algebraic Projective Geometry
  - *Real* is more complex than *Complex*
  - *Projective* is simpler than *Euclidean*
- Dimension 1,2,3
- Lowish Order Polynomials

- Notation, notation, notation
- Lots of Pictures

Pictures? Hartshorne vs. Abraham&Shaw

Why no

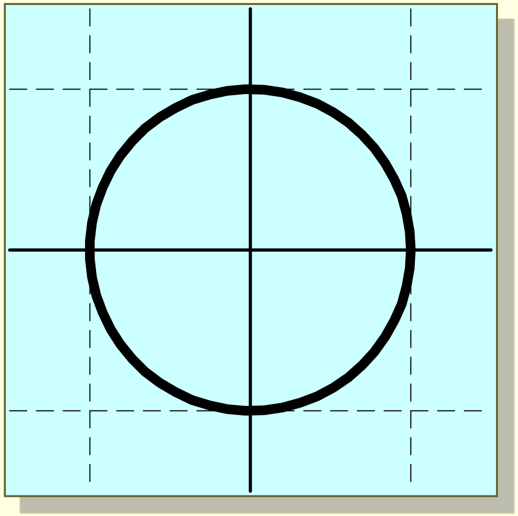
- can fool you
- show only special cases
- hard to generalize to high dimensions
- hard to make
- forces you to think (visualize internally)

Why yes

- intuition
- see patterns
- I am visual thinker (*see* patterns)
- pretty

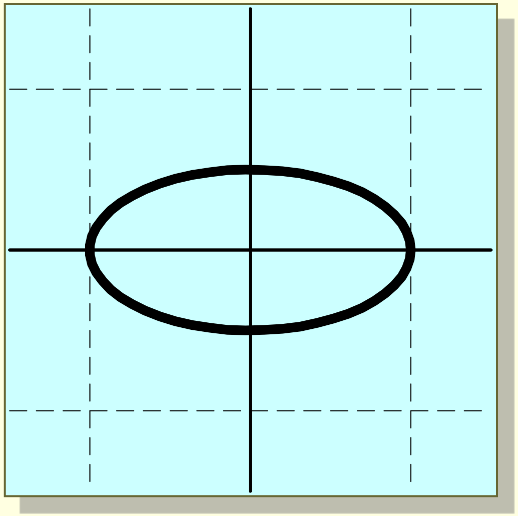
Relation Between Algebra and Geometry

$$X^2 + Y^2 = 1$$



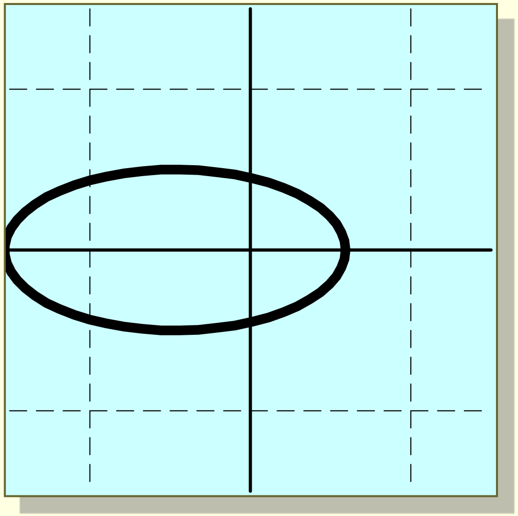
Relation Between Algebra and Geometry

$$X^2 + 4Y^2 = 1$$



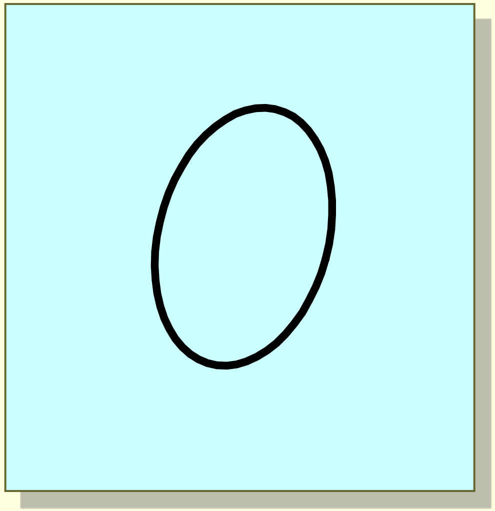
Relation Between Algebra and Geometry

$$X^2 + X + 4Y^2 = 1$$



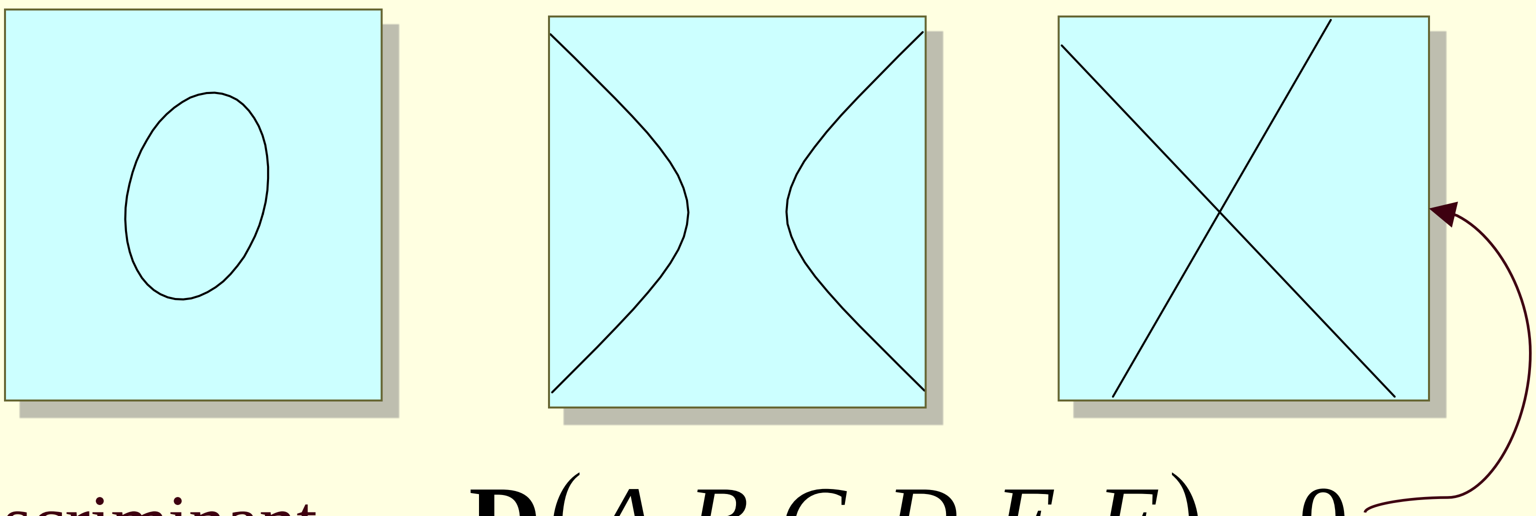
General Quadratic Curve

$$AX^2 + 2BXY + CY^2 + 2DX + 2EY + F = 0$$



Quadratic Curve

$$A X^2 + 2B\,XY + C\,Y^2$$
$$+2D\,X + 2E\,Y + F = 0$$



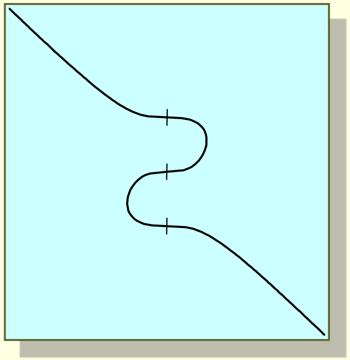
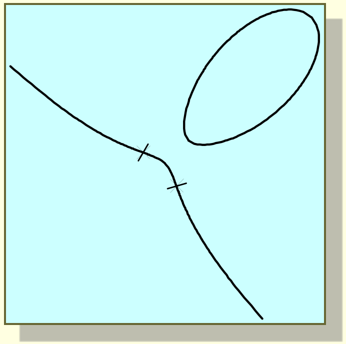
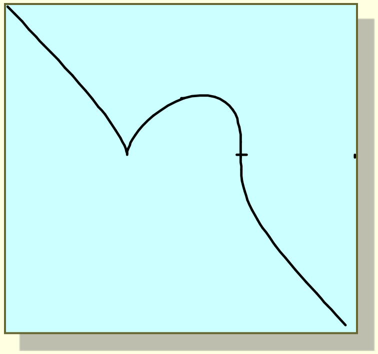
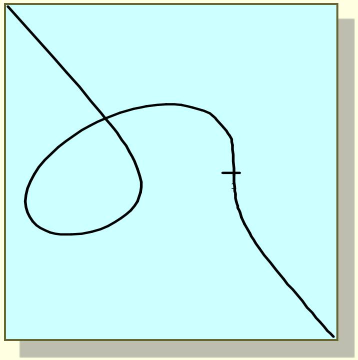
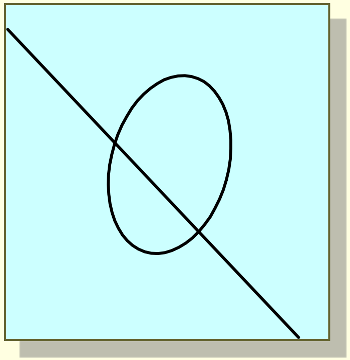
Discriminant

$$\mathbf{D}\left(A, B, C, D, E, F\right) = 0$$

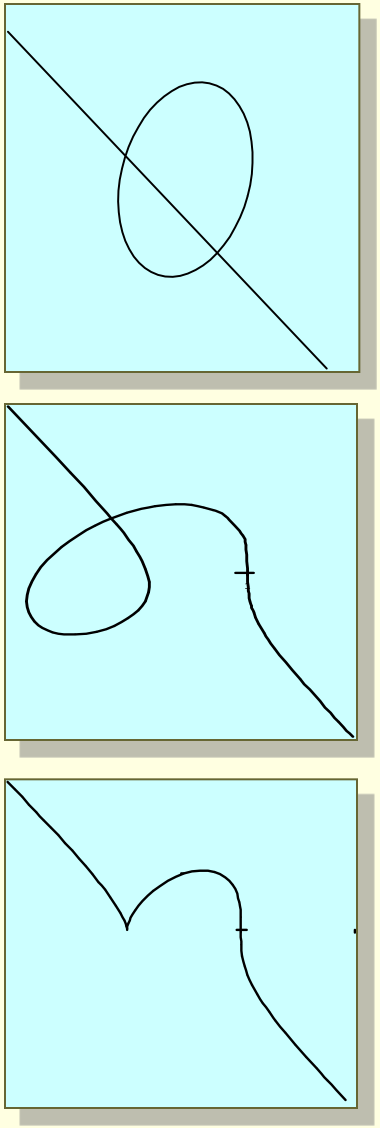
$$\mathbf{D}\left(\ldots\right) = ACF + 2BED - D^2C - E^2A - B^2F$$

Cubic Curve

$$AX^3 + 3BX^2Y + 3CXY^2 + DY^3$$
$$+3EX^2 + 6FXY + 3GY^2$$
$$+3HX + 3JY + K = 0$$

Discriminant of Cubic



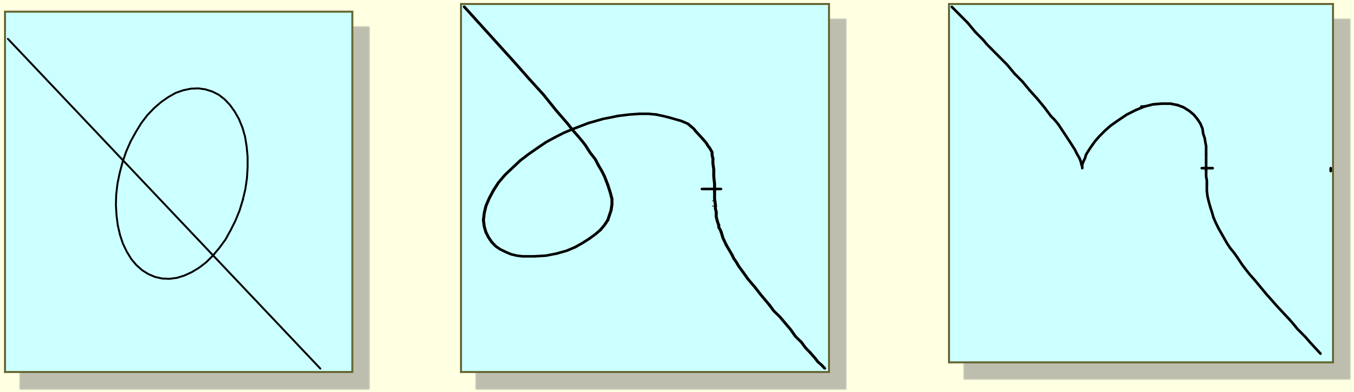
$$\mathbf{D}\left(A, B, C, D, E, F, G, H, J, K\right) = 0$$

G. Salmon (1879):

$$\begin{aligned}\mathbf{D} = {} & A^4D^4K^4 - 12A^4D^3K^3GJ \\ & +36A^4D^2K^2G^2J^2 + 64A^3D^3K^3F^3 \\ & -192A^2D^3K^3F^2BE + 192AD^3K^3FB^2E^2 \\ & -64D^3K^3B^3E^3 + \ldots\end{aligned}$$

D has over 10,000 terms

Discriminant of Cubic



$$\mathbf{D} = 64S^3 + T^2$$

S: degree 4 in *A*…*K*
has 25 terms

T: degree 6 in *A*…*K*
has 103 terms

Want Better Notation

Notation = Creative Abbreviation

$$ab + cd = e$$

$$fb + hd = k$$

$$\begin{bmatrix} a & c \\ f & h \end{bmatrix} \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} e \\ k \end{bmatrix}$$

$$\mathbf{M}\,\mathbf{v} = \mathbf{w}$$

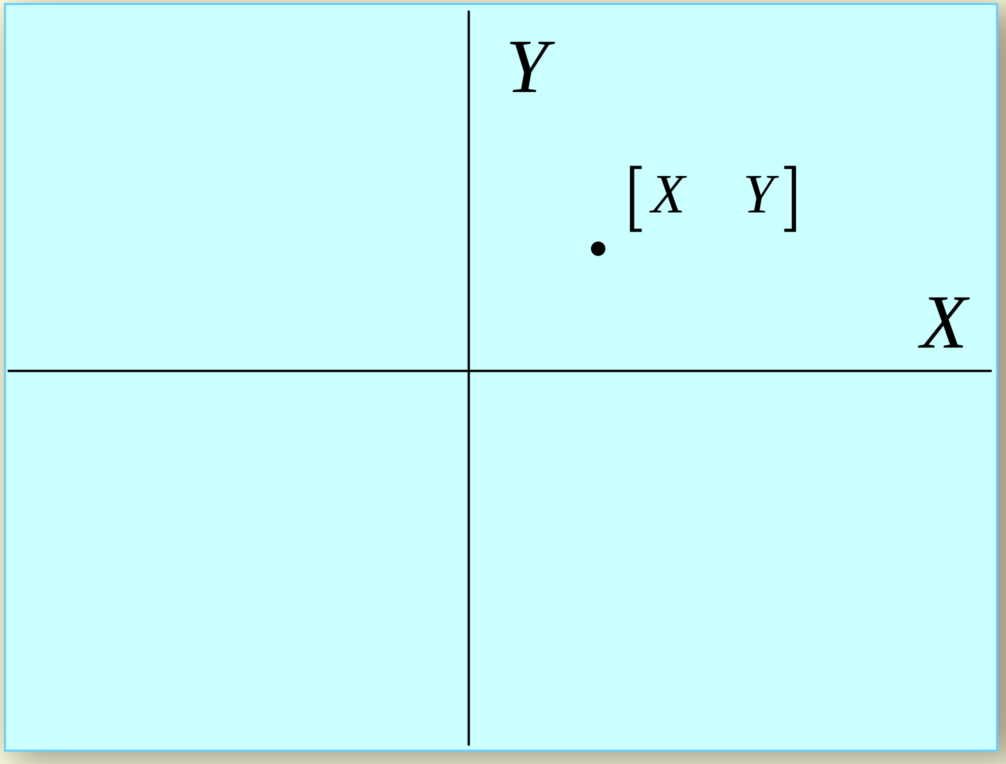
$$\mathbf{N}(\mathbf{M}\,\mathbf{v}) = (\mathbf{N}\,\mathbf{M})\,\mathbf{v}$$

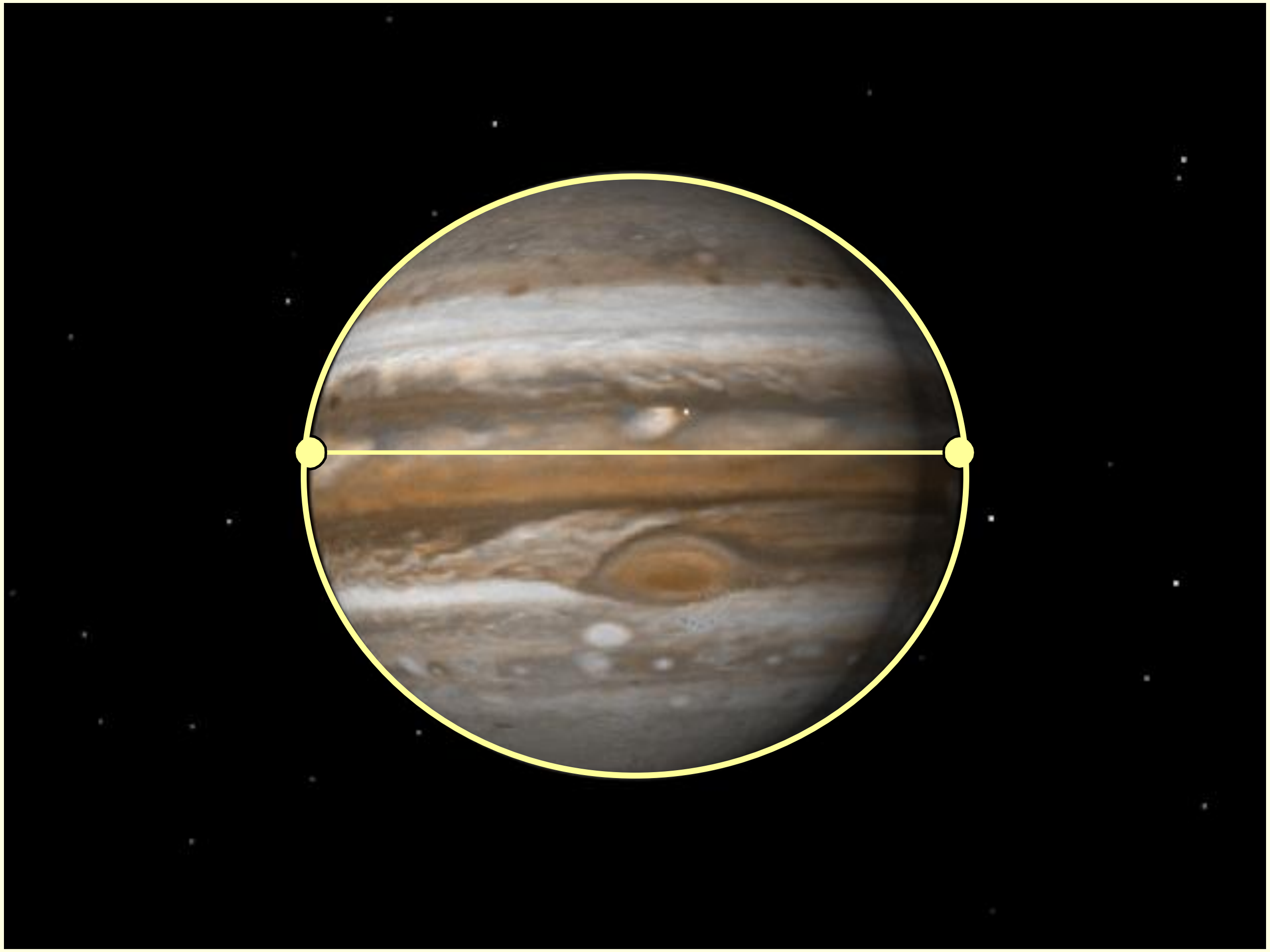
Review of Typical Notation

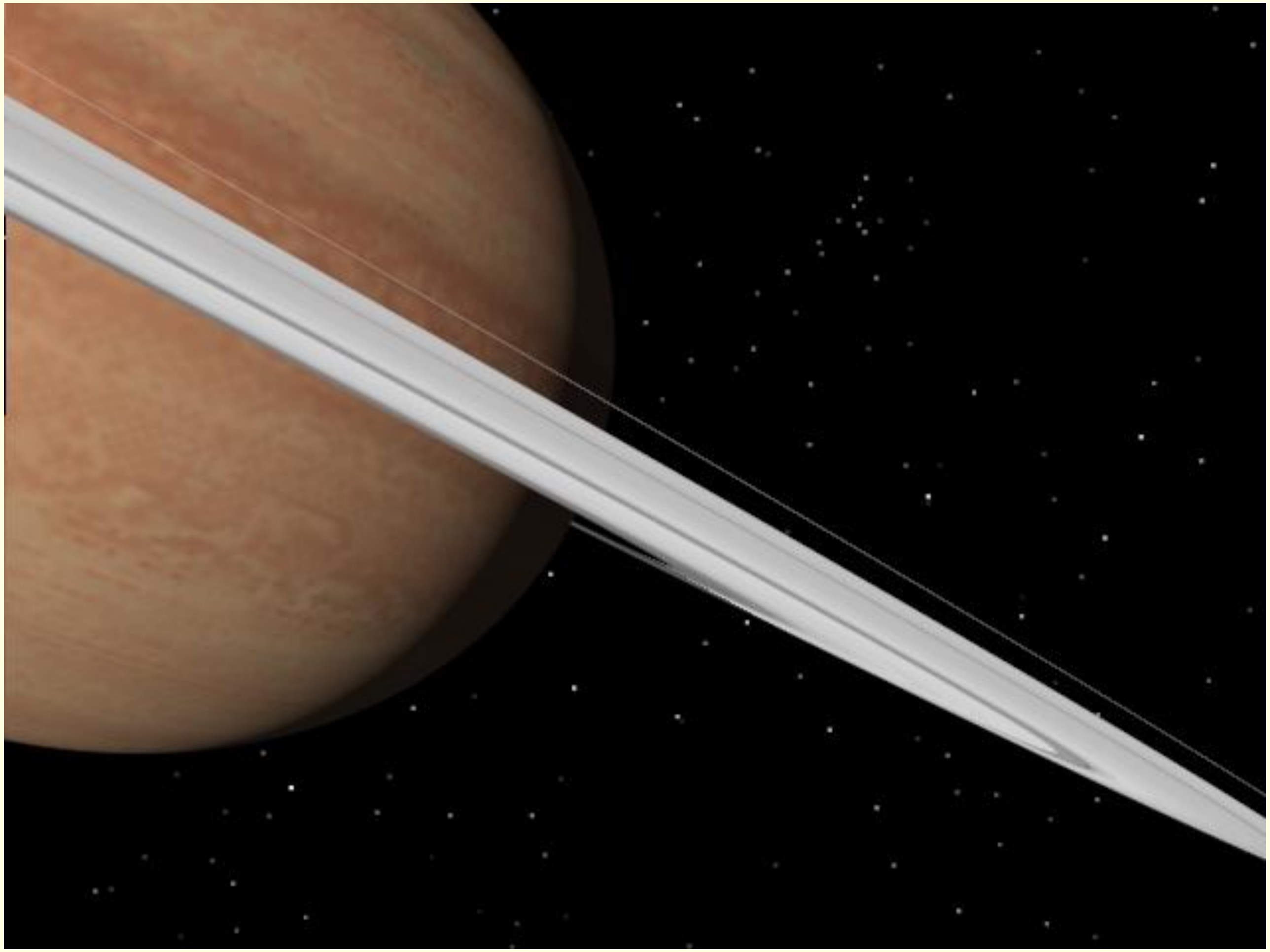
And some snags

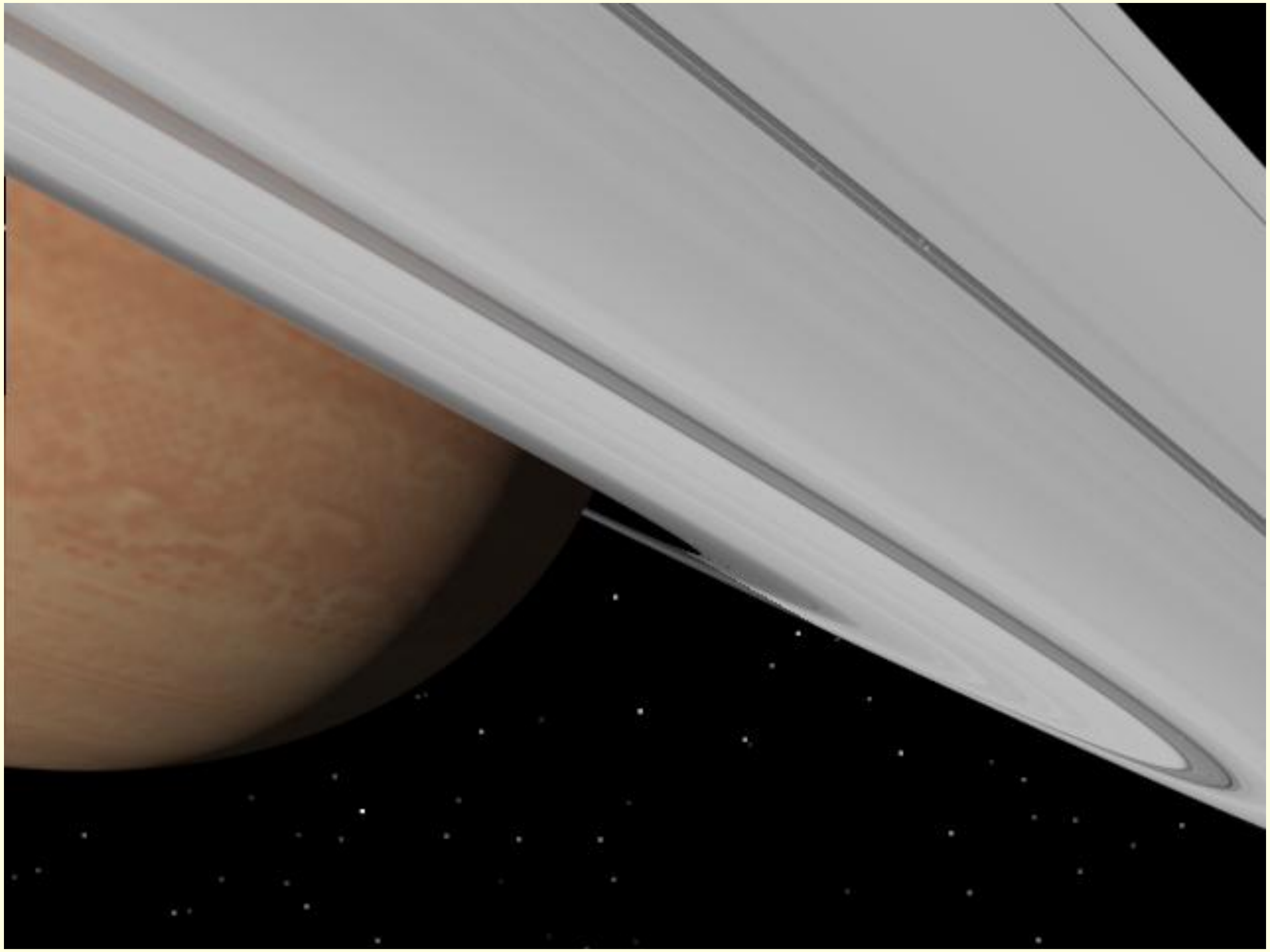
2D Euclidean Geometry

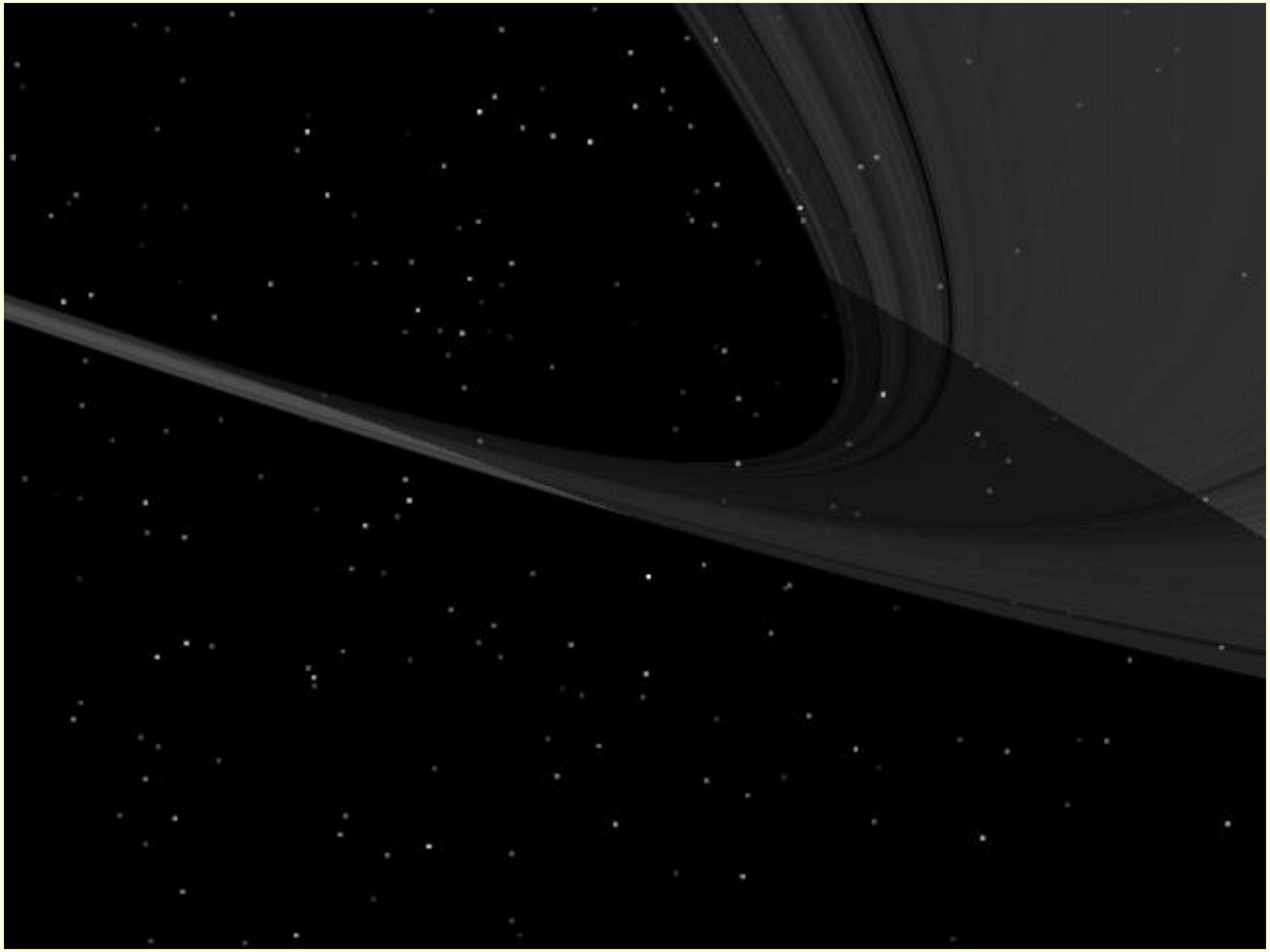
$$\mathbf{P} = \begin{bmatrix} X & Y \end{bmatrix}$$





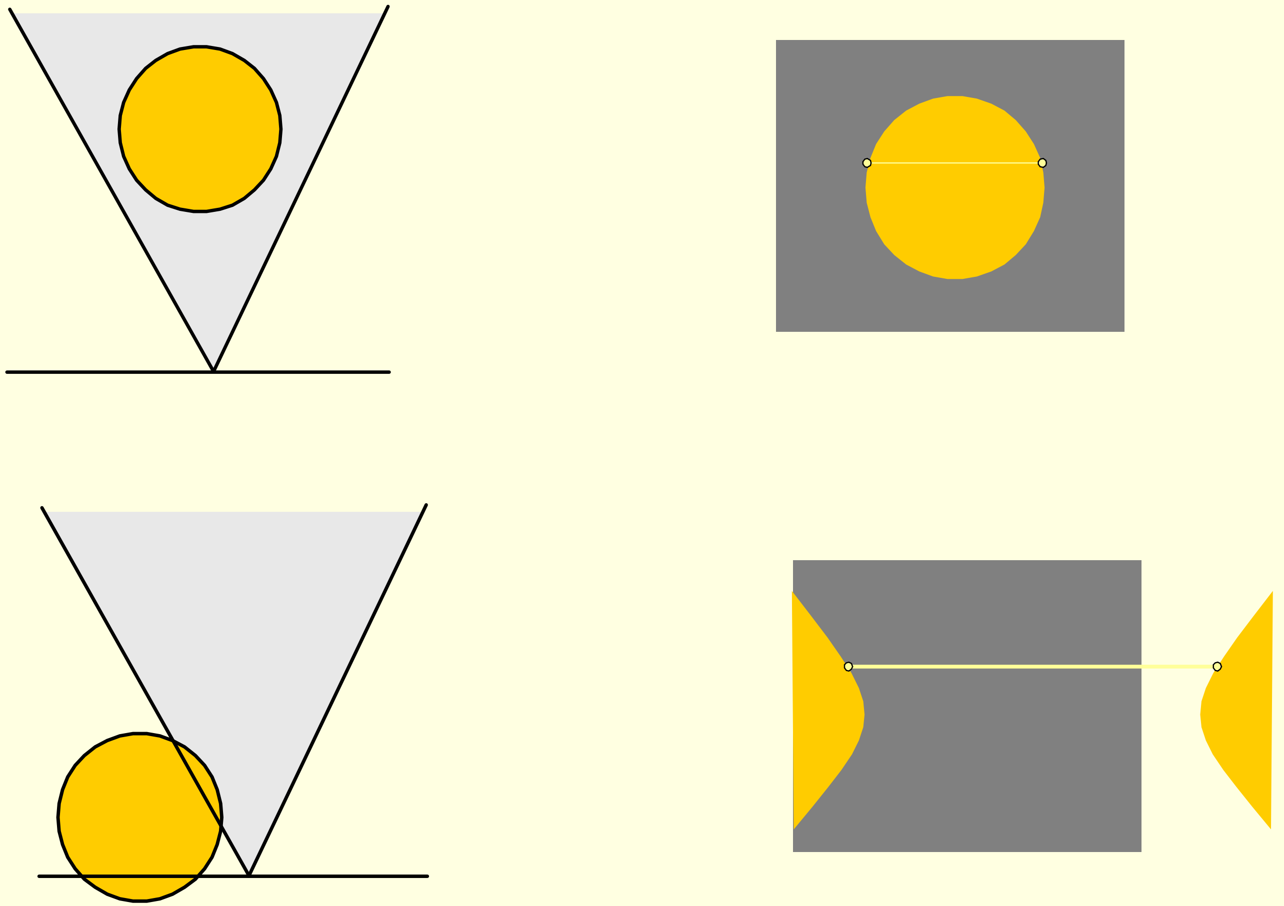






What Went Wrong?

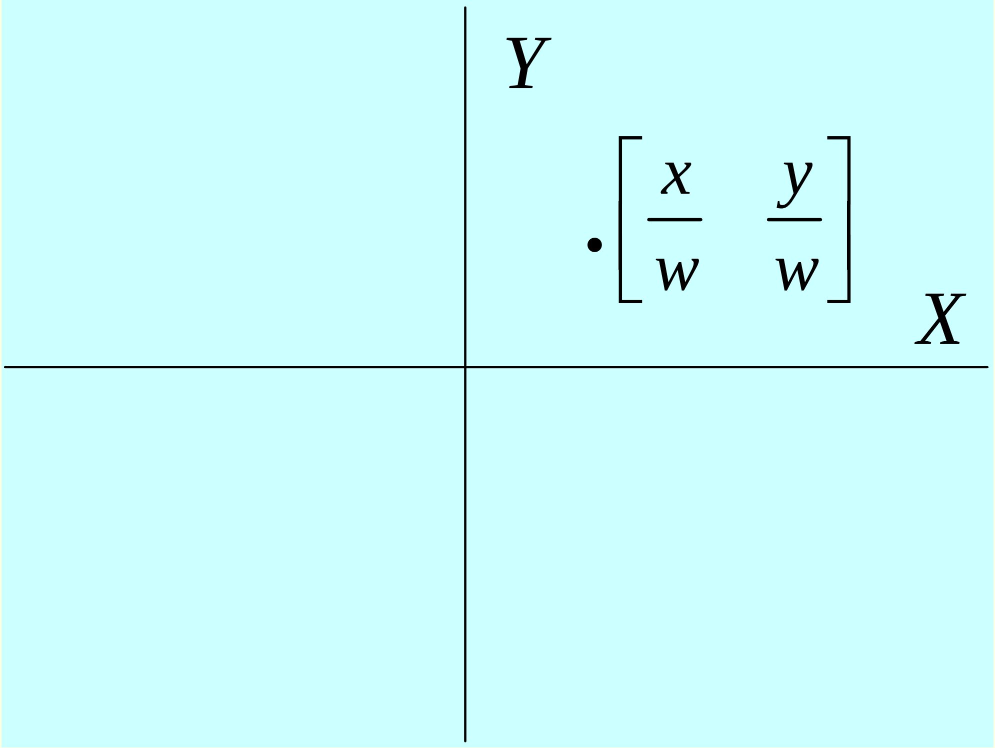
Top View | Front View (Post Perspective)



2D Projective Geometry
3D Algebraic Objects

$$\mathbf{P} = \begin{bmatrix} x & y & w \end{bmatrix}$$

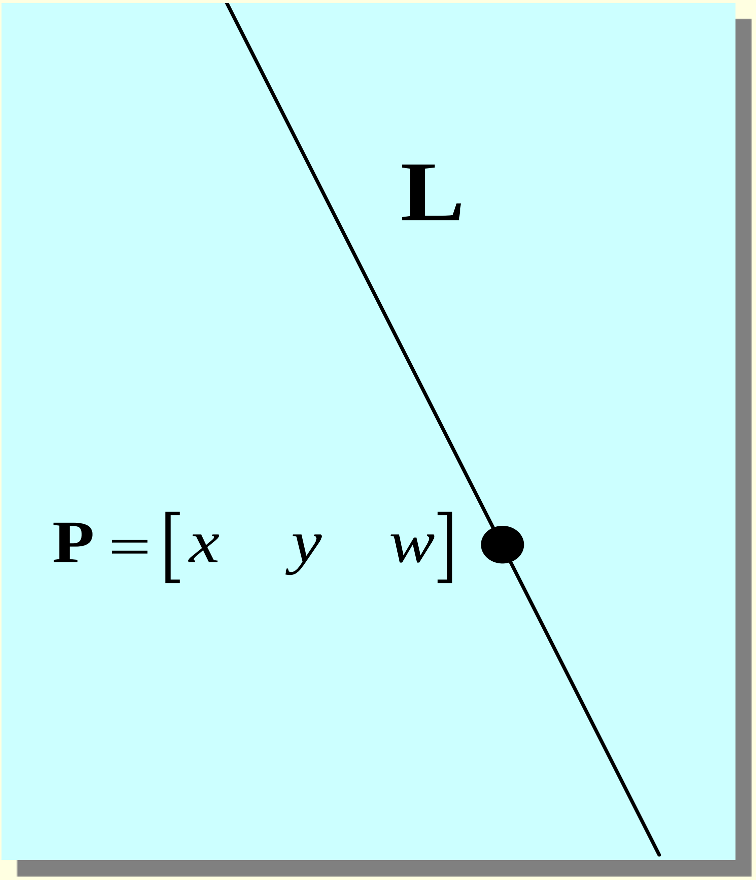
$$\cong \begin{bmatrix} \alpha x & \alpha y & \alpha w \end{bmatrix}$$



Equation of a Line

$$ax + by + cw = 0$$

$$\begin{bmatrix} x & y & w \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

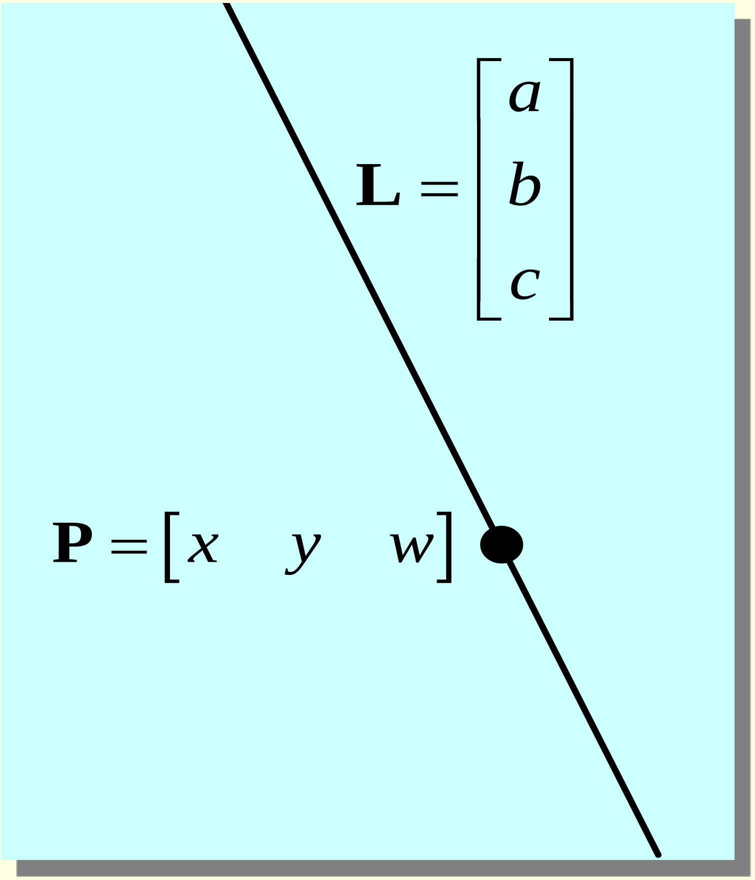


Equation of a Line

$$ax + by + cw = 0$$

$$\begin{bmatrix} x & y & w \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

$$\mathbf{P} \cdot \mathbf{L} = 0$$



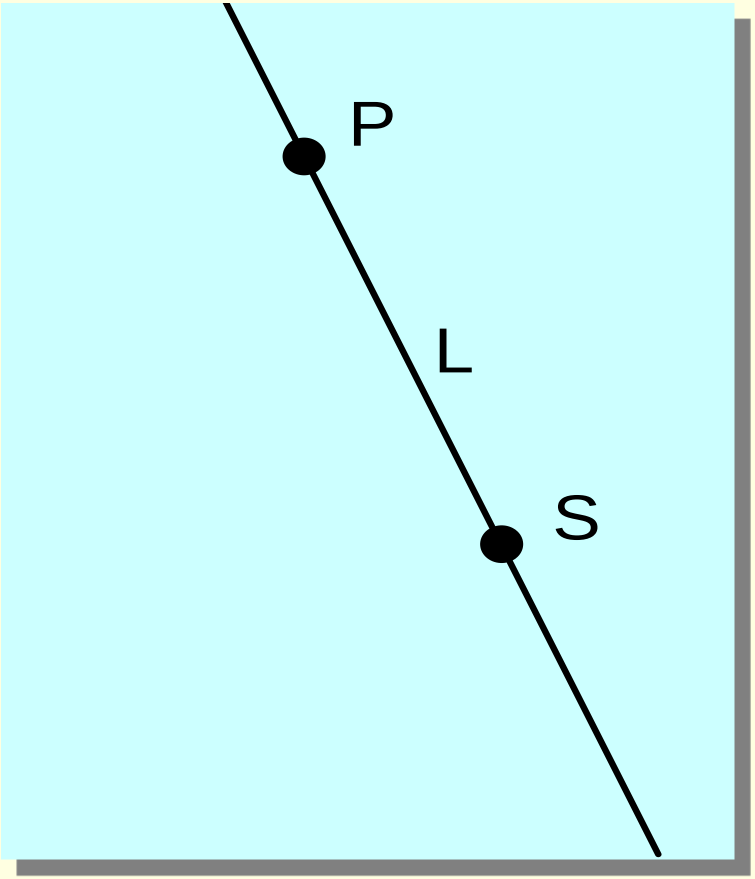
Row/column standardization?

Two Points Make A Line

$$\begin{bmatrix} x_P & y_P & w_P \\ x_S & y_S & w_S \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{matrix} \begin{bmatrix} x_P & y_P & w_P \end{bmatrix} \\ \times \\ \begin{bmatrix} x_S & y_S & w_S \end{bmatrix} \end{matrix}$$

$$\mathbf{L} = \mathbf{P} \times \mathbf{S}$$



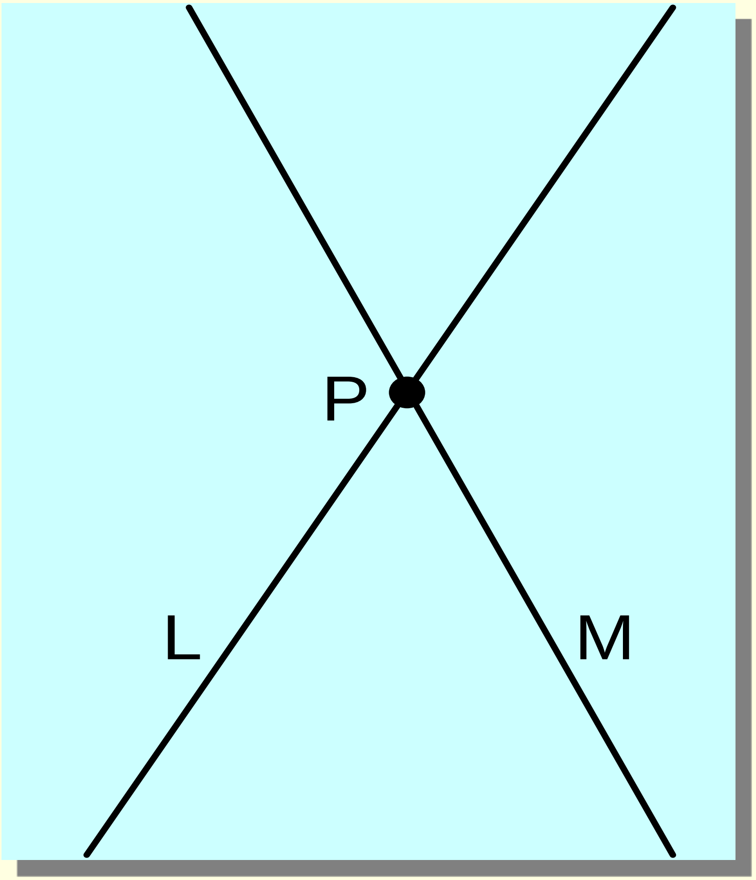
$$a = y_P w_S - w_P y_S, \; b = w_P x_S - x_P w_S, \; c = x_P y_S - y_P x_S$$

Two Lines Make A Point

$$\begin{bmatrix} x & y & w \end{bmatrix} \begin{bmatrix} a_L & a_M \\ b_L & b_M \\ c_L & c_M \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x & y & w \end{bmatrix} = \begin{bmatrix} a_L \\ b_L \\ c_L \end{bmatrix} \times \begin{bmatrix} a_M \\ b_M \\ c_M \end{bmatrix}$$

$$\mathbf{P} = \mathbf{L} \times \mathbf{M}$$



Transforming Points

$$\mathbf{P}\mathbf{T} = \hat{\mathbf{P}}$$

$$\begin{bmatrix} x & y & w \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} = \begin{bmatrix} \hat{x} & \hat{y} & \hat{w} \end{bmatrix}$$

Transforming Lines

$$\mathbf{P} \cdot \mathbf{L} = 0$$

$$\mathbf{P}\left(\mathbf{T}\mathbf{T}^{-1}\right)\mathbf{L} = 0$$

$$\left(\mathbf{P}\mathbf{T}\right)\left(\mathbf{T}^{-1}\mathbf{L}\right) = 0$$

$$\tilde{\mathbf{P}} \cdot \tilde{\mathbf{L}} = 0$$

$$\mathbf{P}\mathbf{T} = \tilde{\mathbf{P}}$$

$$\mathbf{T}^{-1}\mathbf{L} = \tilde{\mathbf{L}}$$

Matrix Adjugate (fka Adjoint)

$$\mathbf{T} = \begin{bmatrix} \cdots R_1 \cdots \\ \cdots R_2 \cdots \\ \cdots R_3 \cdots \end{bmatrix}$$

?

$$\mathbf{T}^* = \begin{bmatrix} \vdots & \vdots & \vdots \\ R_2 \times R_3 & R_3 \times R_1 & R_1 \times R_2 \\ \vdots & \vdots & \vdots \end{bmatrix}$$

$$\mathbf{T}\mathbf{T}^* = \begin{bmatrix} \det \mathbf{T} & 0 & 0 \\ 0 & \det \mathbf{T} & 0 \\ 0 & 0 & \det \mathbf{T} \end{bmatrix} = (\det \mathbf{T}) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Transforming Points and Lines

$$\mathbf{P}\,\mathbf{T} = \tilde{\mathbf{P}} \qquad \begin{bmatrix} x & y & w \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} = \begin{bmatrix} \hat{x} & \hat{y} & \hat{w} \end{bmatrix}$$

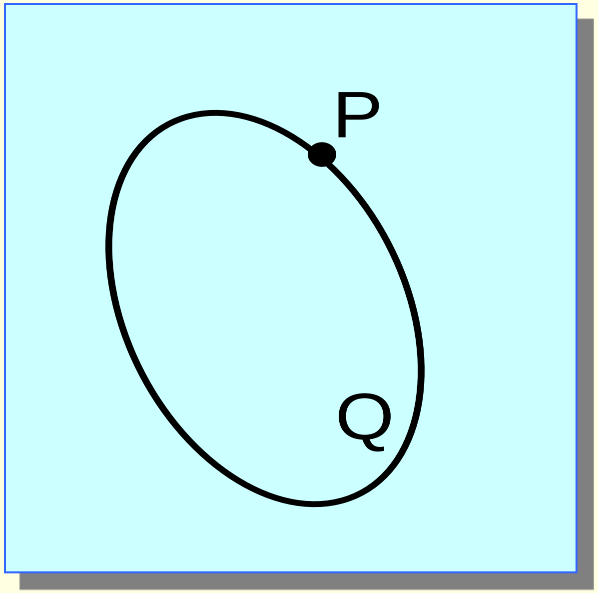
$$\mathbf{T}^{*}\mathbf{L} = \tilde{\mathbf{L}} \qquad \begin{bmatrix} T^{*}{}_{11} & T^{*}{}_{12} & T^{*}{}_{13} \\ T^{*}{}_{21} & T^{*}{}_{22} & T^{*}{}_{23} \\ T^{*}{}_{31} & T^{*}{}_{32} & T^{*}{}_{33} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \hat{a} \\ \hat{b} \\ \hat{c} \end{bmatrix}$$

Point on Quadratic Curve

$$Ax^2 + 2Bxy + 2Cxw$$
$$+Dy^2 + 2Eyw$$
$$+Fw^2 = 0$$

$$\begin{bmatrix} x & y & w \end{bmatrix} \begin{bmatrix} A & B & C \\ B & D & E \\ C & E & F \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = 0$$

$$\mathbf{P} \cdot \mathbf{Q} \cdot \mathbf{P}^T = 0$$



Transforming a Quadratic

$$\mathbf{P}\mathbf{Q}\mathbf{P}^T = 0$$

$$\mathbf{P}\left(\mathbf{T}\mathbf{T}^*\right)\mathbf{Q}\left(\mathbf{T}\mathbf{T}^*\right)^T\mathbf{P}^T = 0$$

$$\left(\mathbf{P}\mathbf{T}\right)\left(\mathbf{T}^*\mathbf{Q}\mathbf{T}^{*T}\right)\left(\mathbf{P}\mathbf{T}\right)^T = 0$$

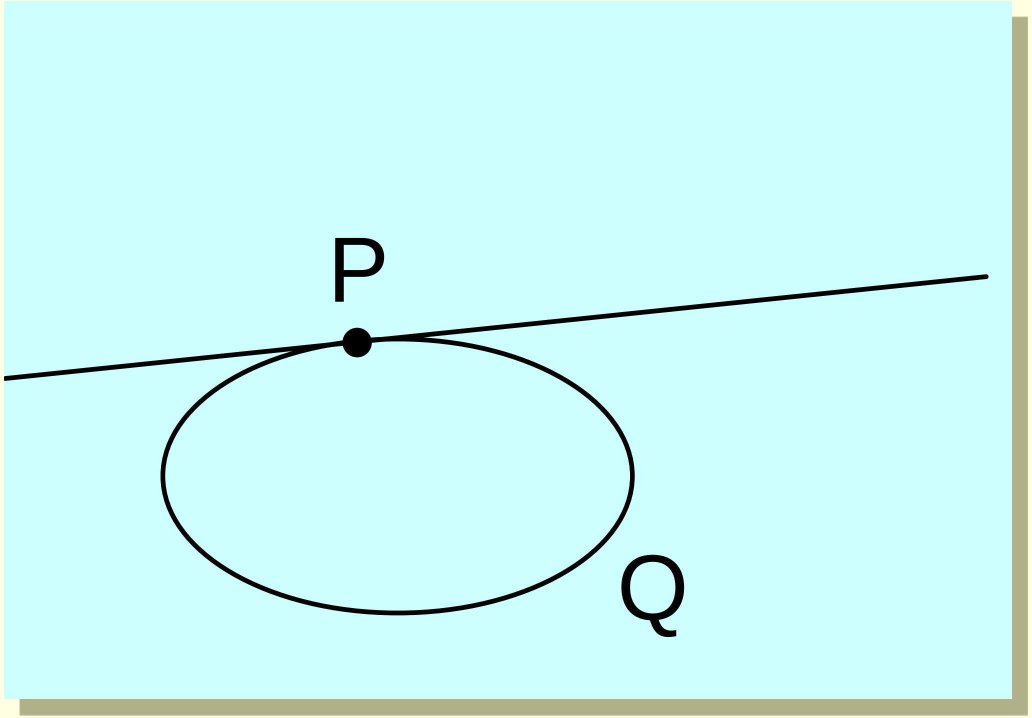
$$\tilde{\mathbf{P}}\tilde{\mathbf{Q}}\tilde{\mathbf{P}}^T = 0$$

$$\mathbf{P}\mathbf{T} = \tilde{\mathbf{P}}$$

$$\mathbf{T}^*\mathbf{Q}\mathbf{T}^{*T} = \tilde{\mathbf{Q}}$$

Given Point, Find Tangent

$$
\begin{aligned}
0 &= \mathbf{P}\mathbf{Q}\mathbf{P}^T \\
&= \mathbf{P} \cdot \left(\mathbf{Q}\mathbf{P}^T\right) \\
&= \mathbf{P} \cdot \mathbf{L}
\end{aligned}
$$

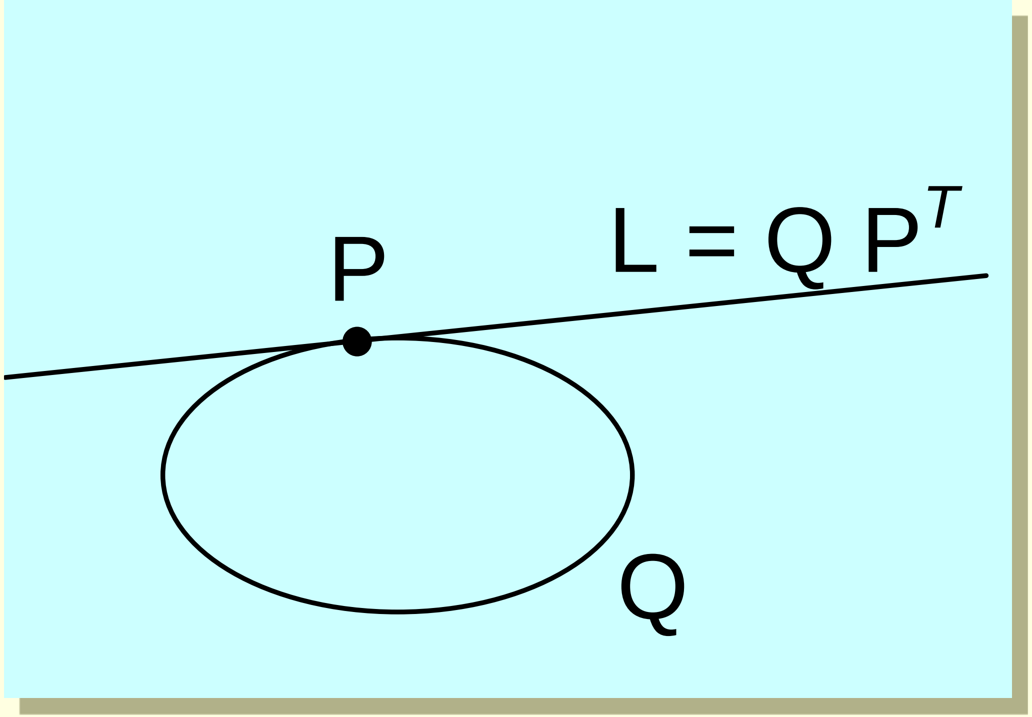


Given Point, Find Tangent

$$0 = \mathbf{P}\mathbf{Q}\mathbf{P}^T$$

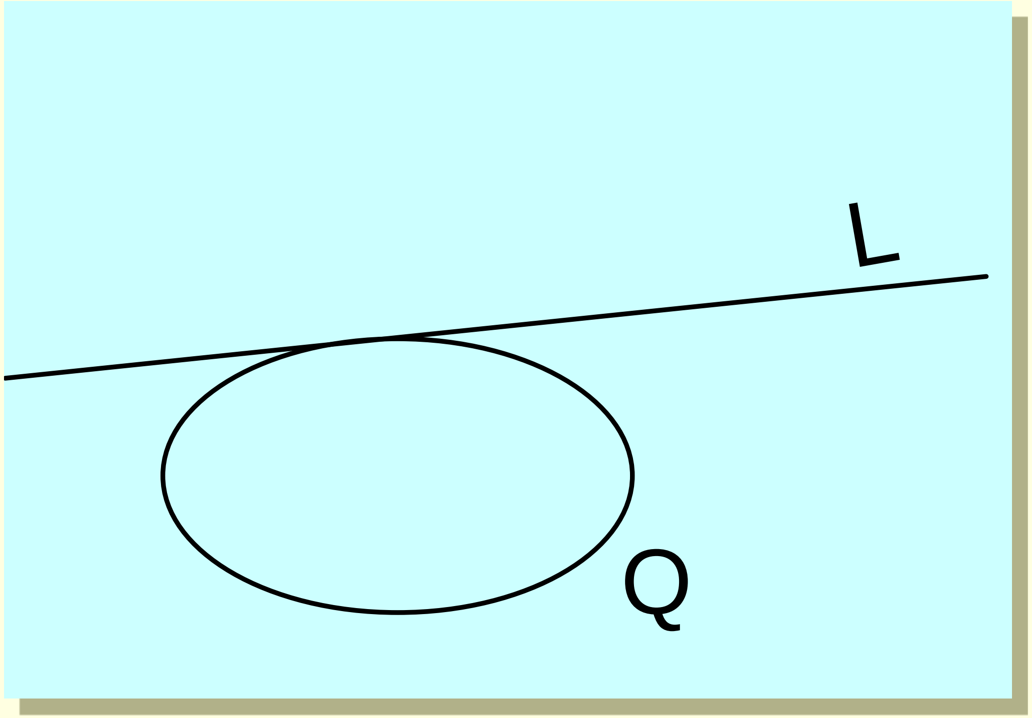
$$= \mathbf{P}\cdot\left(\mathbf{Q}\mathbf{P}^T\right)$$

$$= \mathbf{P}\cdot\mathbf{L}$$



Is a Line Tangent to Q

$$0 = \mathbf{L}^T \mathbf{Q}^* \mathbf{L}$$

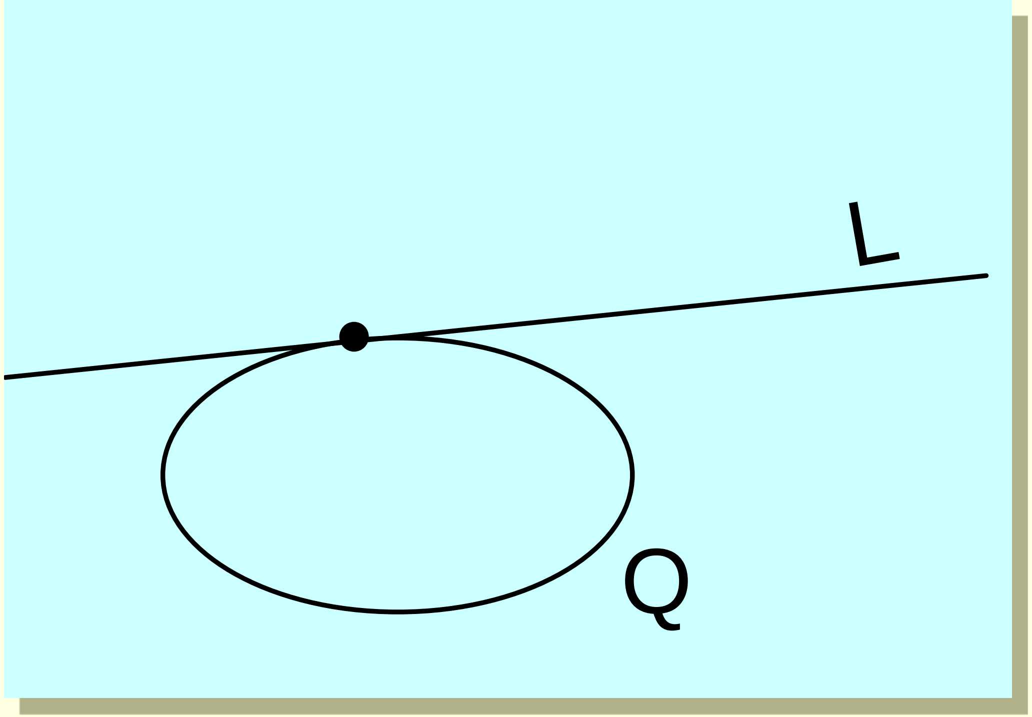


Given Tangent, Find Point

$$0 = \mathbf{L}^T \mathbf{Q}^* \mathbf{L}$$

$$= \left(\mathbf{L}^T \mathbf{Q}^* \right) \mathbf{L}$$

$$= \mathbf{P} \cdot \mathbf{L}$$

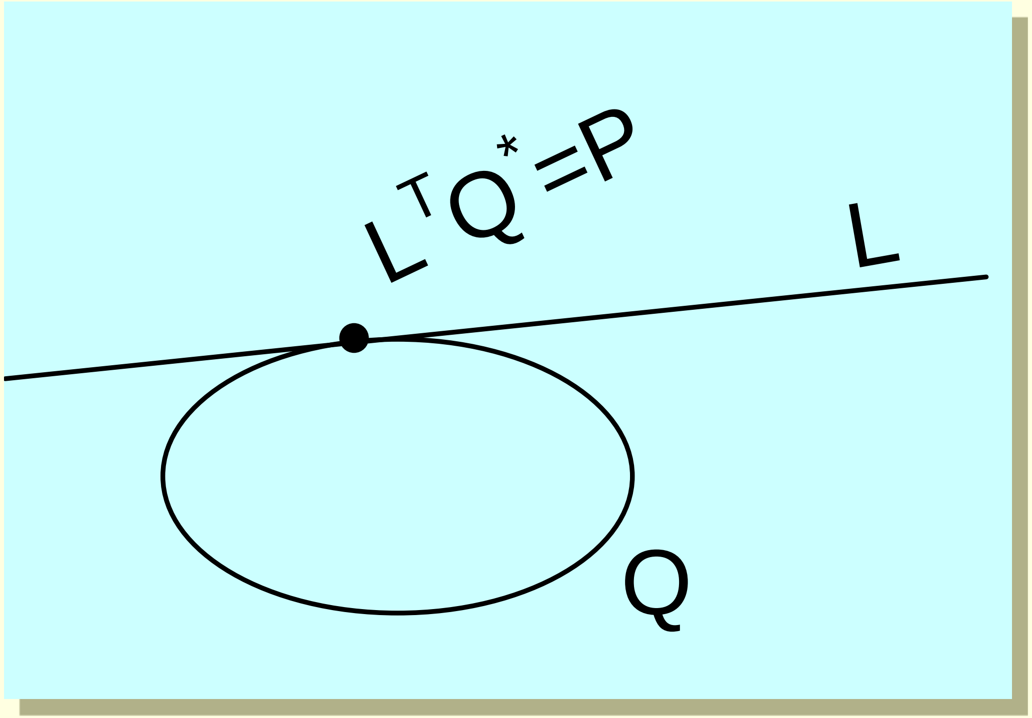


Given Tangent, Find Point

$$0 = \mathbf{L}^T \mathbf{Q}^* \mathbf{L}$$

$$= \left(\mathbf{L}^T \mathbf{Q}^*\right) \mathbf{L}$$

$$= \mathbf{P} \cdot \mathbf{L}$$



Three Kinds of Matrix

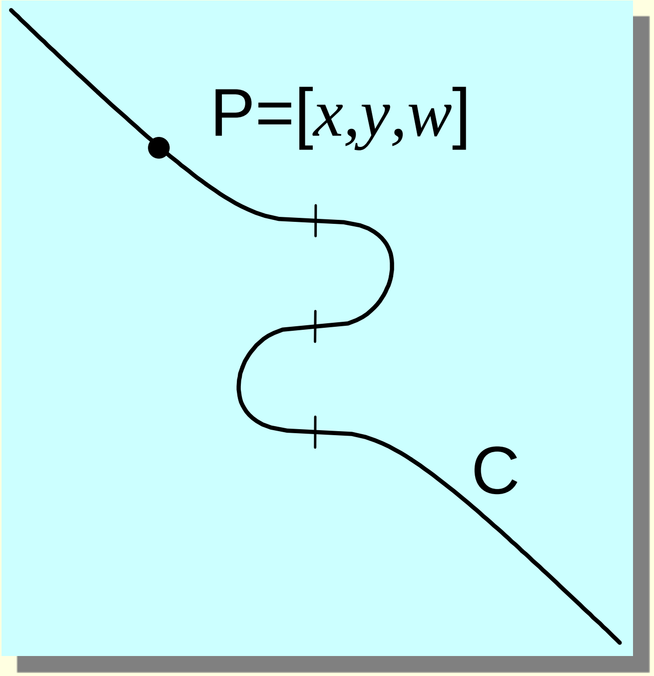
$$[\text{point}] \cdot \mathbf{T} = [\text{point}]$$

$$[\text{point}] \cdot \mathbf{Q} = [\text{line}]^T$$

$$[\text{line}]^T \cdot \mathbf{Q}^* = [\text{point}]$$

Point on Cubic Curve

$$Ax^3 + 3Bx^2y + 3Cxy^2 + Dy^3$$
$$+3Ex^2w + 6Fxyw + 3Gy^2w$$
$$+3Hxw^2 + 3Jyw^2$$
$$+Kw^3 = 0$$



Forms of Cubic Curve Equation

$$Ax^3 + 3Bx^2y + 3Cxy^2 + Dy^3$$
$$+3Ex^2w + 6Fxyw + 3Gy^2w$$
$$+3Hxw^2 + 3Jyw^2$$
$$+Kw^3 = 0$$

$$\left\{ \begin{bmatrix} x & y & w \end{bmatrix} \left[\begin{bmatrix} A & B & E \\ B & C & F \\ E & F & H \end{bmatrix} \begin{bmatrix} B & C & F \\ C & D & G \\ F & G & J \end{bmatrix} \begin{bmatrix} E & F & H \\ F & G & J \\ H & J & K \end{bmatrix} \right] \begin{bmatrix} x \\ y \\ w \end{bmatrix} \right\} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = 0$$

$$\left\{ \mathbf{PCP}^T \right\} \mathbf{P}^T = 0$$

Forms of Cubic Curve Equation

$$\begin{aligned} Ax^3 + 3Bx^2y + 3Cxy^2 + Dy^3 \\ +3Ex^2w + 6Fxyw + 3Gy^2w \\ +3Hxw^2 + 3Jyw^2 \\ +Kw^3 = 0 \end{aligned}$$

$$\left\{ \begin{bmatrix} x & y & w \end{bmatrix} \begin{bmatrix} \begin{bmatrix} A & B & E \\ B & C & F \\ E & F & H \end{bmatrix} & \begin{bmatrix} B & C & F \\ C & D & G \\ F & G & J \end{bmatrix} & \begin{bmatrix} E & F & H \\ F & G & J \\ H & J & K \end{bmatrix} \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} \right\} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = 0$$

$$\sum_{i,j,k} P_i P_j P_k C_{i,j,k} = 0$$

Two Problems With Notation

Row vs. Column Confusion

$$\left[\text{point}\right]\cdot\mathbf{Q}=\left[\text{line}\right]^T$$

Handing More Than Two Indices

$$\mathbf{C}=\begin{bmatrix}\begin{bmatrix}A & B & E\\ B & C & F\\ E & F & H\end{bmatrix} & \begin{bmatrix}B & C & F\\ C & D & G\\ F & G & J\end{bmatrix} & \begin{bmatrix}E & F & H\\ F & G & J\\ H & J & K\end{bmatrix}\end{bmatrix}$$

The Solution

- Steal Notational Tricks from Physics
  - General Relativity
  - Quantum Mechanics
- Tuned to Algberaic Geometry

Old Index Types

$$\mathbf{P} = \begin{bmatrix} P_1 & P_2 & P_3 \end{bmatrix}$$

Row

$$\mathbf{L} = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix}$$

Column

New Index Types

ContraVariant

$$\mathbf{P} = \begin{bmatrix} P^1 & P^2 & P^3 \end{bmatrix}$$

$$\mathbf{L} = \begin{bmatrix} L_1 & L_2 & L_3 \end{bmatrix}$$

CoVariant

The Multiplication Machine

$$\mathbf{P} \cdot \mathbf{L} = \begin{bmatrix} P_1 & P_2 & P_3 \end{bmatrix} \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix}$$

$$= P^1 L_1 + P^2 L_2 + P^3 L_3$$

$$= \sum_i P^i L_i$$

$$= P^\alpha L_\alpha$$

Einstein Index Notation

Three Kinds of Matrix

T^i_j — Mixed

Q_{ij} — Pure covariant

$\left(Q^*\right)^{ij}$ — Pure contravariant

Three Kinds of Matrix

$$P^j T_j^i = \tilde{P}^i$$

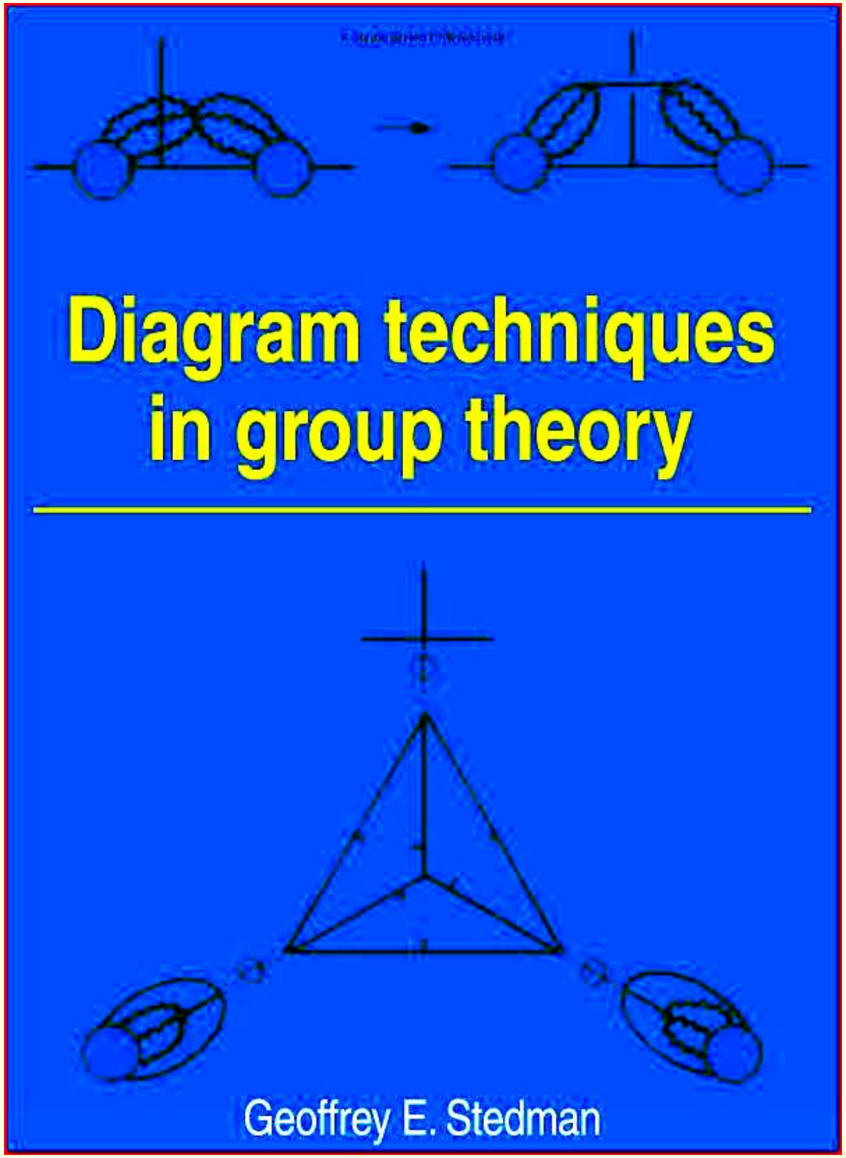
$$P^i Q_{ij} = L_j$$

$$L_i \left(Q^*\right)^{ij} = P^j$$

General Tensor Contraction

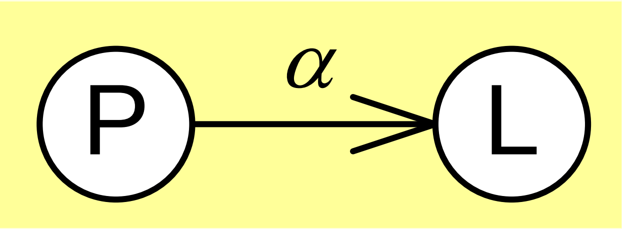
$$F_{ij}^{k} H_{km}^{lu} R^{j} S_{u} = W_{im}^{l}$$

Taking More Ideas from Physics



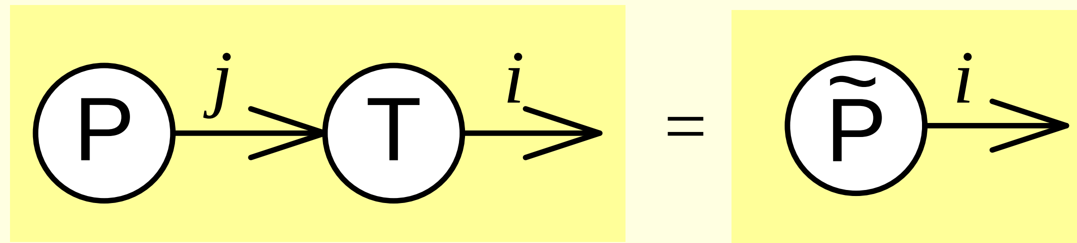
Writing Tensor Contraction in Diagram Form

$$P^{\alpha} L_{\alpha}$$

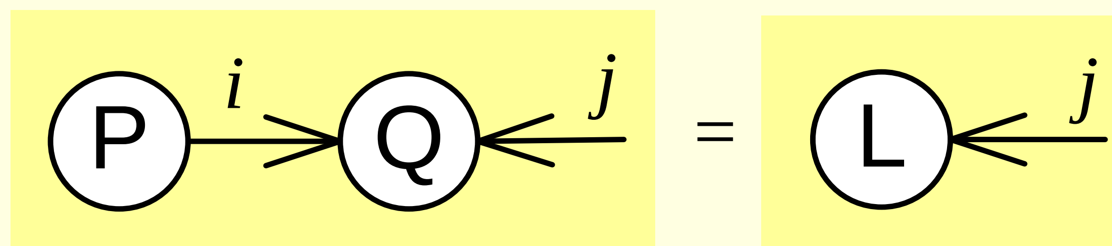


Three Kinds of Matrix

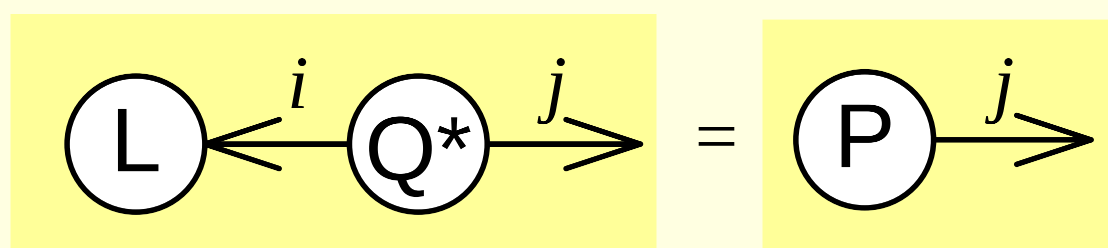
$$P^j T_j^i = \hat{P}^i$$



$$P^i Q_{ij} = L_j$$

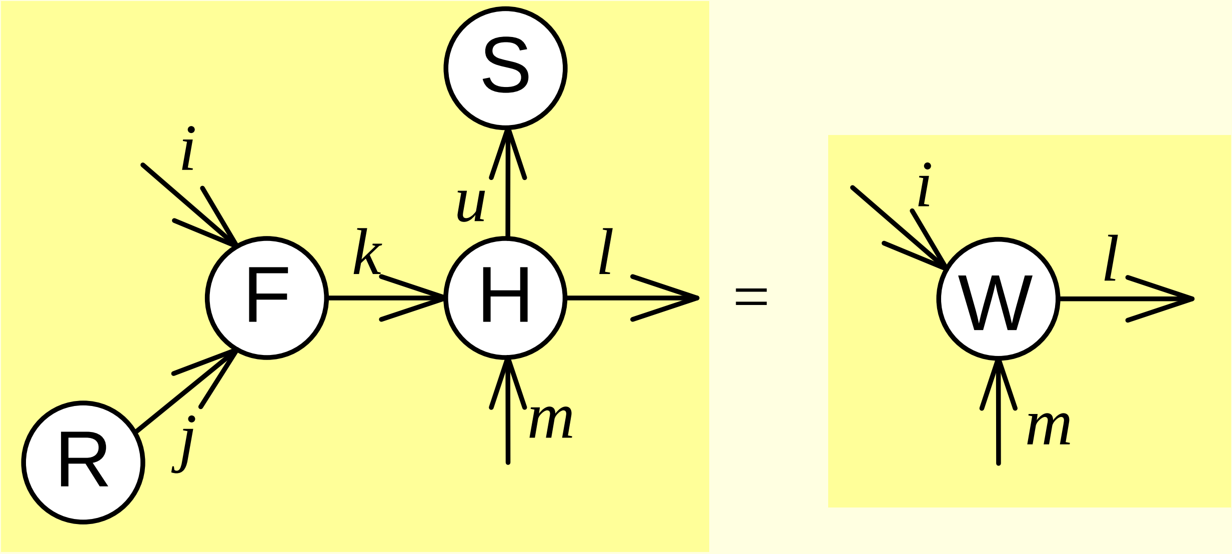


$$L_i \left(Q^*\right)^{ij} = P^j$$



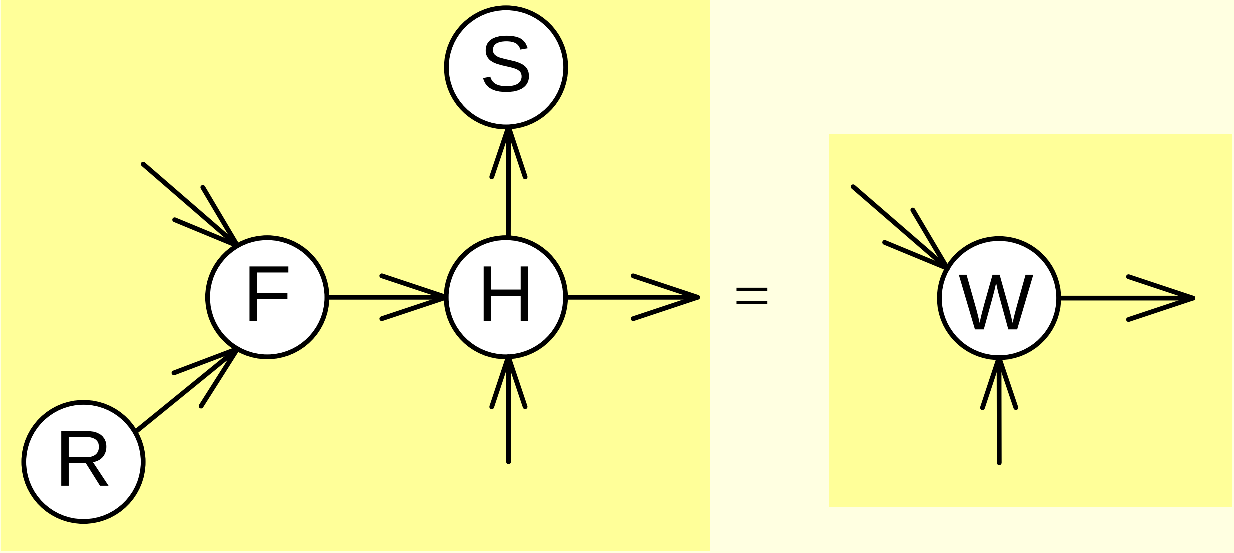
General Tensor Contraction

$$F_{ij}^{k} H_{km}^{lu} R^{j} S_{u} = W_{im}^{l}$$



Don’t need index labels

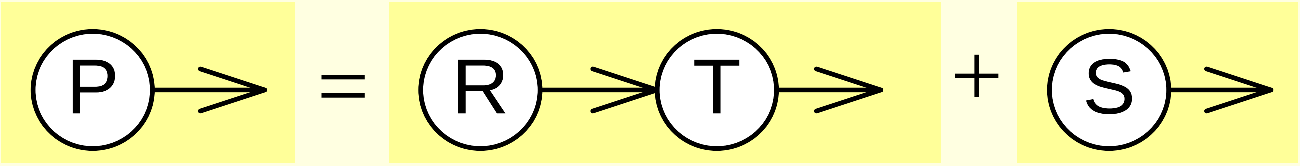
Just be careful about matching dangling arcs



Sum of Terms

$$\mathbf{P} = \mathbf{R}\mathbf{T} + \mathbf{S}$$

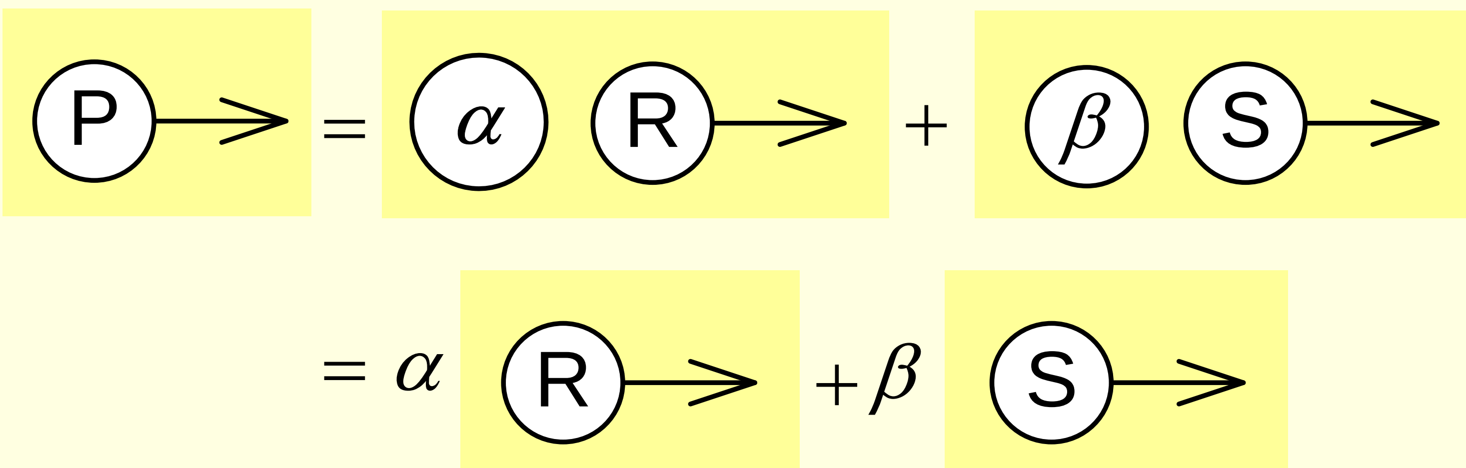
$$P^i = R^j T^i_j + S^i$$



Consistent type evident

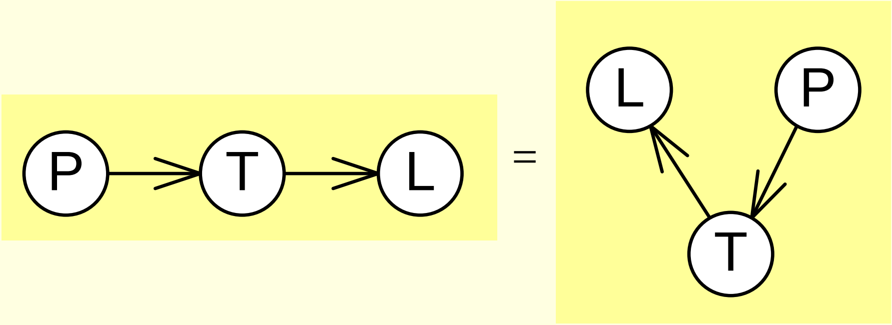
Scalar Product

$$\mathbf{P} = \alpha\mathbf{R} + \beta\mathbf{S}$$



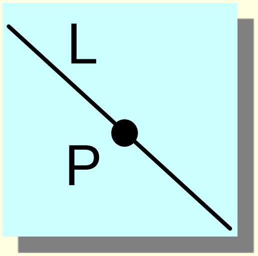
Only Connectivity Matters

Rearranging internal arcs/nodes doesn't change value



Now Back To Geometry

Point on a Line

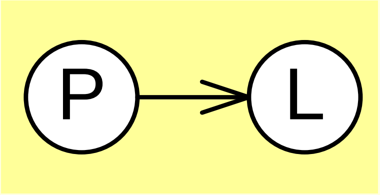


$$ax + by + cw = 0$$

$$\begin{bmatrix} x & y & w \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

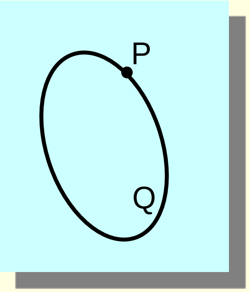
$$\mathbf{P} \cdot \mathbf{L} = 0$$

$$P^i L_i = 0$$



$= 0$

Point on a Quadratic Curve

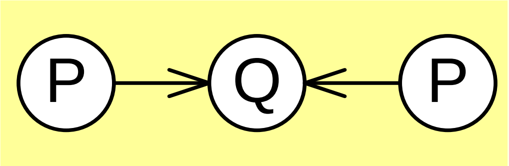


$$Ax^2 + 2Bxy + 2Cxw + Dy^2 + 2Eyw + Fw^2 = 0$$

$$\begin{bmatrix} x & y & w \end{bmatrix} \begin{bmatrix} A & B & C \\ B & D & E \\ C & E & F \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = 0$$

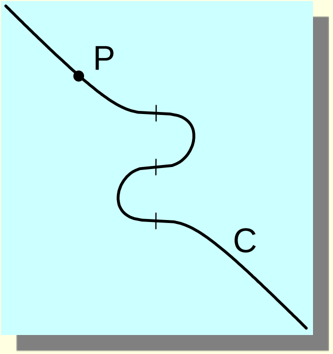
$$\mathbf{P} \cdot \mathbf{Q} \cdot \mathbf{P}^T = 0$$

$$P^i Q_{ij} P^j = 0$$



= 0

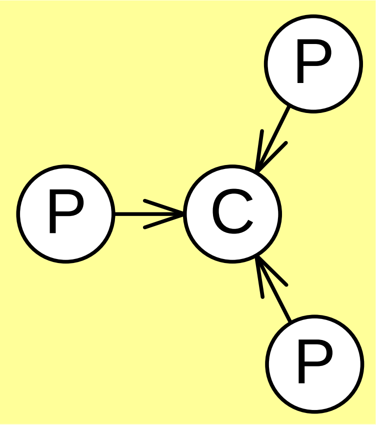
Point on a Cubic Curve



$$Ax^3 + 3Bx^2y + 3Cxy^2 + Dy^3$$
$$+3Ex^2w + 6Fxyw + 3Gyw^2$$
$$+2Hxw^2 + 3Jyw^2$$
$$+Kw^3 = 0$$

$$\left\{ \begin{bmatrix} x & y & w \end{bmatrix} \left[\begin{bmatrix} A & B & E \\ B & C & F \\ E & F & H \end{bmatrix} \begin{bmatrix} B & C & F \\ C & D & G \\ F & G & J \end{bmatrix} \begin{bmatrix} E & F & H \\ F & G & J \\ H & J & K \end{bmatrix} \right] \begin{bmatrix} x \\ y \\ w \end{bmatrix} \right\} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = 0$$

$$P^i P^j P^k C_{ijk} = 0$$

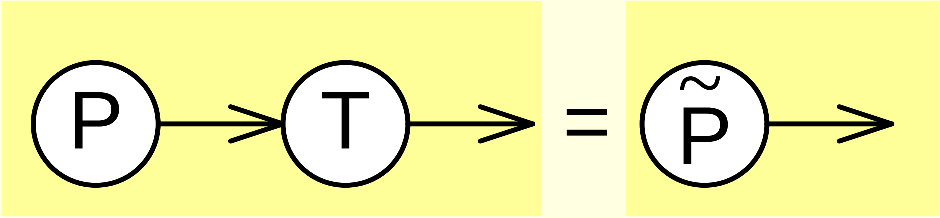


$= 0$

Transforming a Point

$$\mathbf{P}\,\mathbf{T} = \tilde{\mathbf{P}}$$

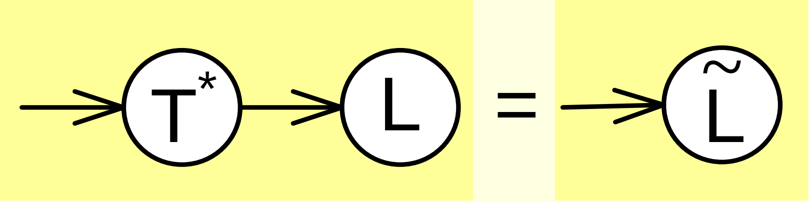
$$P^i T_i^{\,j} = \tilde{P}^j$$



Transforming a Line

$$\left(\mathbf{T}^{*}\right)\mathbf{L} = \tilde{\mathbf{L}}$$

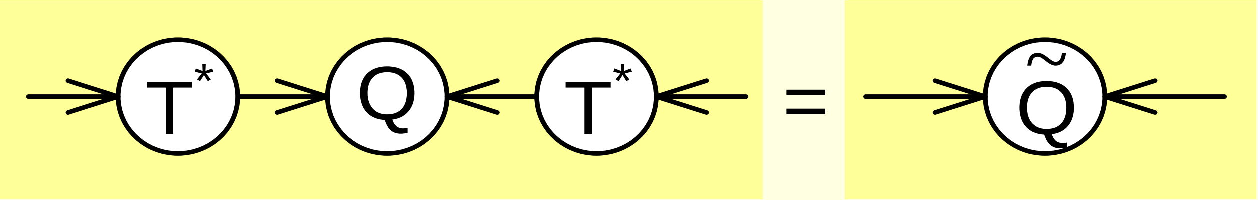
$$\left(T^{*}\right)_{j}^{i} L_{i} = \tilde{L}_{j}$$



Transforming A Quadratic Curve

$$\left(\mathbf{T}^*\right)\mathbf{Q}\left(\mathbf{T}^*\right)^T = \tilde{\mathbf{Q}}$$

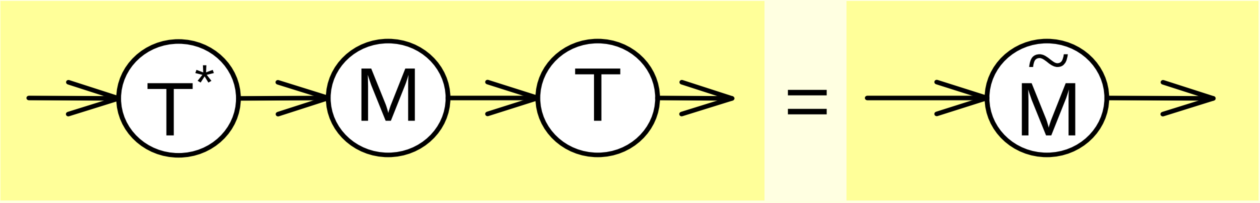
$$\left(T^*\right)_k^i Q_{ij} \left(T^*\right)_l^j = \tilde{Q}_{kl}$$



Transforming A Transformation

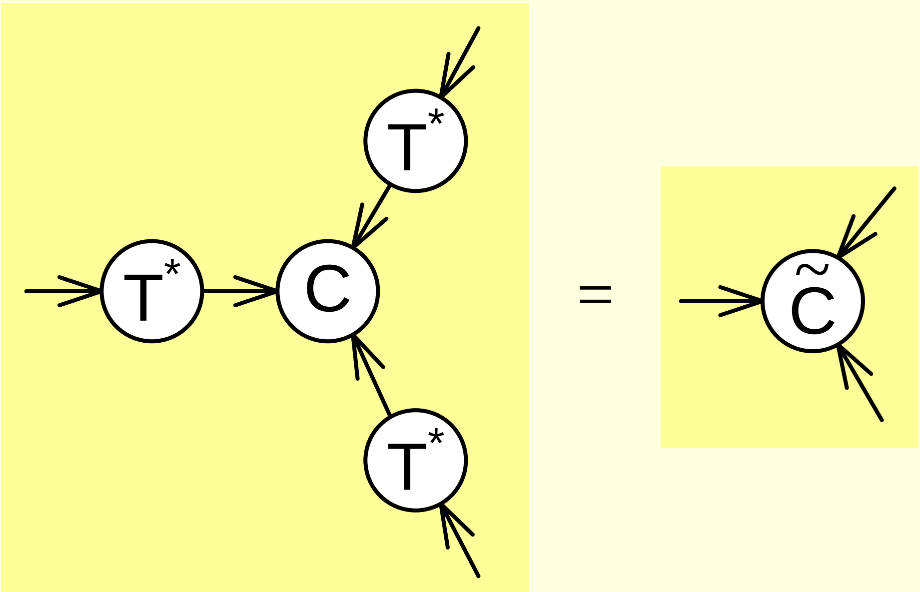
$$\mathbf{T}^{*}\mathbf{M}\mathbf{T} = \tilde{\mathbf{M}}$$

$$\left(T^{*}\right)_{k}^{i} M_{i}^{\,j} \left(T\right)_{j}^{l} = \tilde{M}_{k}^{l}$$

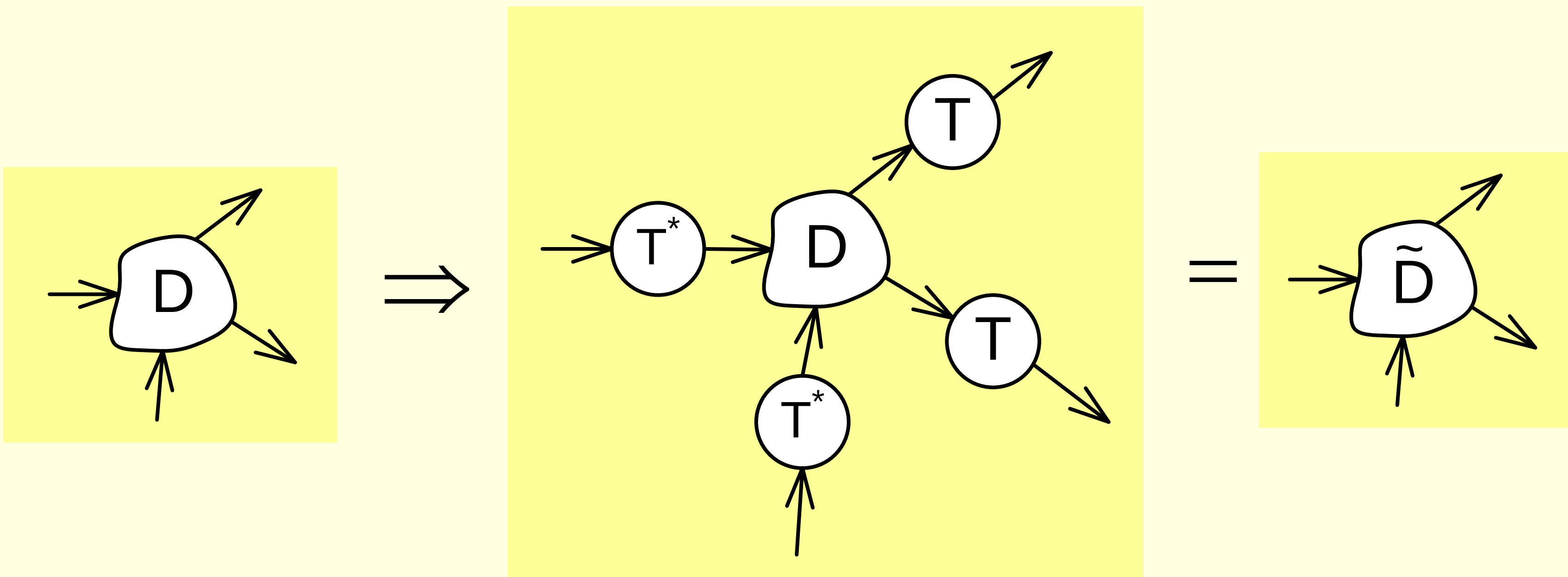


Transforming a Cubic Curve

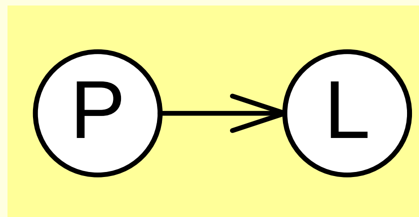
$$\left(T^*\right)_l^i \left(T^*\right)_m^j \left(T^*\right)_n^k C_{ijk} = \tilde{C}_{lmn}$$



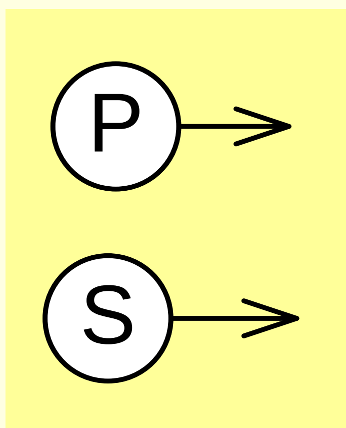
General Transformation Rule



Dot and Cross Product

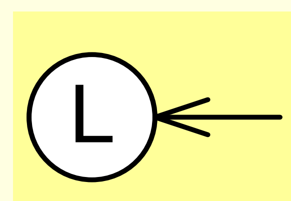


$= s$



?

becomes



Levi-Civita Epsilon

$$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = +1$$

$$\varepsilon_{321} = \varepsilon_{132} = \varepsilon_{213} = -1$$

$$\varepsilon_{ijk} = 0 \quad \text{otherwise}$$

$$\varepsilon = \begin{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{bmatrix}$$

Cross Product

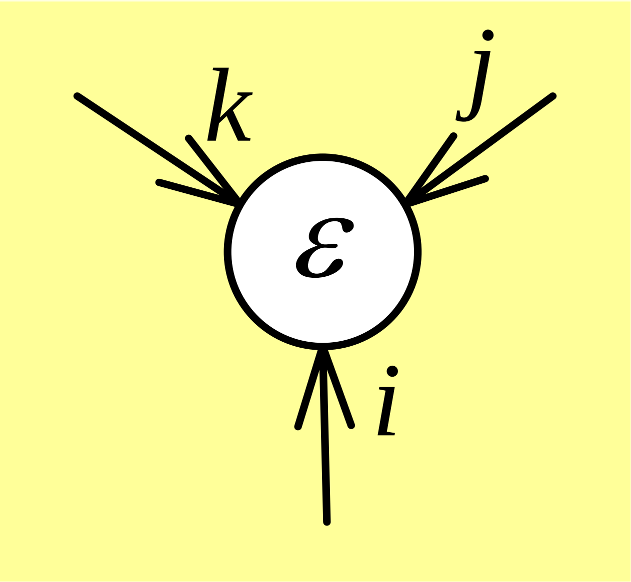
$$\begin{bmatrix} x_P & y_P & w_P \end{bmatrix} \left[\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right] \begin{bmatrix} x_S \\ y_S \\ w_S \end{bmatrix} =$$

$$\begin{bmatrix} y_P w_S - w_P y_S & w_P x_S - x_P w_S & y_P x_S - x_P y_S \end{bmatrix}$$

$$P^i S^j \varepsilon_{ijk} = L_k$$

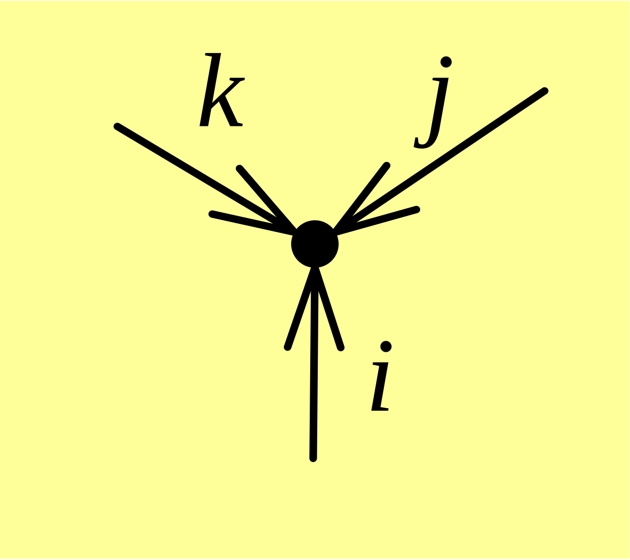
Levi-Civita Epsilon Diagram

$$\varepsilon_{ijk}$$



Levi-Civita Epsilon Diagram

$$\varepsilon_{ijk}$$

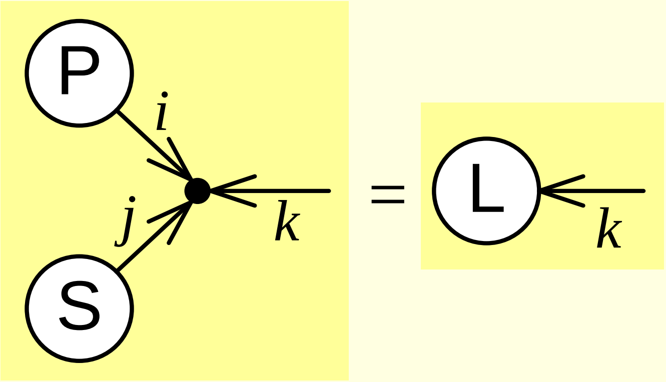


Cross Product

$$\begin{bmatrix} P^1 & P^2 & P^3 \end{bmatrix} \times \begin{bmatrix} S^1 & S^2 & S^3 \end{bmatrix} = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix}$$

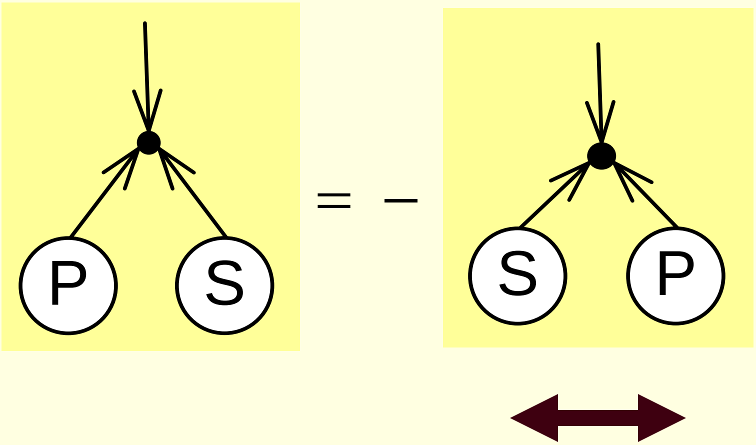
$$\mathbf{P} \times \mathbf{S} = \mathbf{L}$$

$$P^i S^j \varepsilon_{ijk} = L_k$$



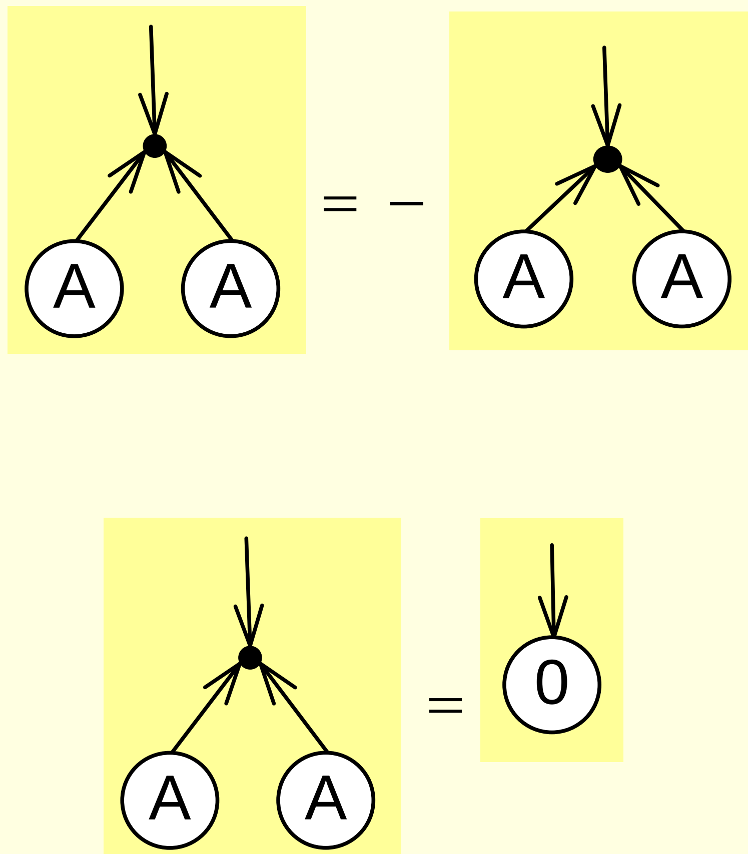
Anti-Symmetry and Epsilon

$$\mathbf{P} \times \mathbf{S} = -(\mathbf{S} \times \mathbf{P})$$



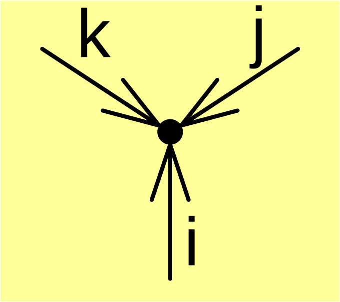
Mirror Reflections flip sign

AxA=0

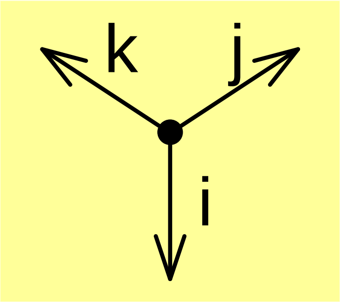


Two Types of Epsilon

COvariant ε_{ijk}



CONTRAvariant ε^{ijk}

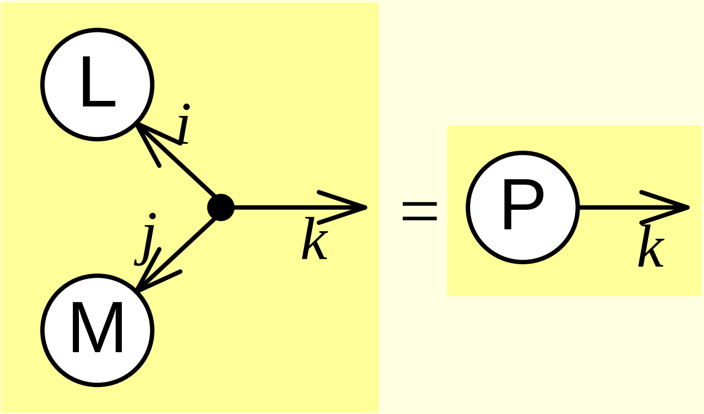


The Other Cross Product

$$\begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} \times \begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix} = \begin{bmatrix} P^1 & P^2 & P^3 \end{bmatrix}$$

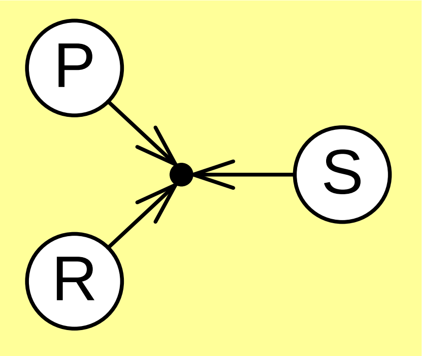
$$\mathbf{L} \times \mathbf{M} = \mathbf{P}$$

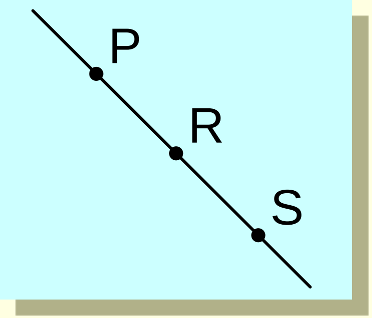
$$L_i M_j \varepsilon^{ijk} = P^k$$



Triple Product

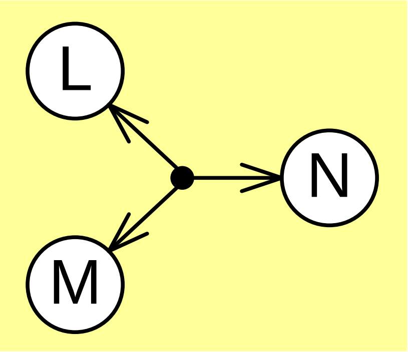
$$\mathbf{P}\times\mathbf{R}\cdot\mathbf{S}=\mathbf{R}\times\mathbf{S}\cdot\mathbf{P}=\mathbf{S}\times\mathbf{P}\cdot\mathbf{R}=$$

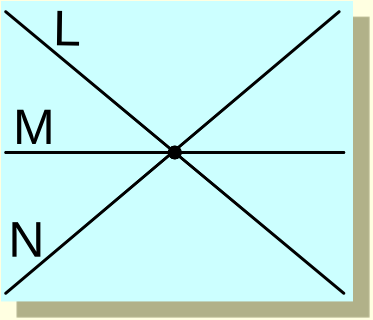




$$=\left[\mathbf{PRS}\right]$$

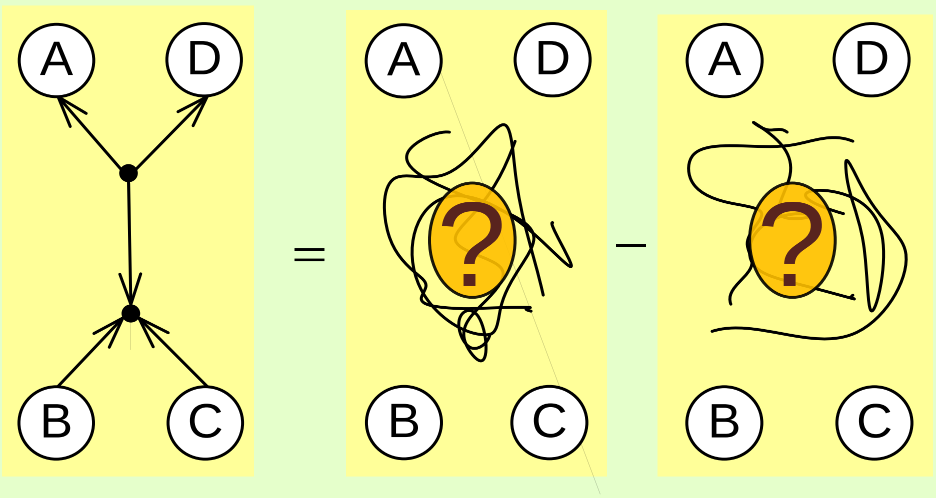
$$\mathbf{L}\times\mathbf{M}\cdot\mathbf{N}=\mathbf{M}\times\mathbf{N}\cdot\mathbf{L}=\mathbf{N}\times\mathbf{L}\cdot\mathbf{M}=$$



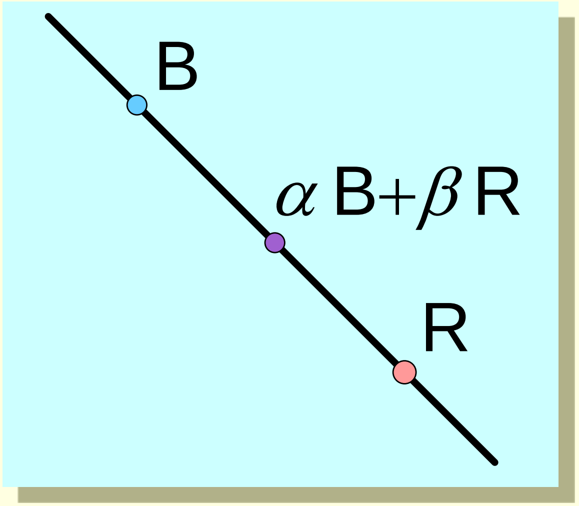


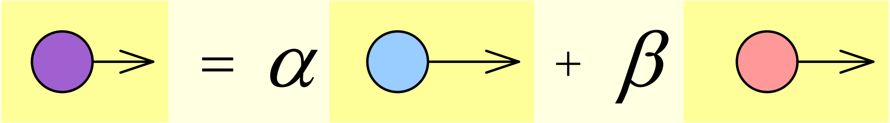
$$=\left[\mathbf{LMN}\right]$$

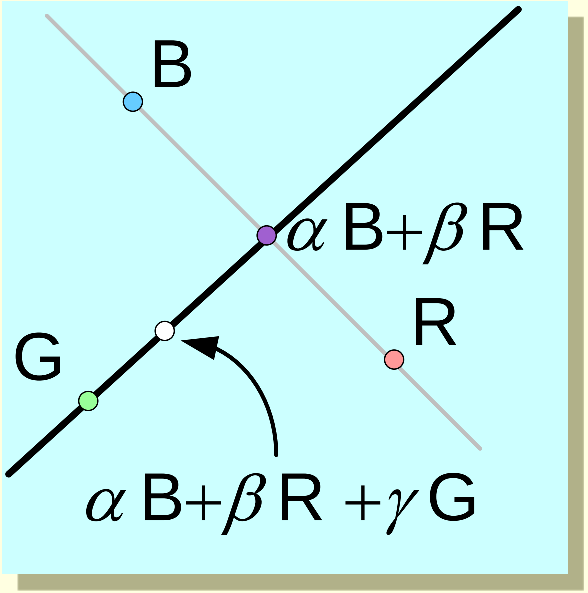
Generating Algebraic Relations Between Diagrams

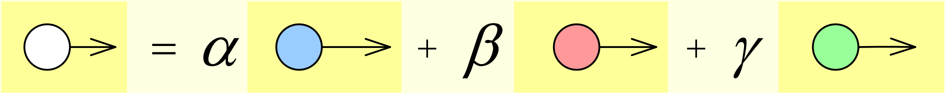


Linear Combinations of Points

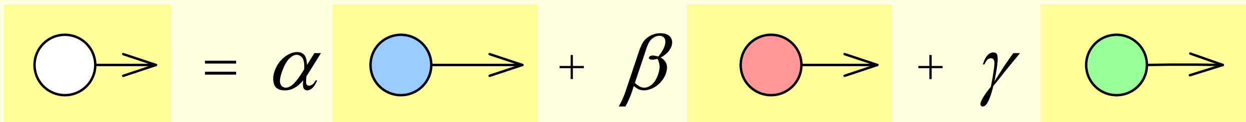








Linear Combinations of Points



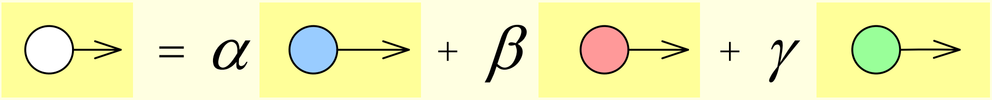
$$\begin{bmatrix} \cdots \mathbf{W} \cdots \end{bmatrix} = \begin{bmatrix} \alpha & \beta & \gamma \end{bmatrix} \begin{bmatrix} \cdots \mathbf{B} \cdots \\ \cdots \mathbf{R} \cdots \\ \cdots \mathbf{G} \cdots \end{bmatrix}$$

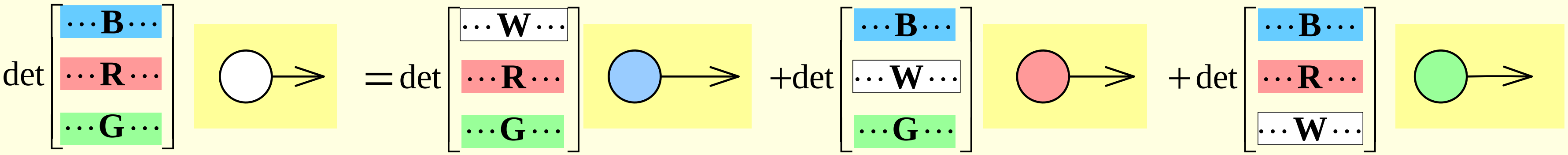
Cramer's Rule

$$\alpha = \frac{\det \begin{bmatrix} \cdots \mathbf{W} \cdots \\ \cdots \mathbf{R} \cdots \\ \cdots \mathbf{G} \cdots \end{bmatrix}}{\det \begin{bmatrix} \cdots \mathbf{B} \cdots \\ \cdots \mathbf{R} \cdots \\ \cdots \mathbf{G} \cdots \end{bmatrix}}, \quad \beta = \frac{\det \begin{bmatrix} \cdots \mathbf{B} \cdots \\ \cdots \mathbf{W} \cdots \\ \cdots \mathbf{G} \cdots \end{bmatrix}}{\det \begin{bmatrix} \cdots \mathbf{B} \cdots \\ \cdots \mathbf{R} \cdots \\ \cdots \mathbf{G} \cdots \end{bmatrix}}, \quad \gamma = \frac{\det \begin{bmatrix} \cdots \mathbf{B} \cdots \\ \cdots \mathbf{R} \cdots \\ \cdots \mathbf{W} \cdots \end{bmatrix}}{\det \begin{bmatrix} \cdots \mathbf{B} \cdots \\ \cdots \mathbf{R} \cdots \\ \cdots \mathbf{G} \cdots \end{bmatrix}},$$

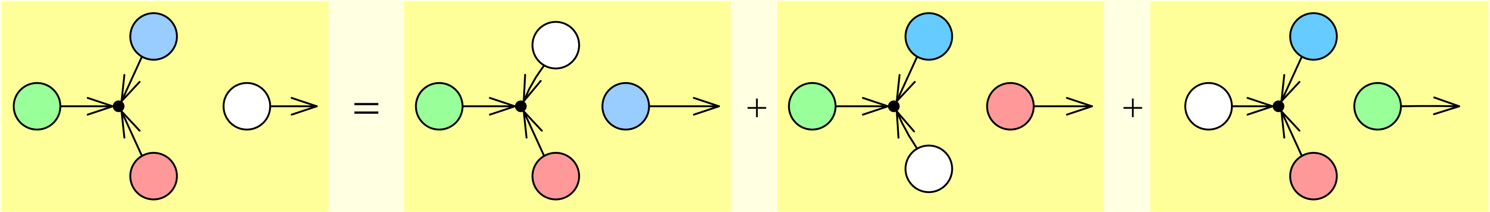
Basic Linear Relationship

$$\alpha = \frac{\det\begin{bmatrix} \cdots \mathbf{W} \cdots \\ \cdots \mathbf{R} \cdots \\ \cdots \mathbf{G} \cdots \end{bmatrix}}{\det\begin{bmatrix} \cdots \mathbf{B} \cdots \\ \cdots \mathbf{R} \cdots \\ \cdots \mathbf{G} \cdots \end{bmatrix}}, \qquad \beta = \frac{\det\begin{bmatrix} \cdots \mathbf{B} \cdots \\ \cdots \mathbf{W} \cdots \\ \cdots \mathbf{G} \cdots \end{bmatrix}}{\det\begin{bmatrix} \cdots \mathbf{B} \cdots \\ \cdots \mathbf{R} \cdots \\ \cdots \mathbf{G} \cdots \end{bmatrix}}, \qquad \gamma = \frac{\det\begin{bmatrix} \cdots \mathbf{B} \cdots \\ \cdots \mathbf{R} \cdots \\ \cdots \mathbf{W} \cdots \end{bmatrix}}{\det\begin{bmatrix} \cdots \mathbf{B} \cdots \\ \cdots \mathbf{R} \cdots \\ \cdots \mathbf{G} \cdots \end{bmatrix}},$$



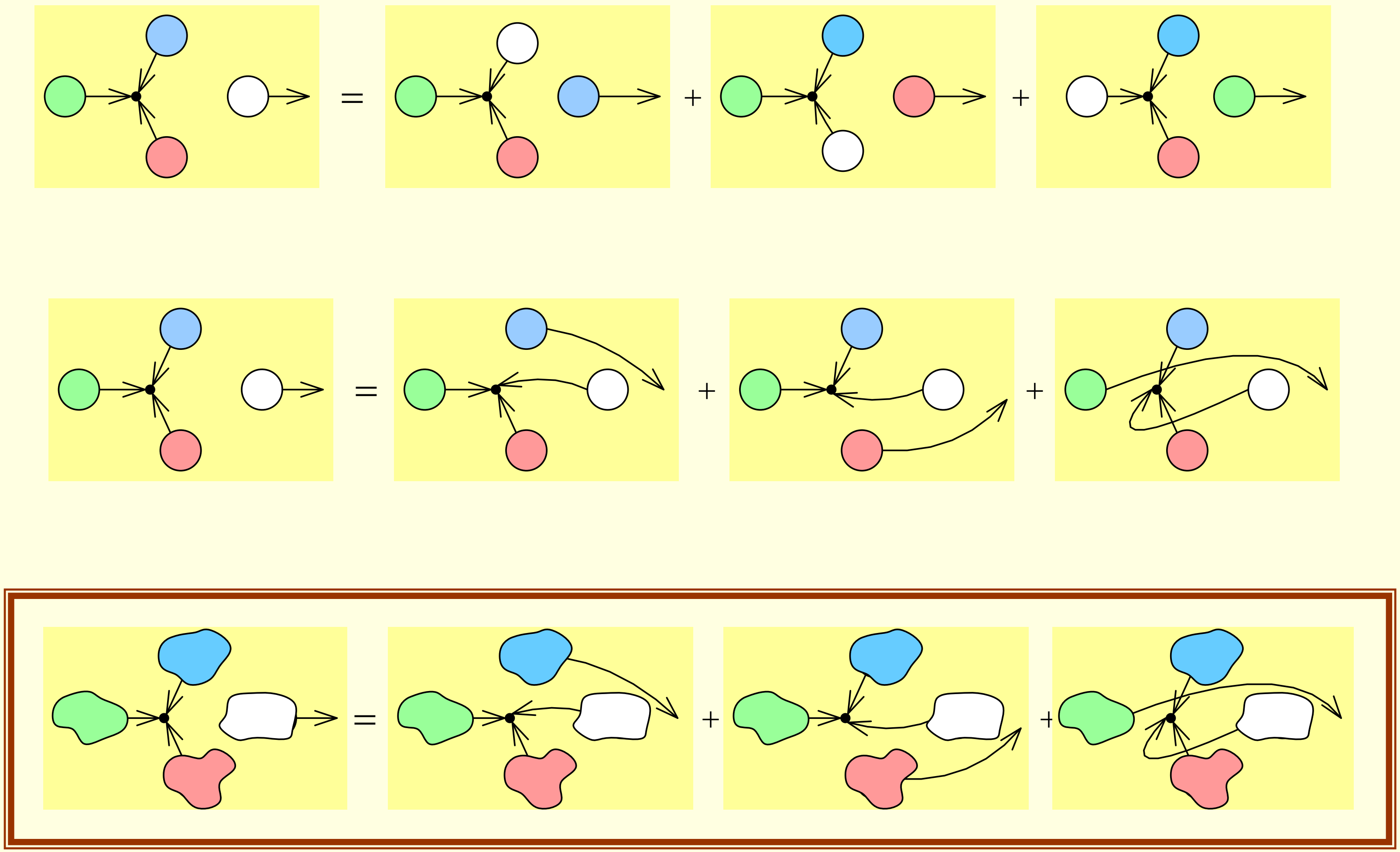


Grassman-Plucker relation



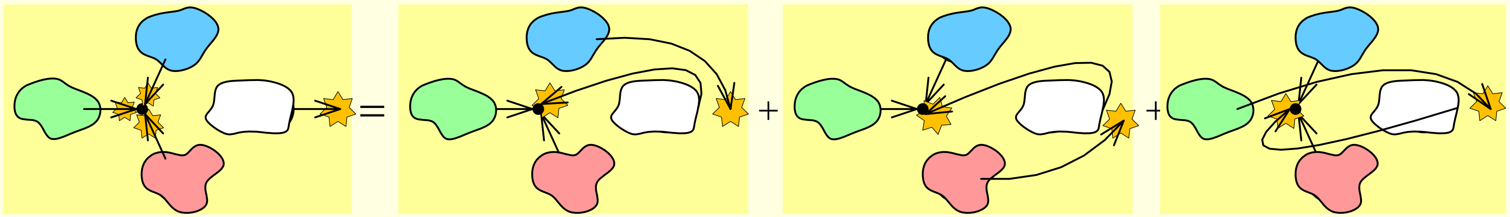
Note Symmetry

Arc Swapping Identity

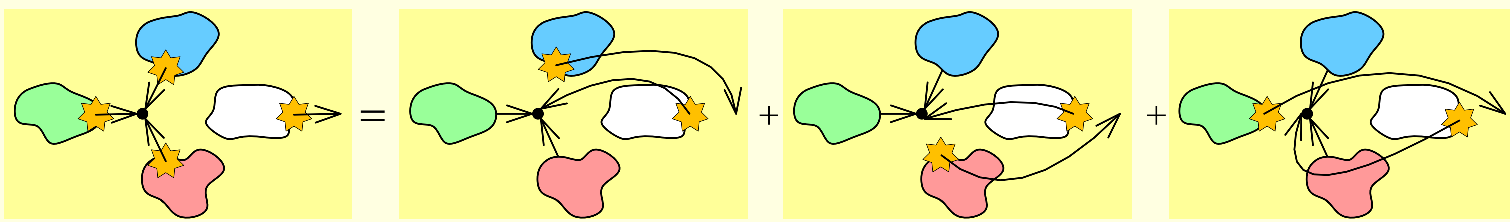


Arc Swapping Identity - Variations

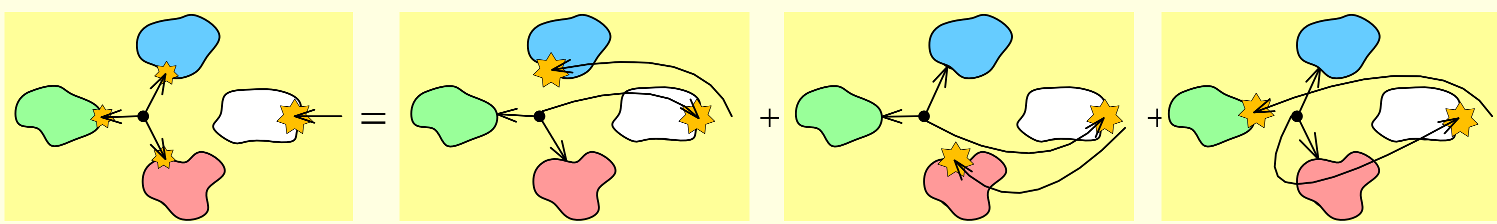
Swap heads



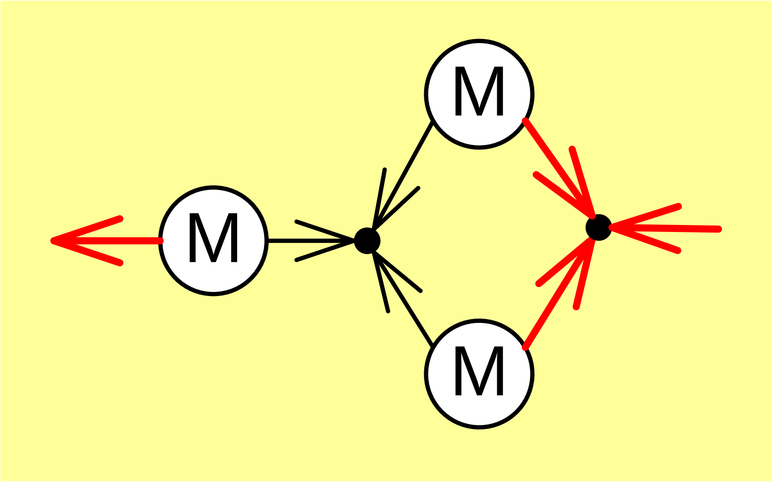
Swap tails



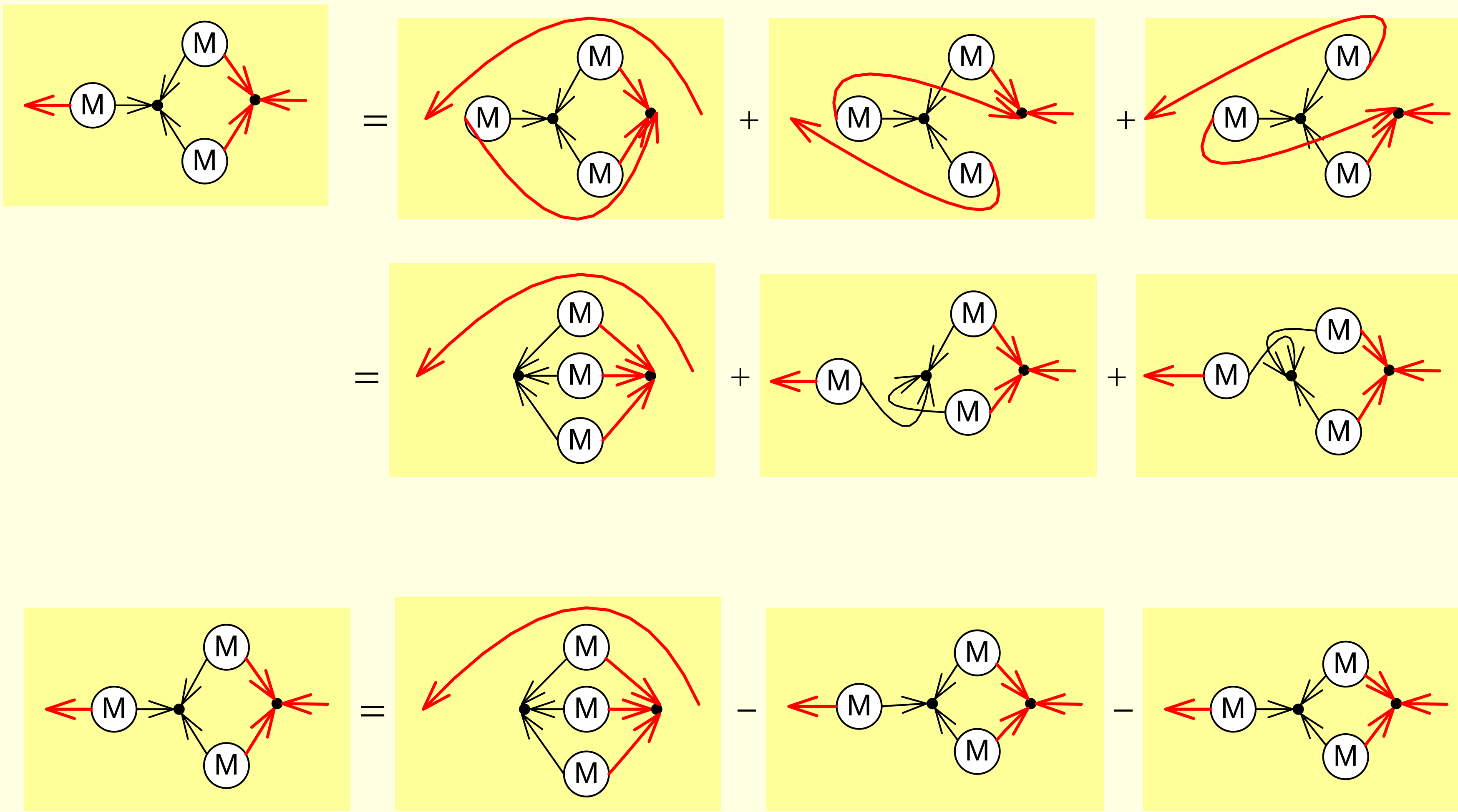
Swap heads (dual)



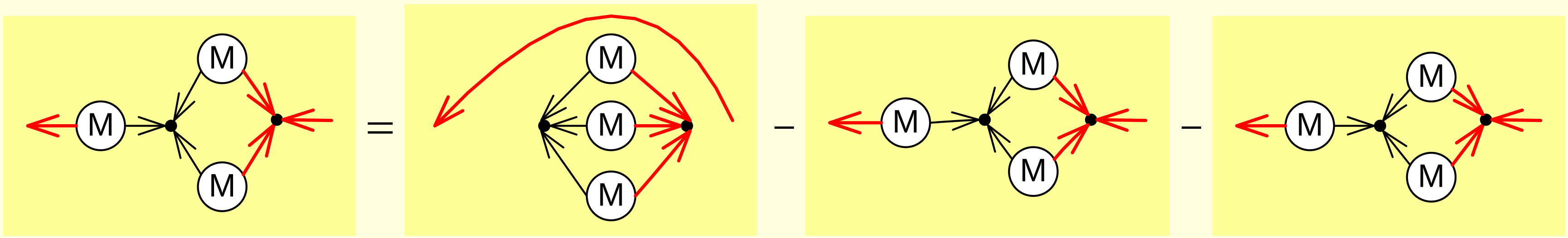
An Application of Arc Swapping

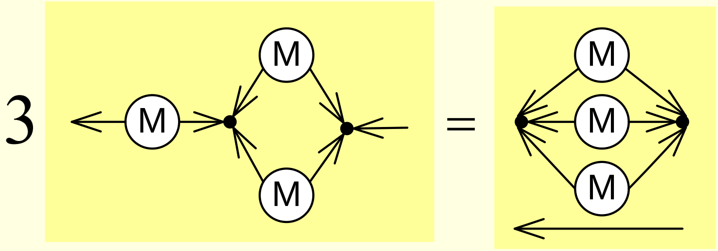


An Application of Arc Swapping



An Application of Arc Swapping

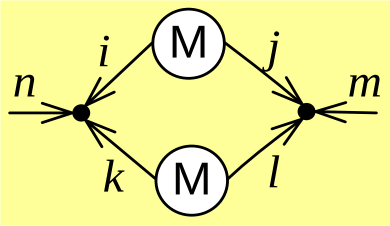




Compare with: $$\mathbf{M}\mathbf{M}^{*} = \left(\det \mathbf{M}\right)\mathbf{I}$$

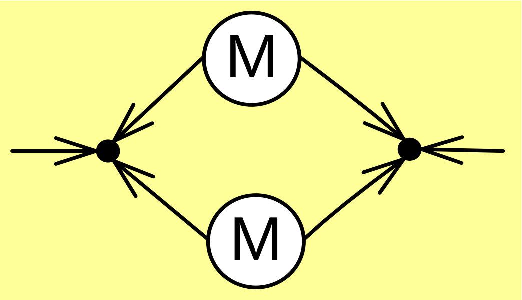
Relation of Diagram to Adjugate

$$D_{mn} = M^{ij} M^{kl} \varepsilon_{ikn} \varepsilon_{ljm} =$$

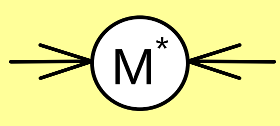


Example element:

$$D_{11} = \begin{pmatrix} M^{33} M^{22} \varepsilon_{321} \varepsilon_{231} \\ + M^{22} M^{33} \varepsilon_{231} \varepsilon_{321} \\ + M^{23} M^{32} \varepsilon_{231} \varepsilon_{231} \\ + M^{32} M^{23} \varepsilon_{321} \varepsilon_{321} \end{pmatrix} = \begin{pmatrix} -M^{33} M^{22} \\ -M^{22} M^{33} \\ +M^{23} M^{32} \\ +M^{32} M^{23} \end{pmatrix} = -2\left(M^{22} M^{33} - M^{32} M^{23} \right)$$



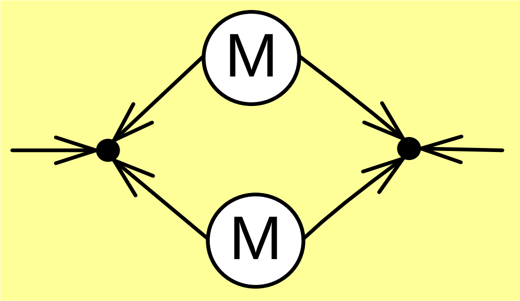
$= -2$



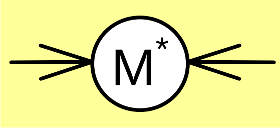
Constant factor

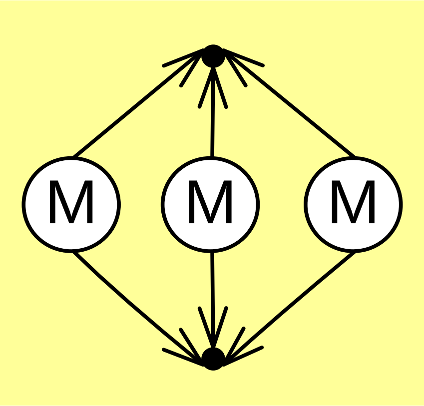
Actual adjugate

Adjugate and Determinant



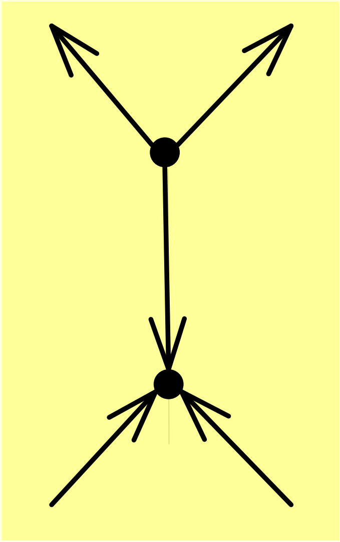
$= -2$



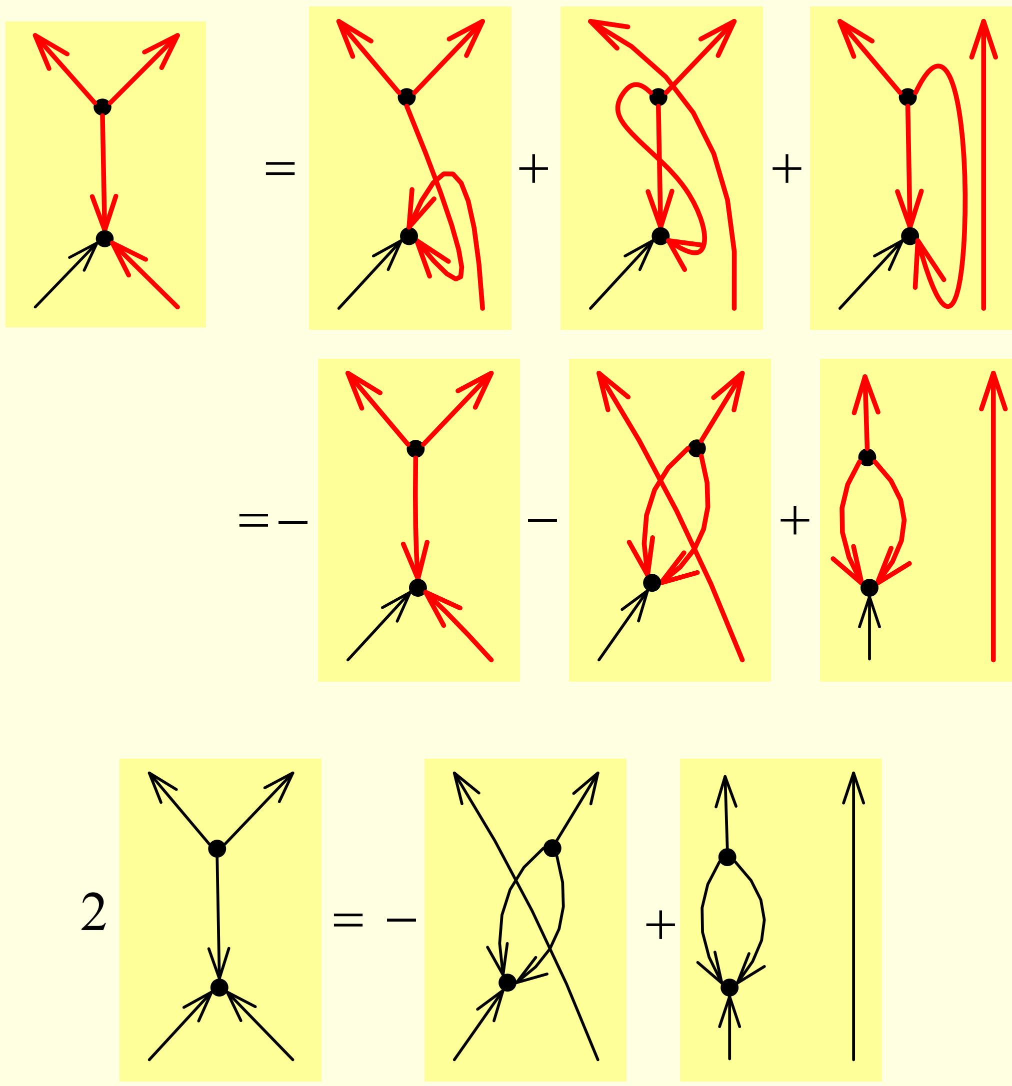


$= -6 \det \mathbf{M}$

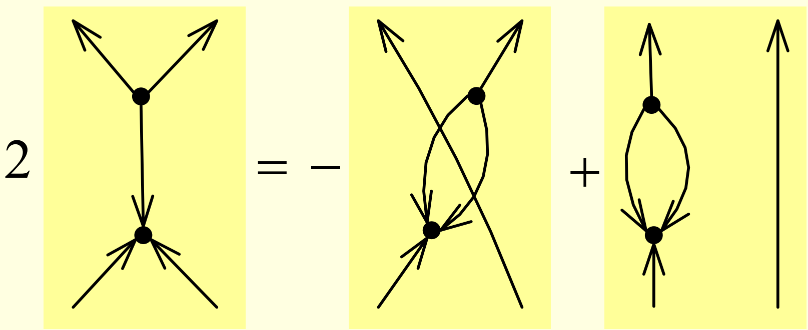
Another Arc Swap Application

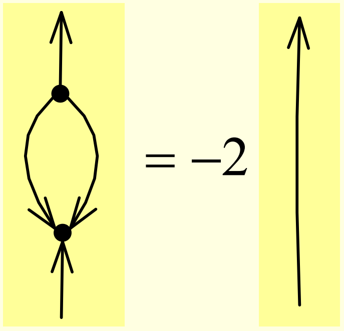


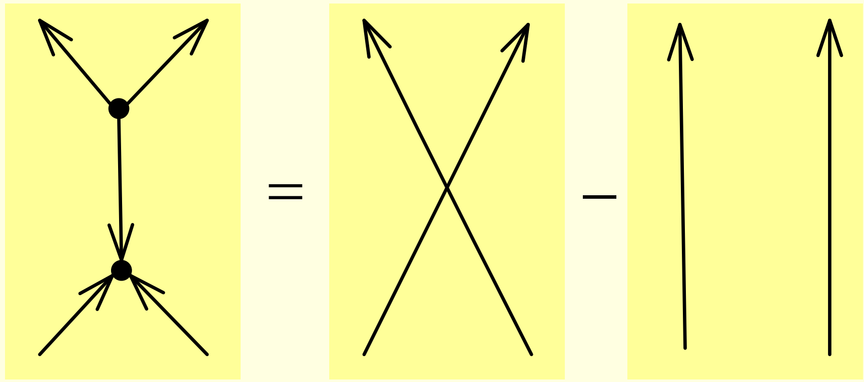
Another Arc Swap Application



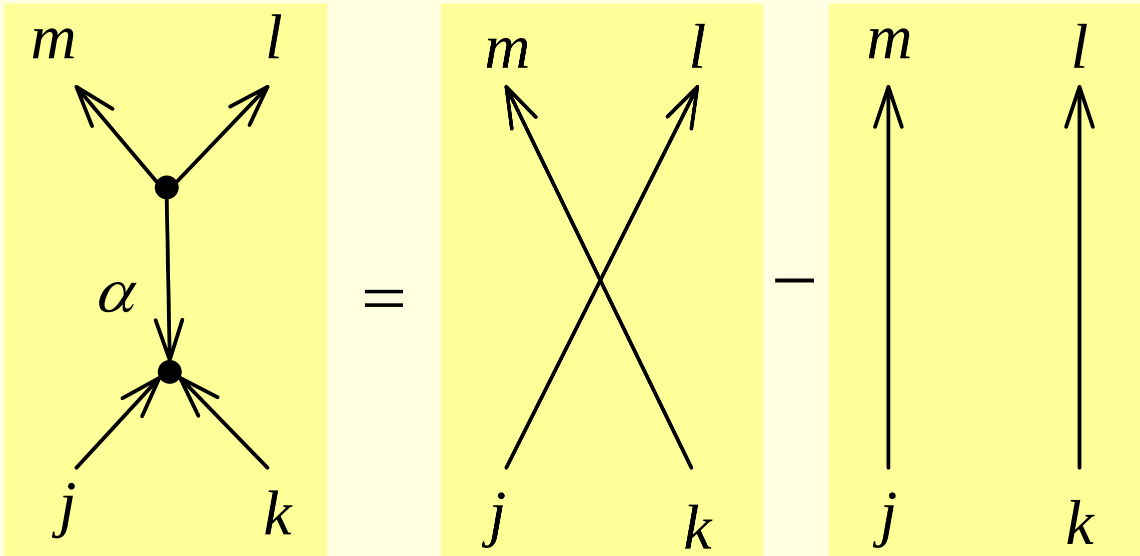
Another Arc Swap Application



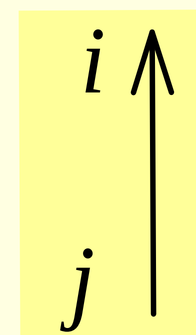




Epsilon-Delta Rule

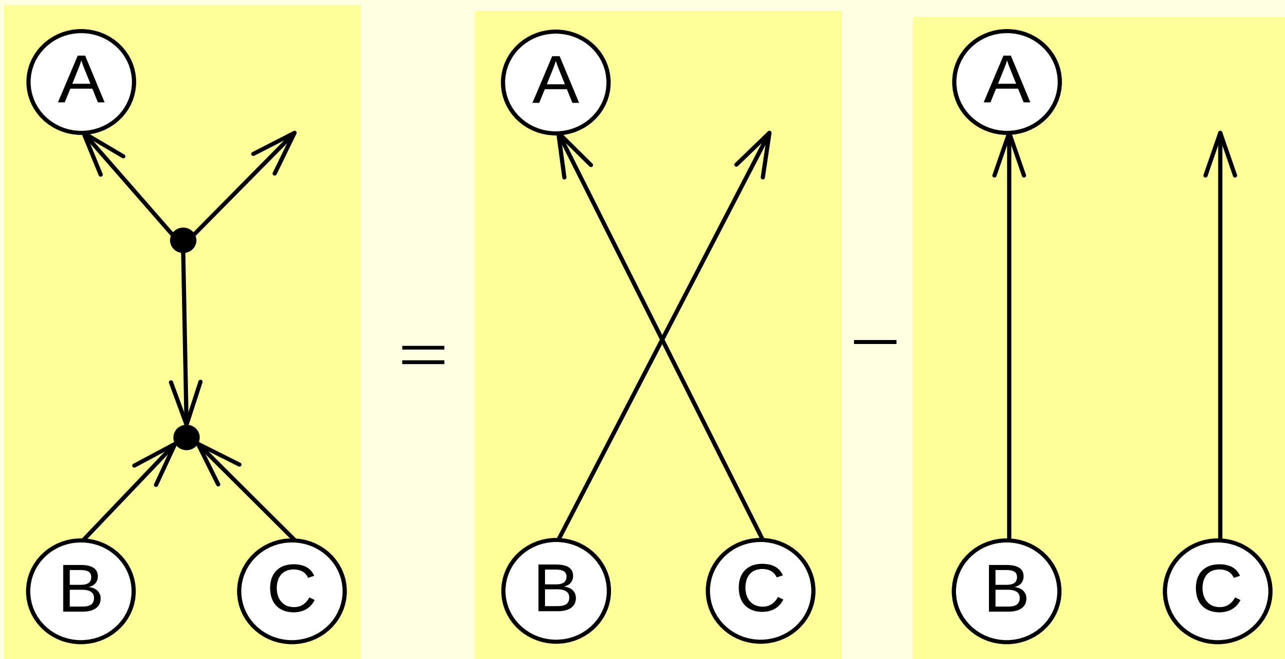


$$\delta^i_j =$$



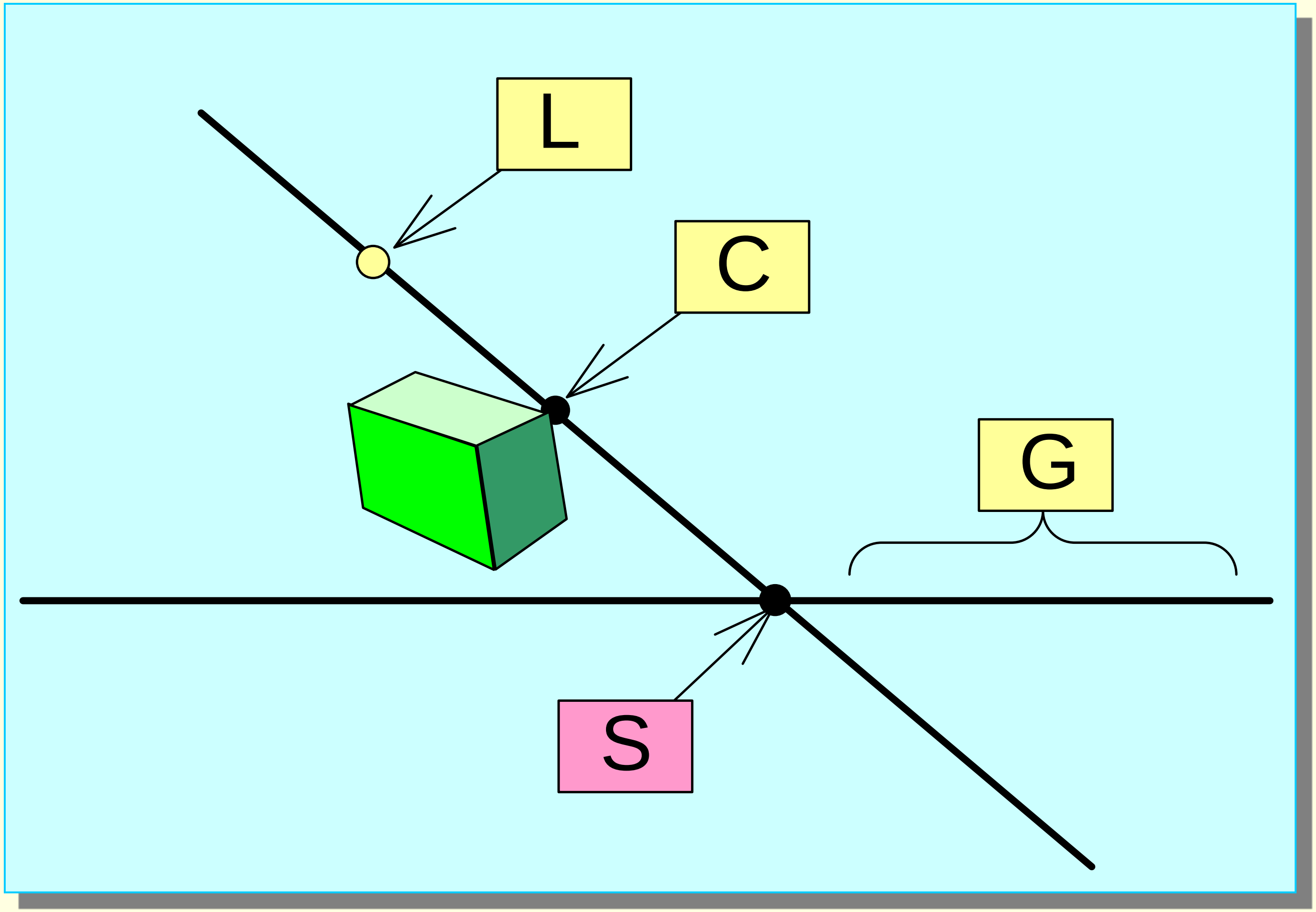
$$\varepsilon_{\alpha jk}\varepsilon^{\alpha lm} = \delta^l_j \delta^m_k - \delta^m_j \delta^l_k$$

Algebraic Interpretation

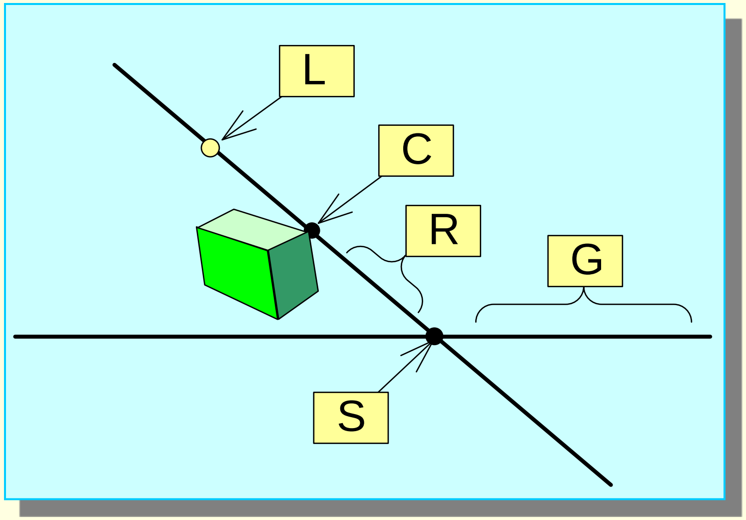


$$A \times (B \times C) = (A \cdot C)B - (A \cdot B)C$$

Projection from L thru C onto G

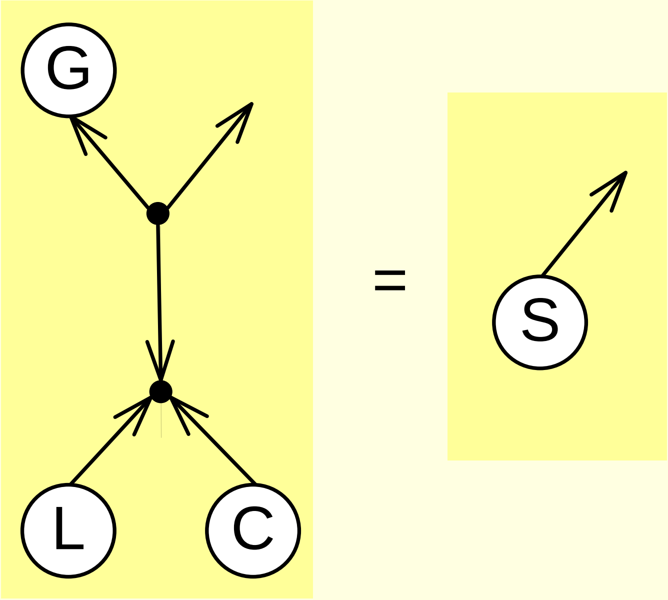


Projection from L thru C onto G



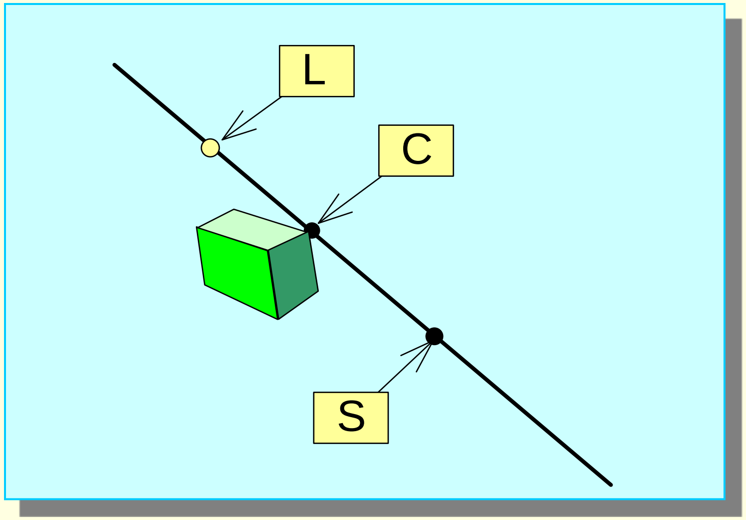
$$\mathbf{L} \times \mathbf{C} = \mathbf{R}$$

$$\mathbf{G} \times \mathbf{R} = \mathbf{S}$$



Projection from L thru C onto G

$$\mathbf{S} = \alpha \mathbf{L} + \beta \mathbf{C}$$



Projection from L thru C onto G

$$\mathbf{S} = \alpha\mathbf{L} + \beta\mathbf{C}$$

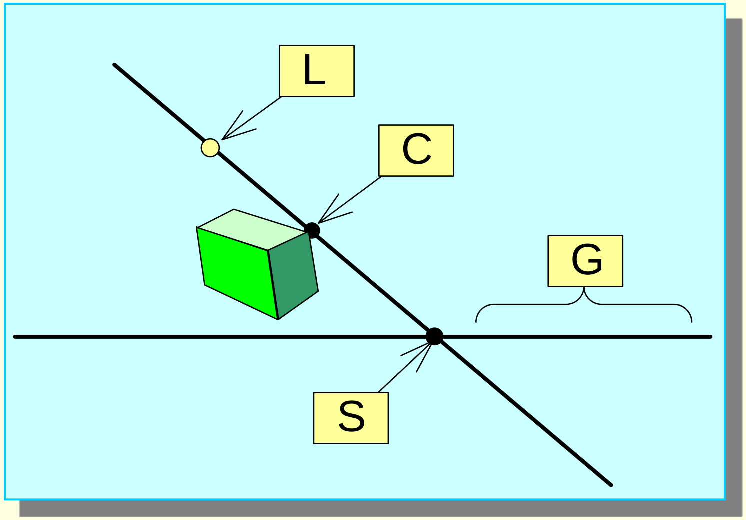
$$\mathbf{S} \cdot \mathbf{G} = 0$$

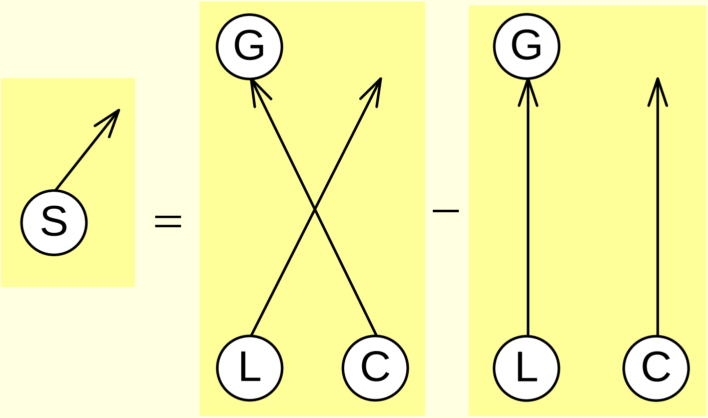
$$0 = \alpha(\mathbf{L} \cdot \mathbf{G}) + \beta(\mathbf{C} \cdot \mathbf{G})$$

$$\alpha = (\mathbf{C} \cdot \mathbf{G})$$

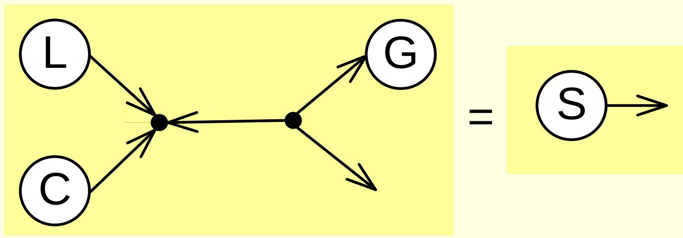
$$\beta = -(\mathbf{L} \cdot \mathbf{G})$$

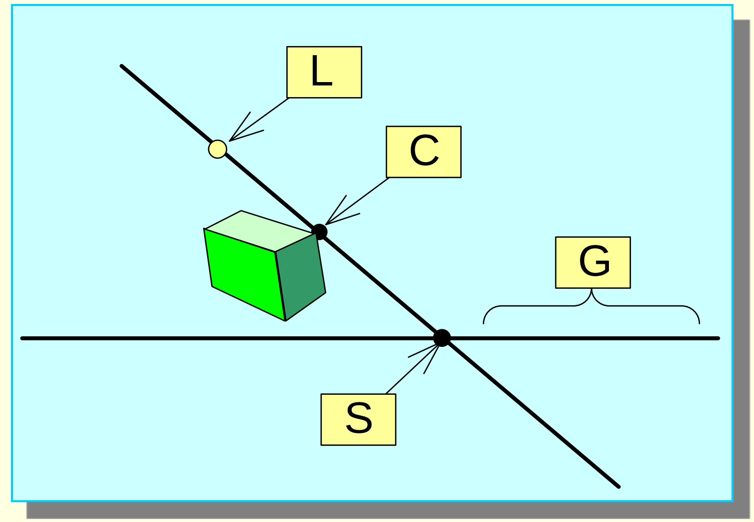
$$\mathbf{S} = (\mathbf{C} \cdot \mathbf{G})\mathbf{L} - (\mathbf{L} \cdot \mathbf{G})\mathbf{C}$$

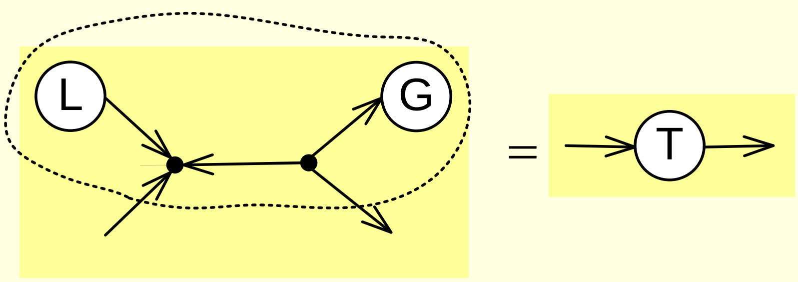


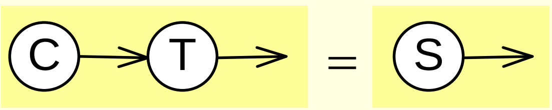


Shadow Projection Matrix



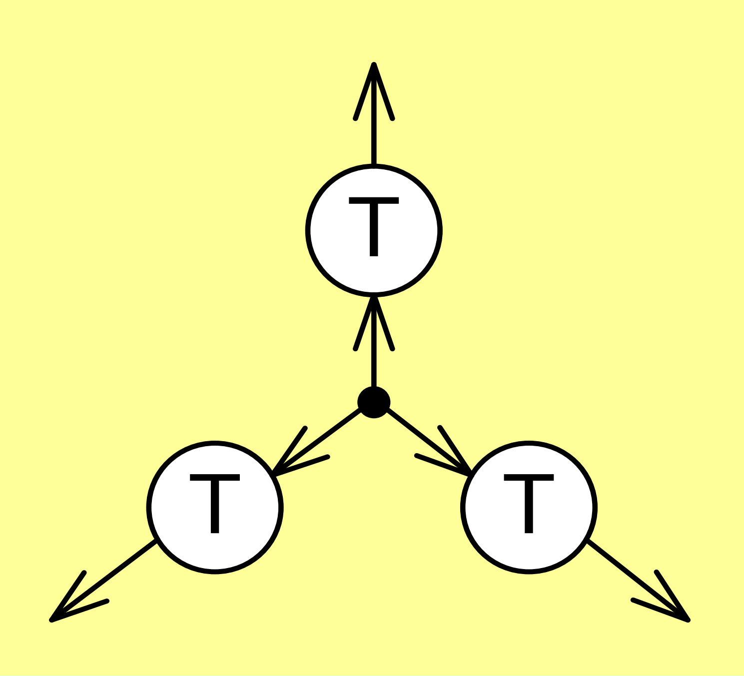




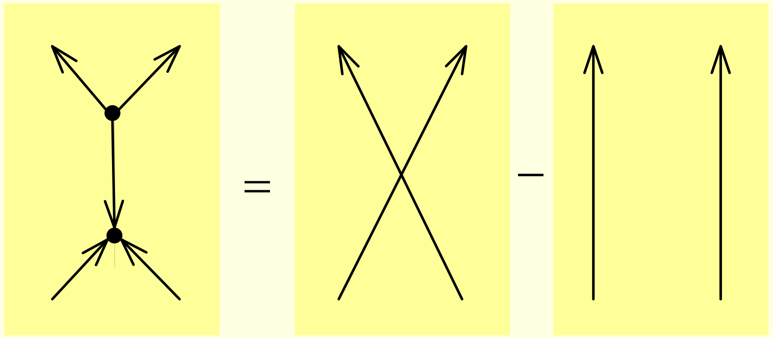


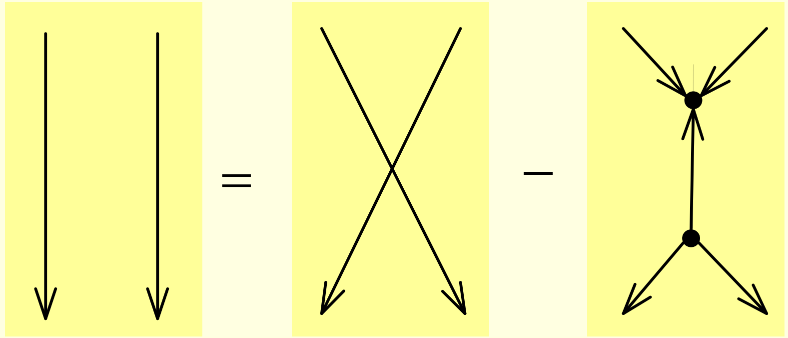
An Important Identity

What is transformed Epsilon?

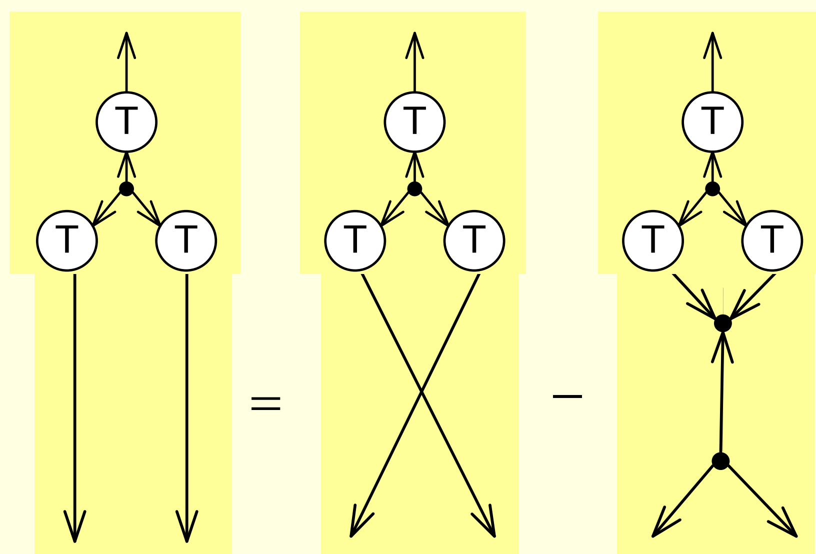


Use Modification of EpsDel

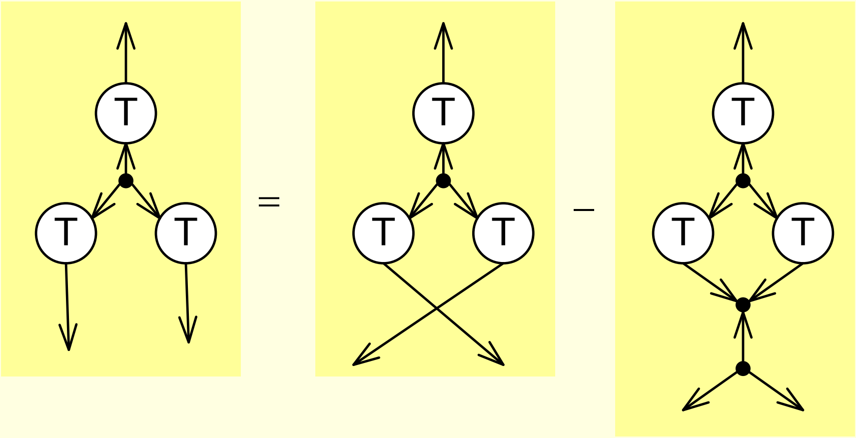


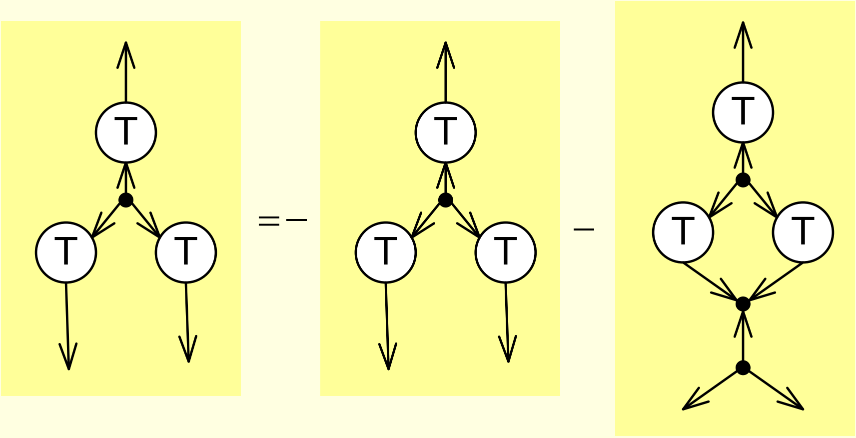


Apply to Transformed Epsilon

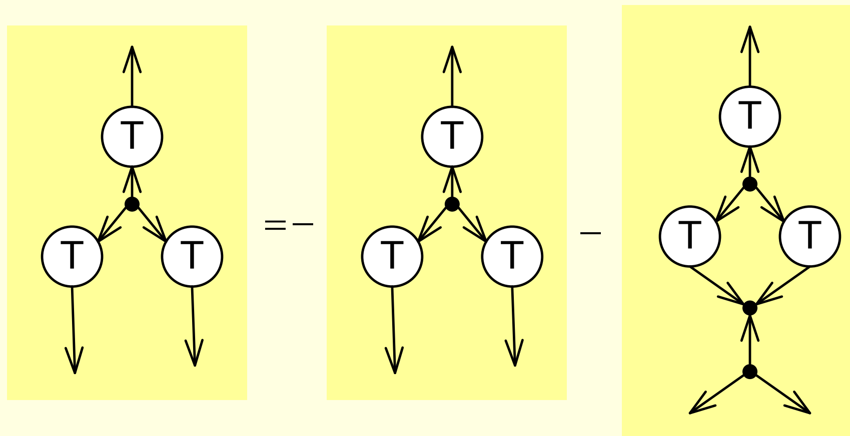


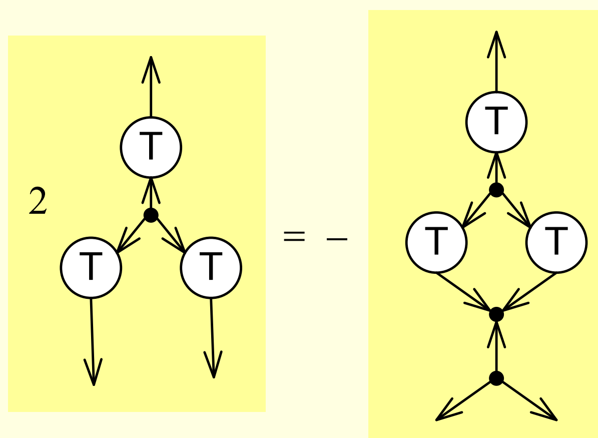
Mirror Reflection = Change Sign



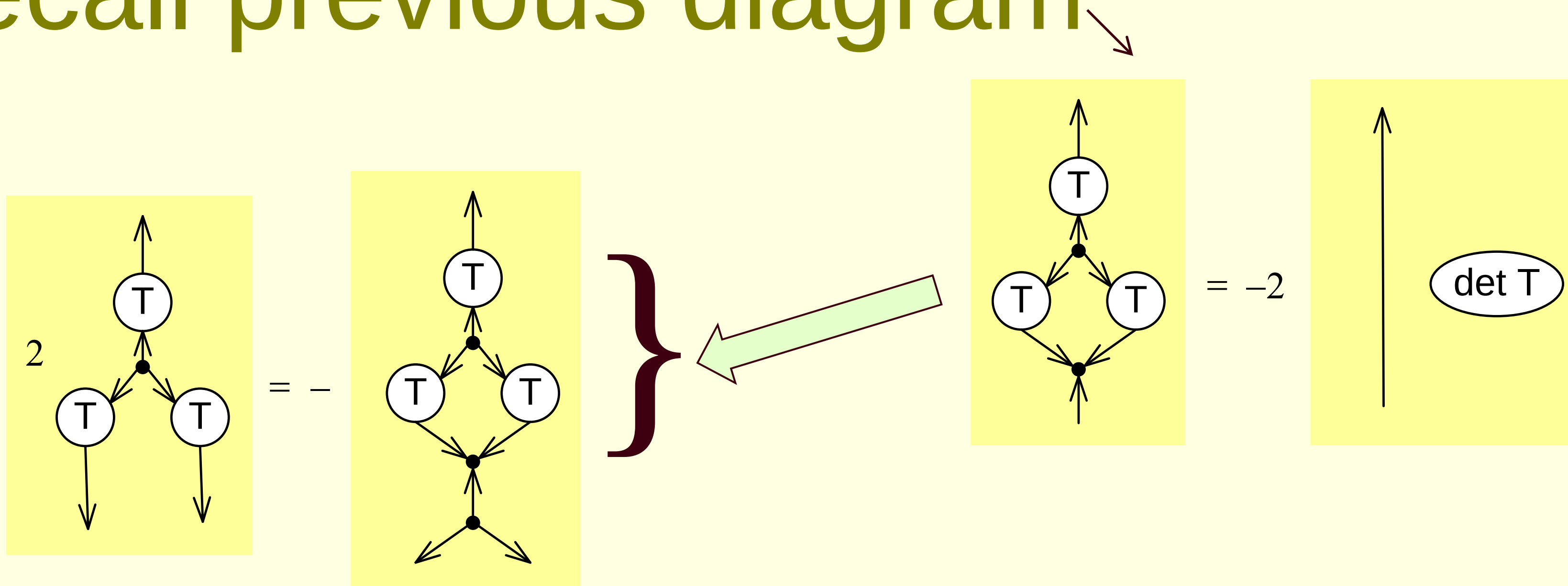


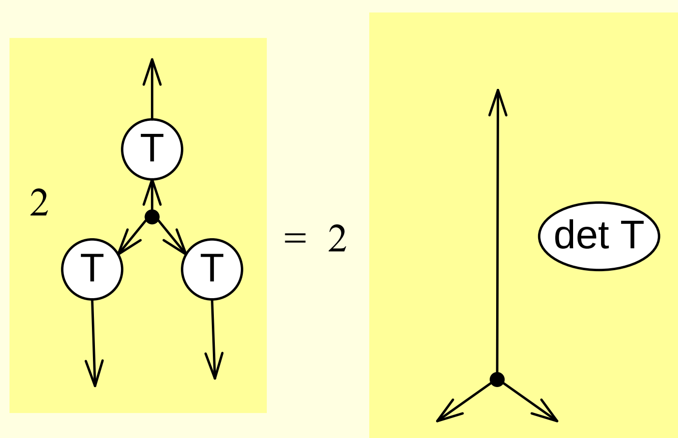
Move over = sign





Recall previous diagram





An Important Identity

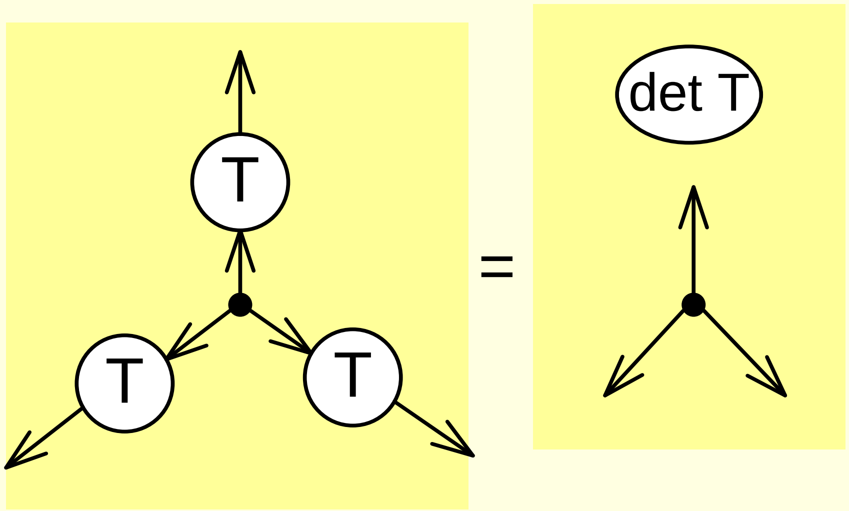
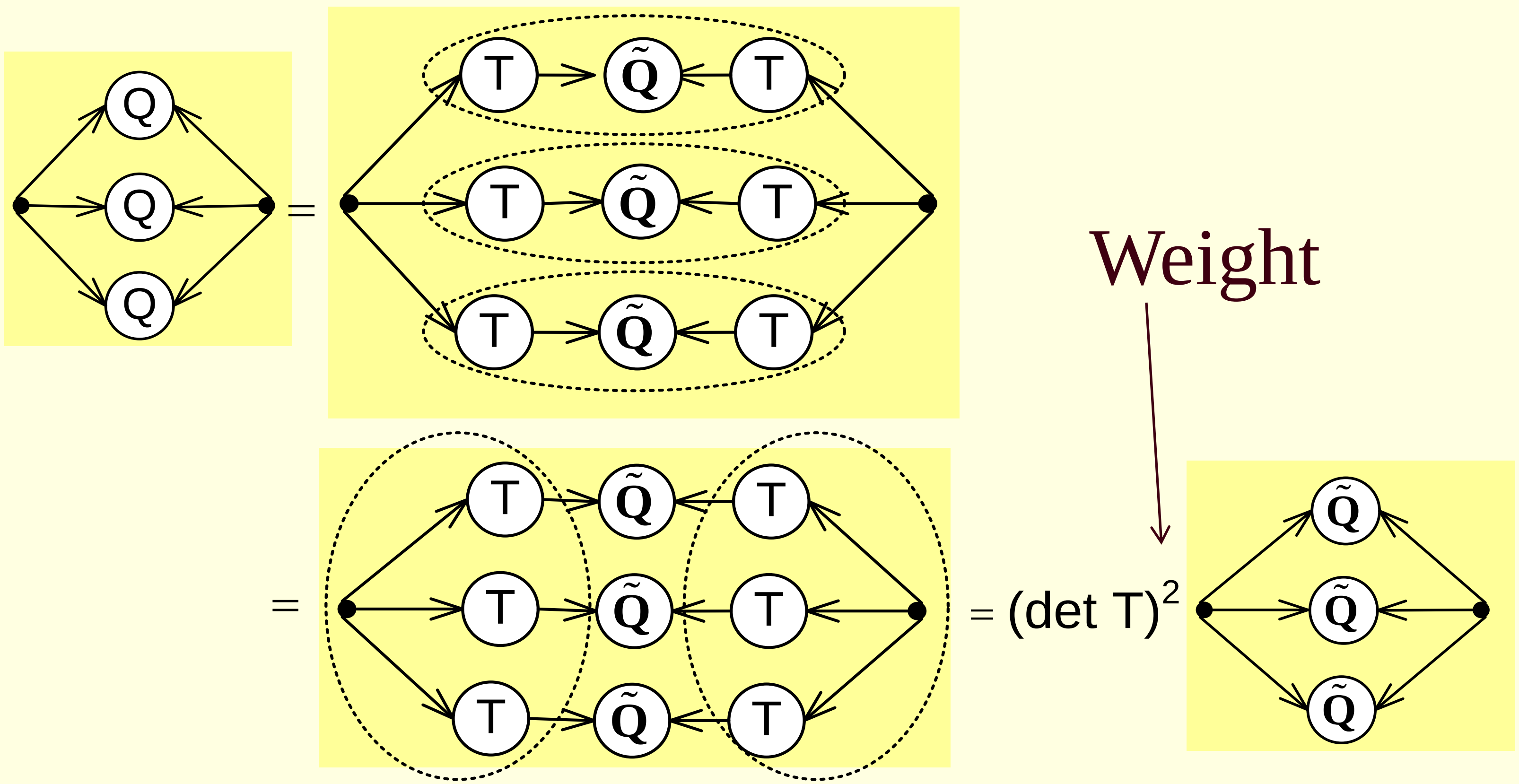


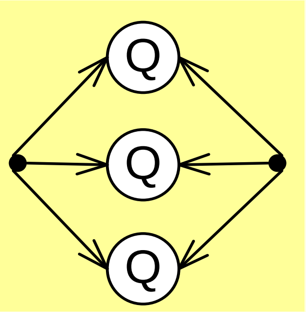
Diagram of Transformed Quadratic Determinant



MAJOR PUNCHLINE

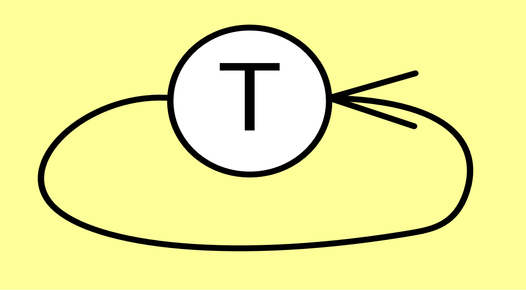
Of all the Gazillion possible polynomials in the coefficients

Tensor Diagrams express only those that represent Invariant Properties

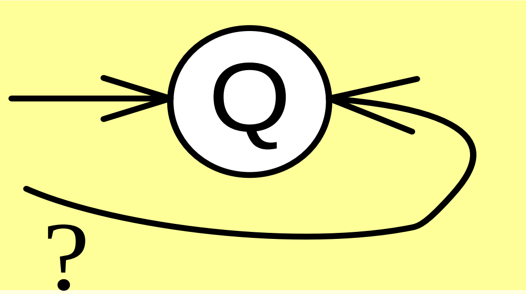


Trace of Matrix

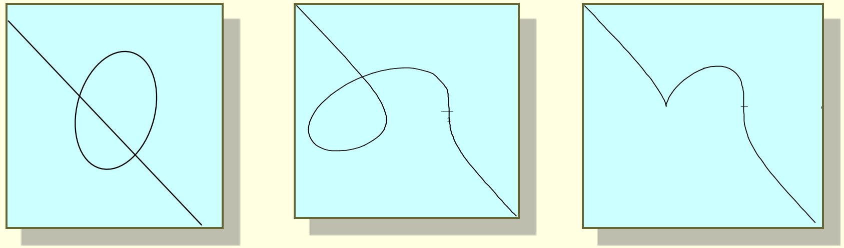
$$\mathrm{trace}\,\mathbf{T} = \sum_i T_i^i =$$

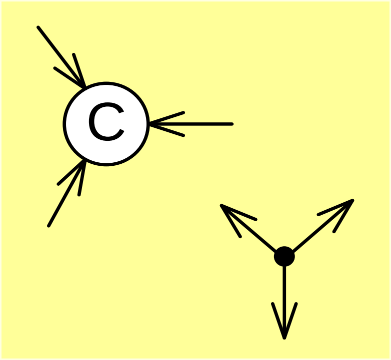


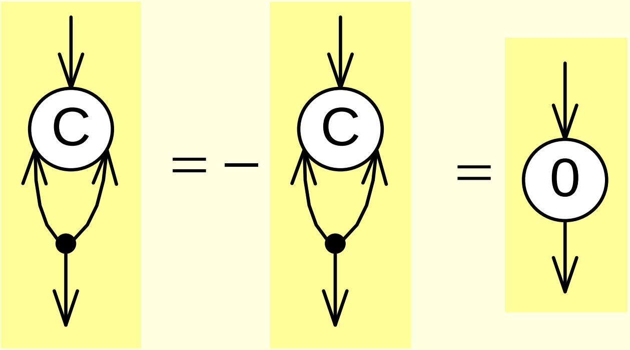
$$\mathrm{trace}\,\mathbf{Q} = \sum_i Q_{ii} =$$

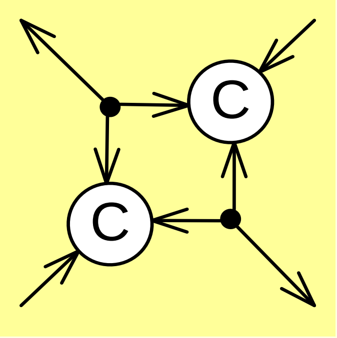


Discriminant of Cubic

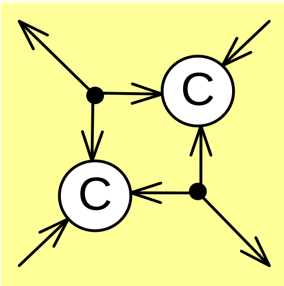


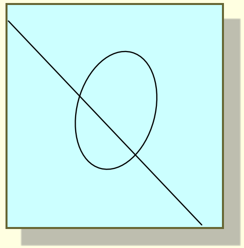


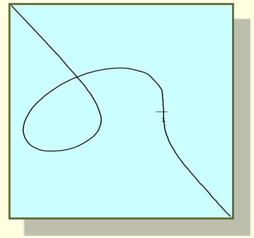


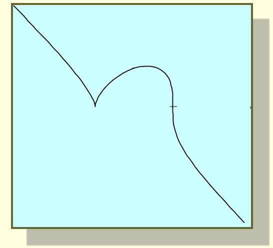


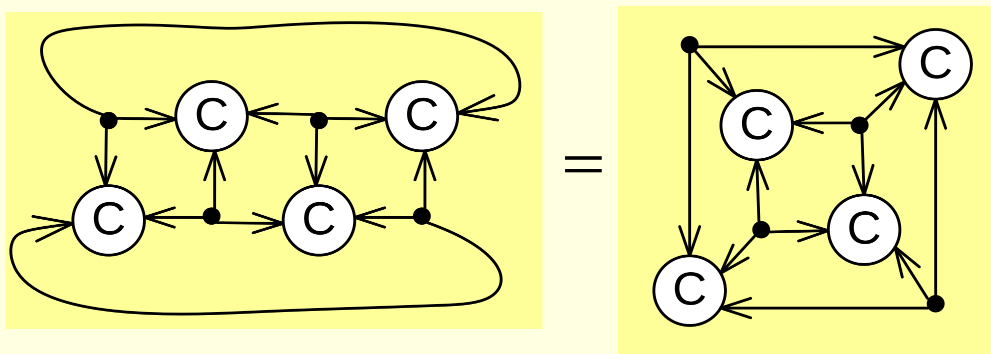
Discriminant of Cubic

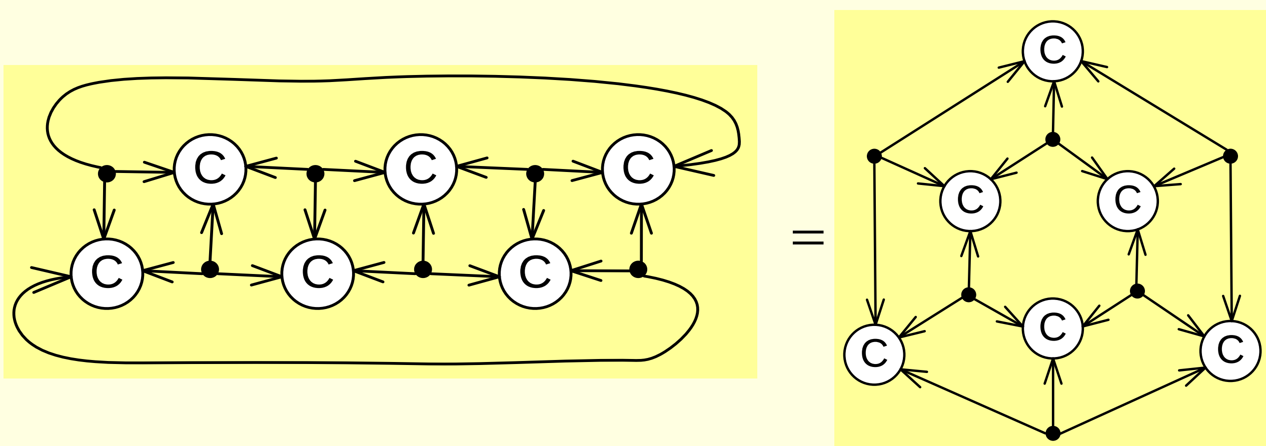




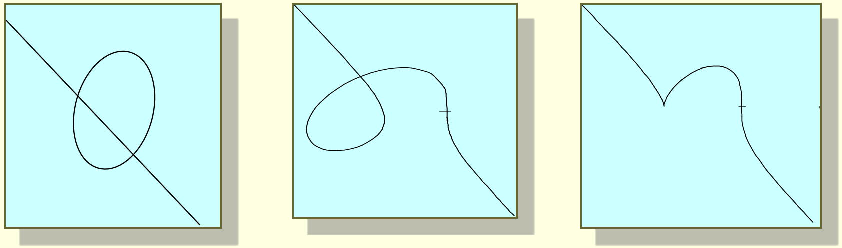








Discriminant of Cubic

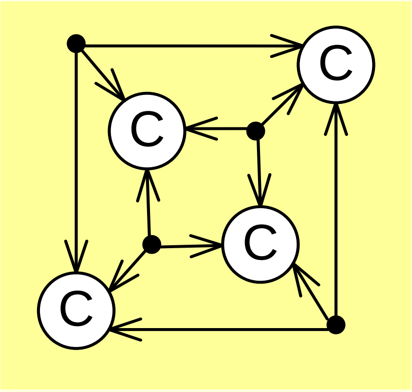


$$\mathbf{D} = 64S^3 + T^2$$

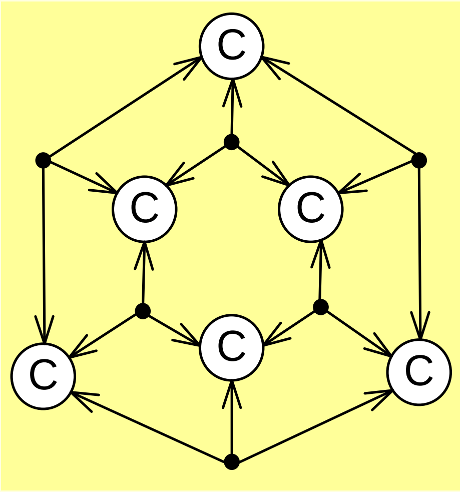
S: degree 4 in $A \ldots K$
has 25 terms

T: degree 6 in $A \ldots K$
has 103 terms

$S = -\frac{1}{24}$

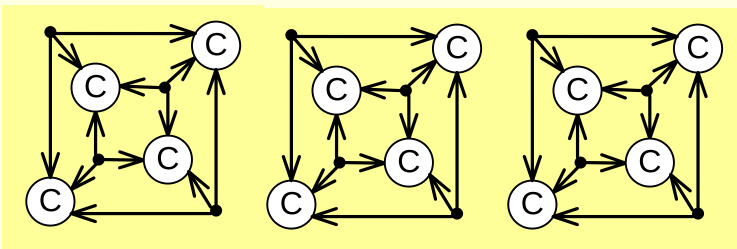
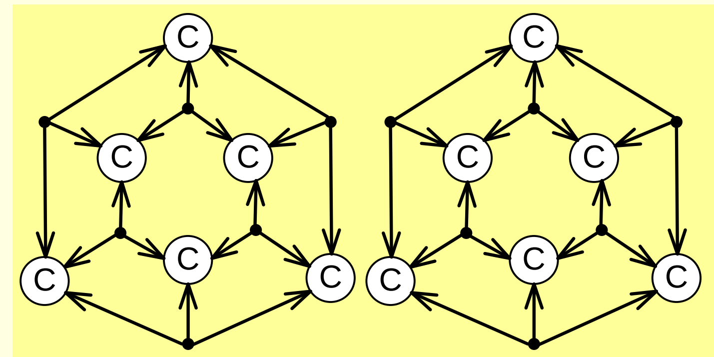


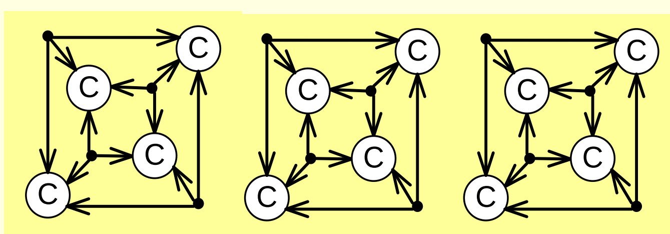
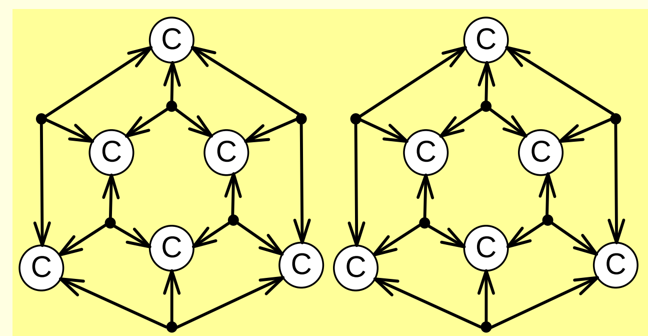
$T = -\frac{1}{6}$



“Phase space” of cubics

$$\mathbf{D} = 64S^3 + T^2$$

$\mathbf{D} =$  -6 

$\{\alpha, \beta\} = \Big\{$  ,  $\Big\}$

History of Diagrammatic Invariant Notation

1878 Sylvester & Clifford

1885 Kempe

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1989 Olver & Shakiban

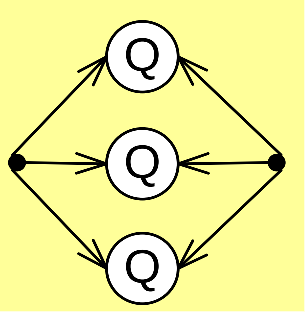
1990 Stedman

1992-2007 Blinn

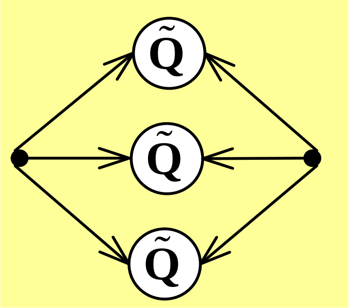
2011 Richter-Gebert

Effect of Changes

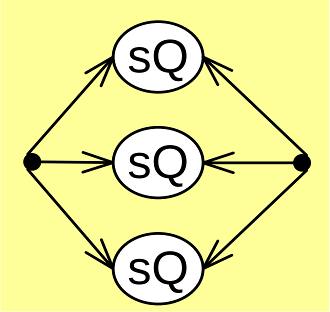
Geometric Transformation



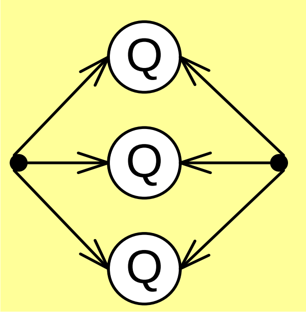
$= (\det T)^2$



Homogeneous Scale

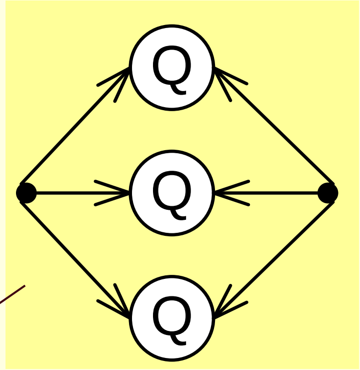


$= s^3$



What Stays Constant?

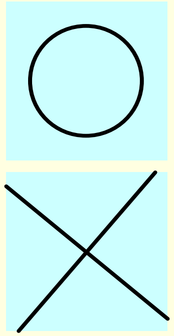
Zeroness



Odd number of nodes

$\neq 0$

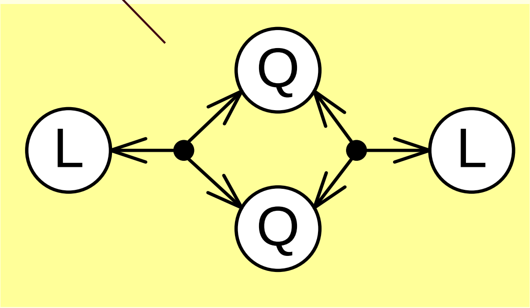
$= 0$



$$= ACF + 2BED - D^2C - E^2A - B^2F$$

Even number of nodes

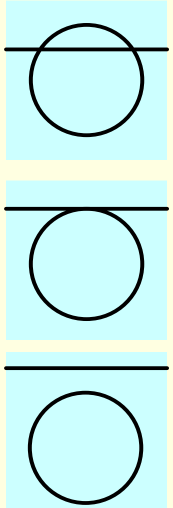
Sign



< 0

$= 0$

> 0



Where do we go from here

Other Dimensions

Polynomials in P^1

2D algebra

$$f\left(x,w\right)=Ax^2+Bxw+Cw^2$$

Curves in P^2

3D algebra

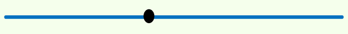
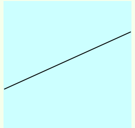
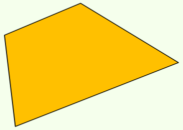
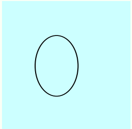
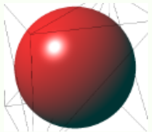
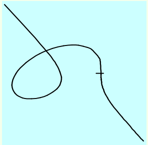
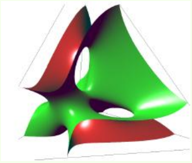
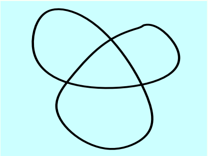
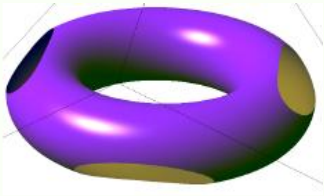
$$f\left(x,y,w\right)=Dx^2+Eyw+Fw^2+\ldots$$

Surfaces in P^3

4D algebra

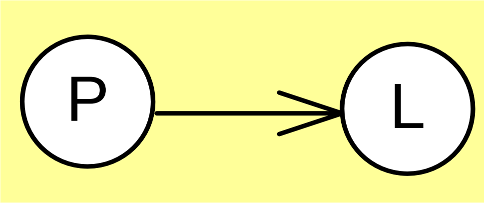
$$f\left(x,y,z,w\right)=G\,x^2+H\,yw+J\,zw+\ldots$$

The Grid

	2D=P^1 Point sets on line	3D=P^2 Curves in plane	4D=P^3 Surfaces in space
LINEAR			
QUADRATIC			
CUBIC			
QUARTIC			
etc			

Order of traversal?

Other Dimensions

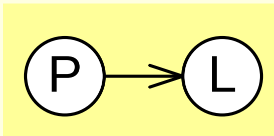


$$2D: \quad ax + bw$$

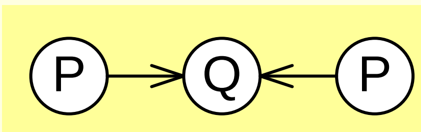
$$3D: \quad ax + by + cw$$

$$4D: \quad ax + by + cz + dw$$

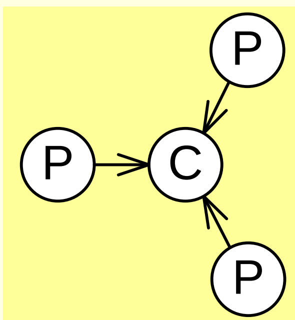
Same Across Dimensionality



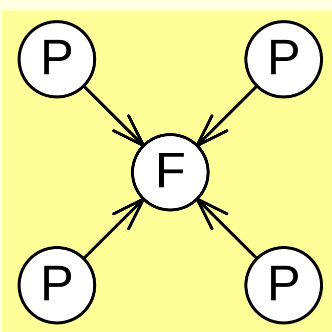
$= 0$



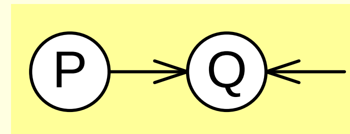
$= 0$



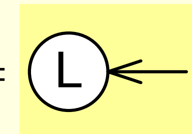
$= 0$

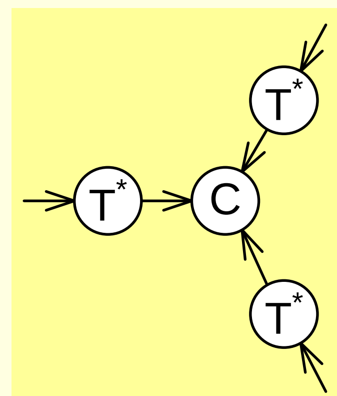


$= 0$

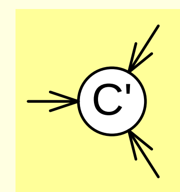


$=$

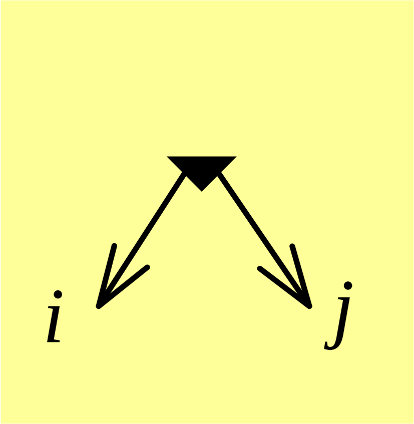




$=$



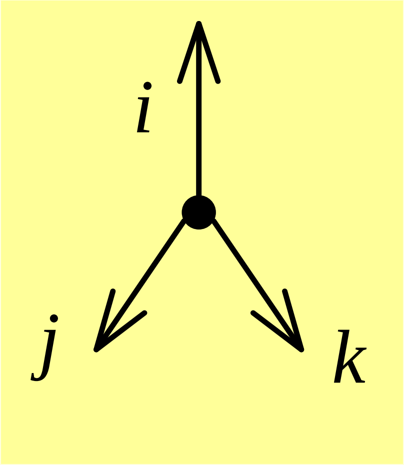
Dimensionality and Epsilon



ε^{ij}

2D algebra

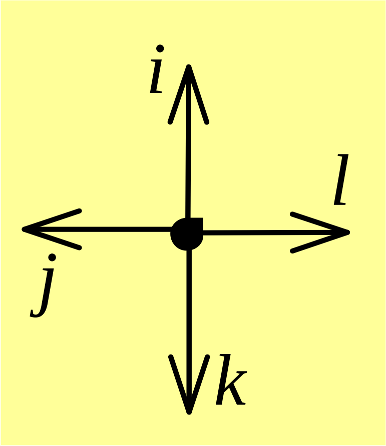
1D geometry



ε^{ijk}

3D algebra

2D geometry



ε^{ijkl}

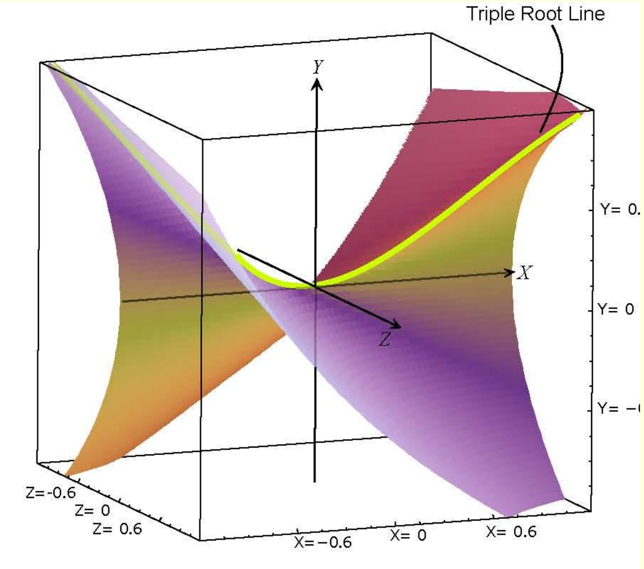
4D algebra

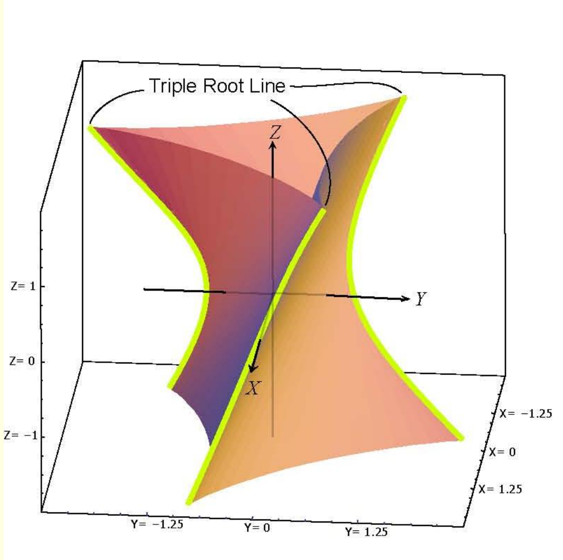
3D geometry

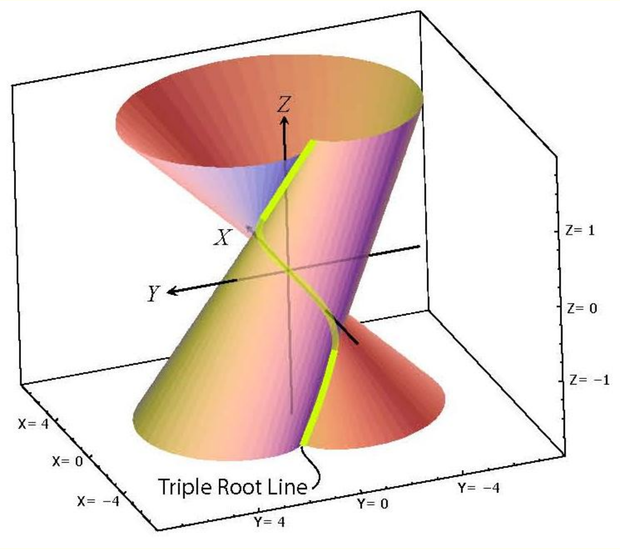
Previews of Coming Attractions

Discriminant Surface

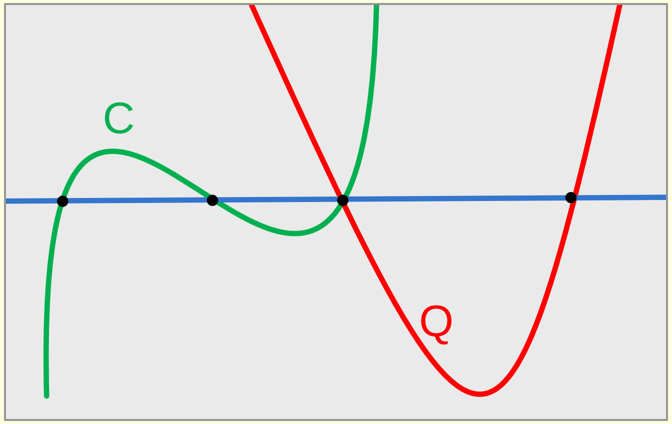
$$-A^2D^2+6ABCD-4AC^3-4B^3D+3B^2C^2=0$$



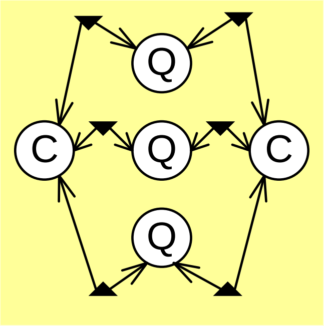




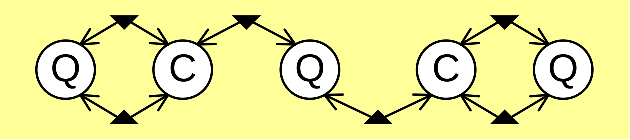
Resultants



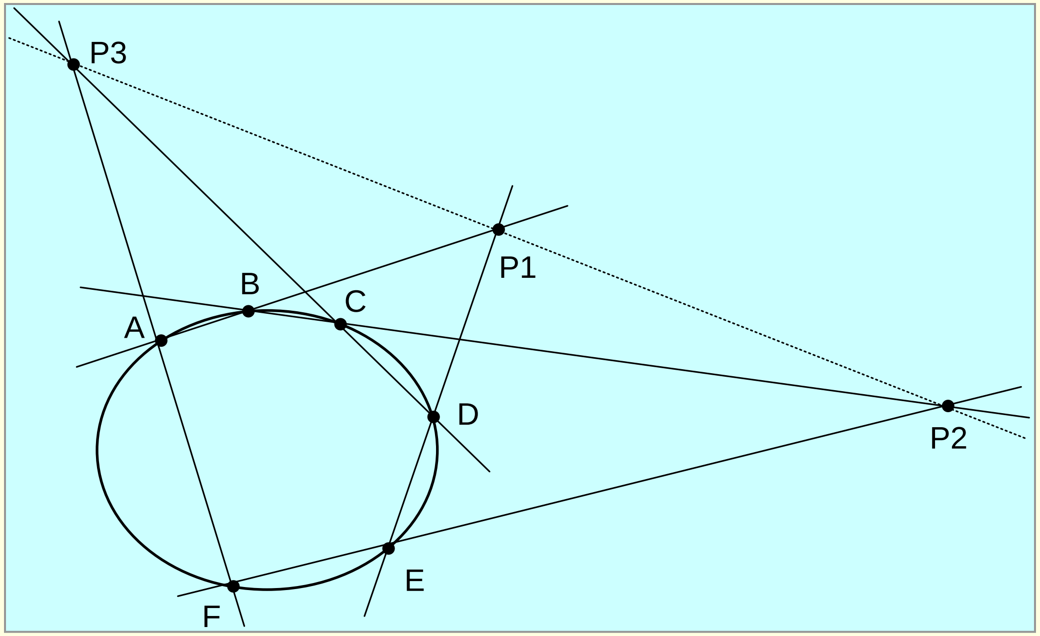
$-8\rho(Q,C) = 4$

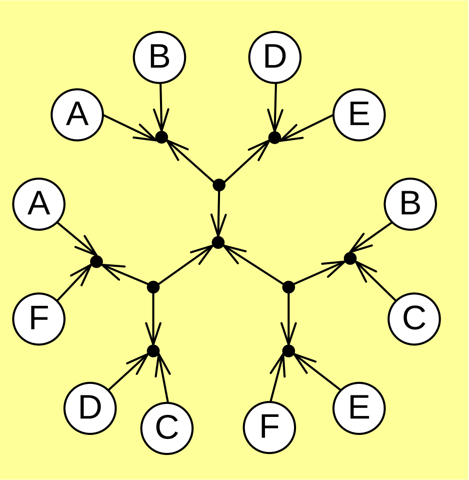


$+3$



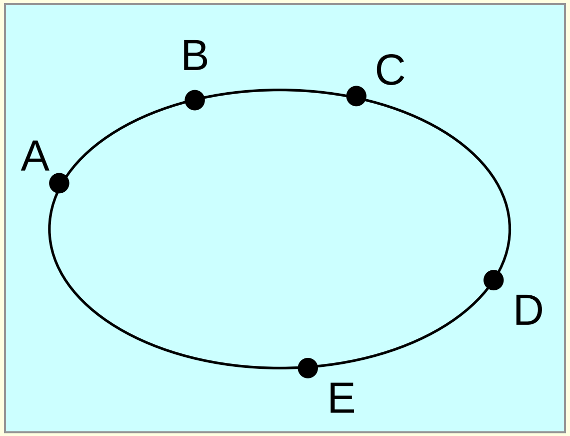
Theorem of Pascal

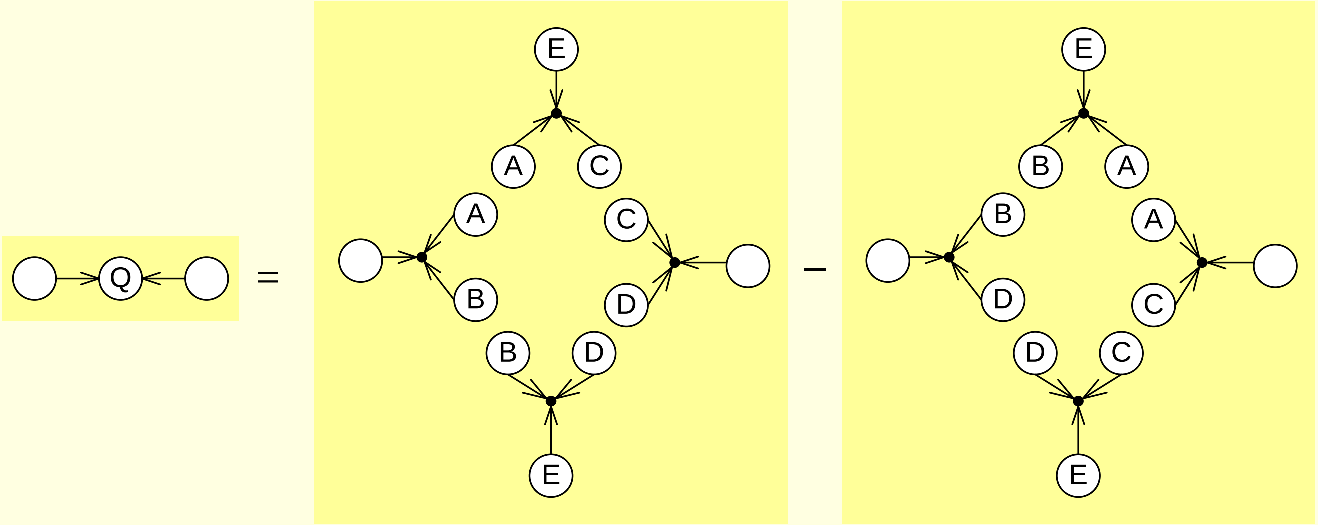




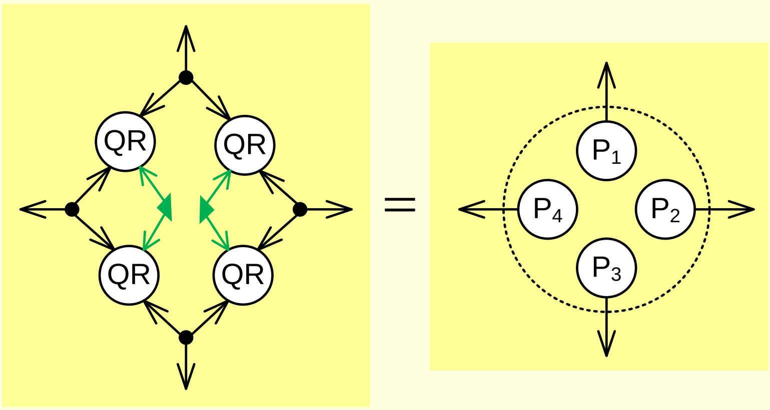
= 0

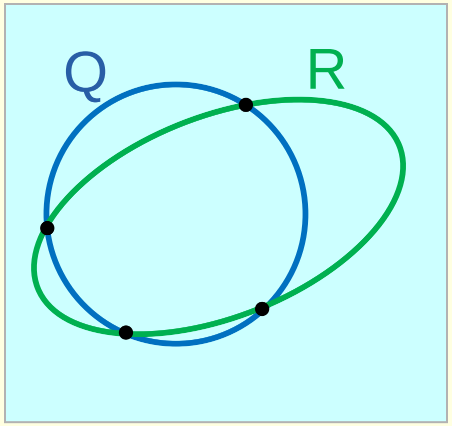
5 Points Determine a Quadratic

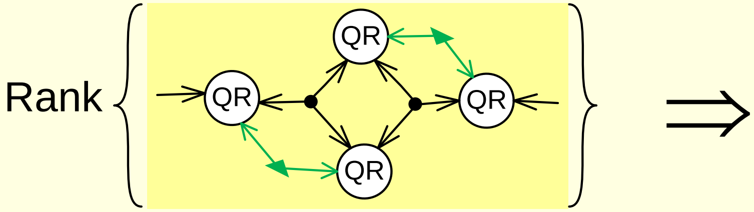


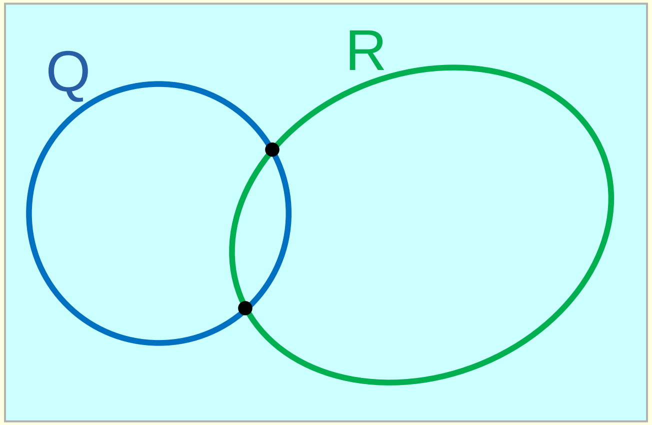


Intersecting Two Quadratic Curves

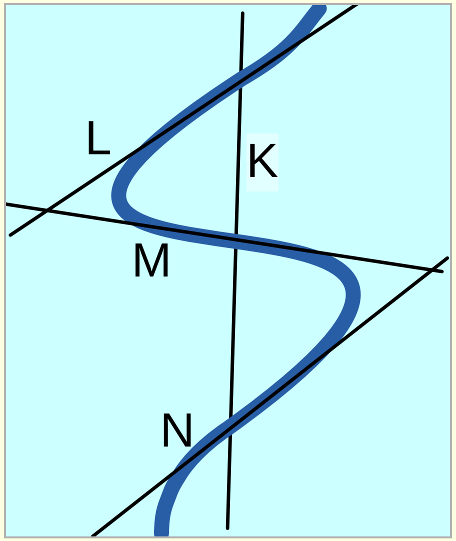


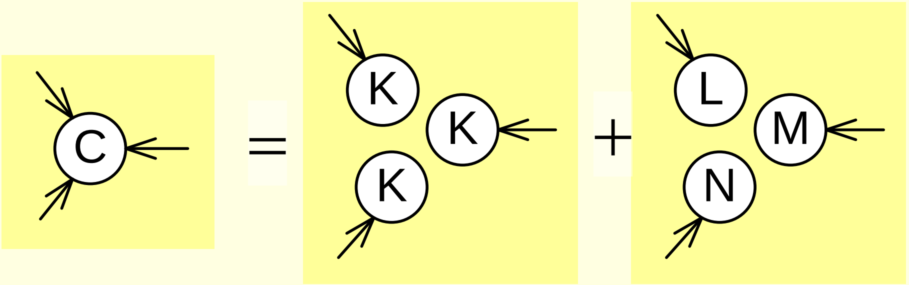






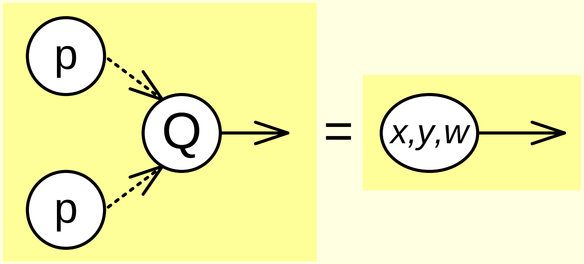
Analyzing Cubic Curves

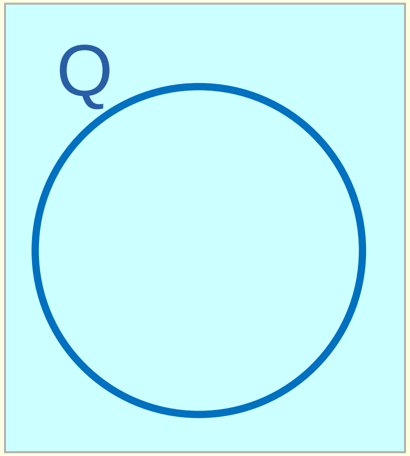




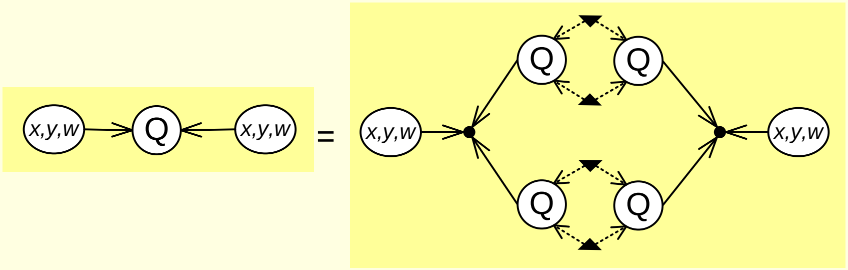
Parametric Curves

Parametric

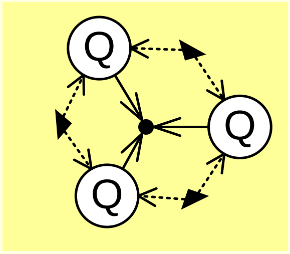




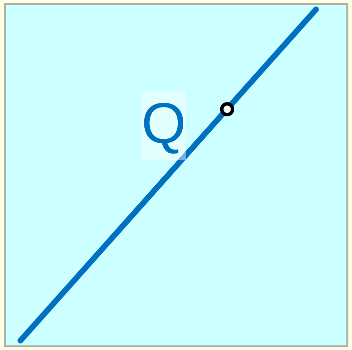
Implicit



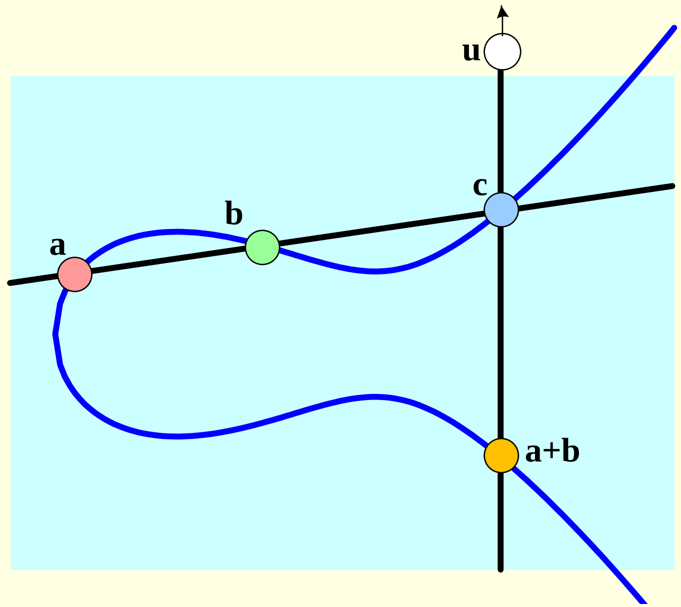
Degeneracy:
Base Point if

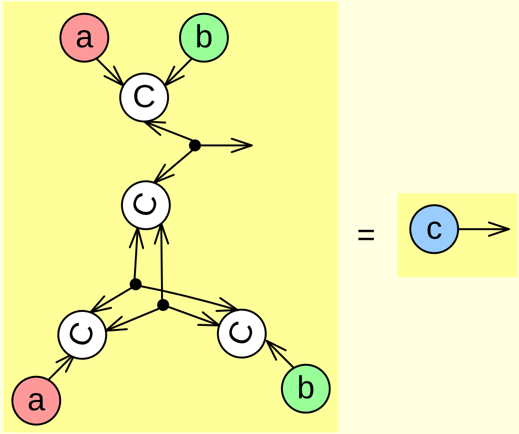


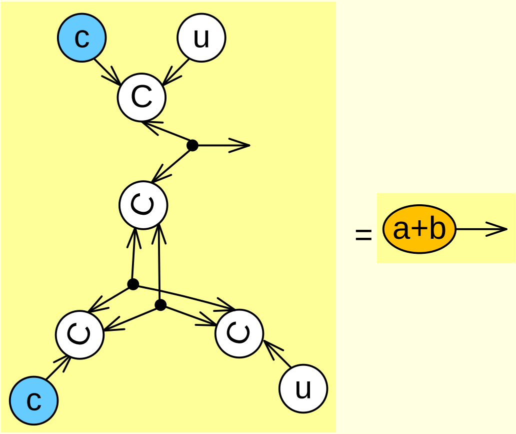
= 0



Group Structure of Cubic

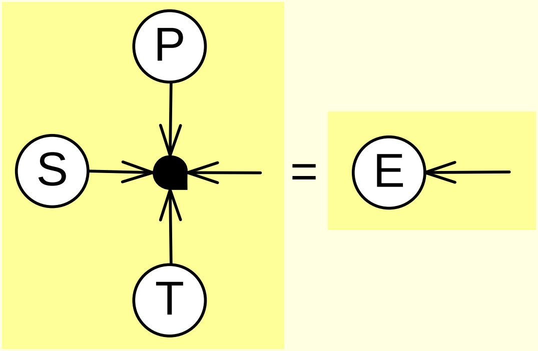




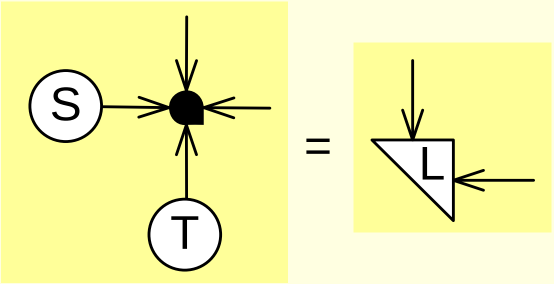


Three Dimensional Projective Geometry

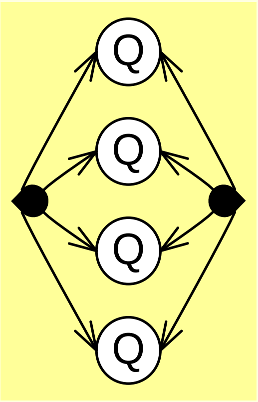
3 Points = A Plane



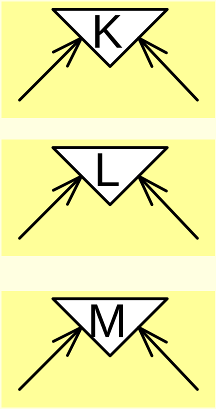
2 Points = A Line

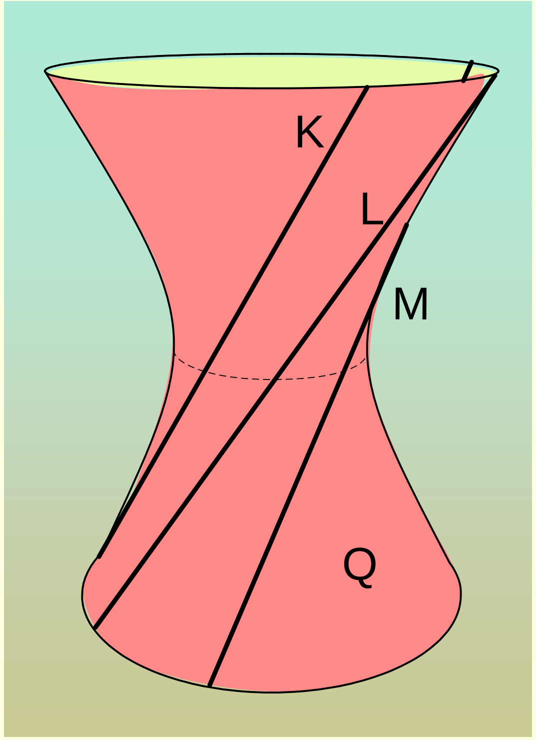


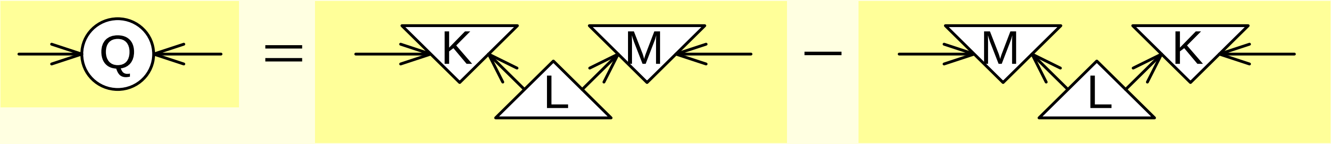
Discriminant of Quadric



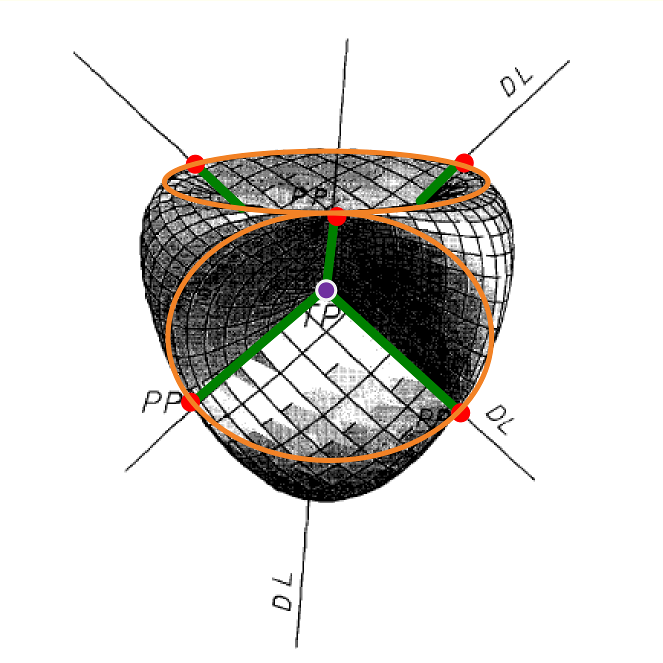
Three Skew Lines



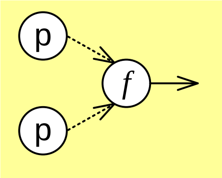




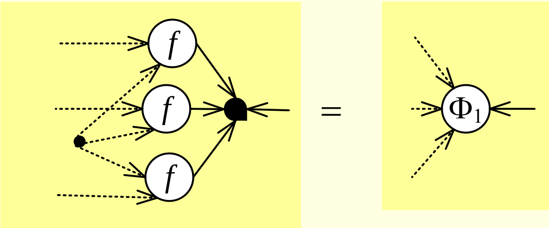
Steiner Surfaces



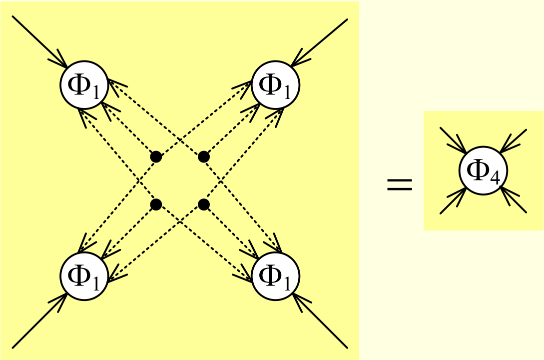
Parametric



Tangent



Implicit



Tensor Diagrams

- Keep Track of CoVariant/ContraVariant Pairings
- Represent Higher Order Curves Nicely
- Express Only Invariant Quantities
- Allow for Algebra on These Quantities
- Are coordinate free
- Allow us to feel really cool at sharing notation with Einstein and Feynman

More Information

www.JimBlinn.com