CSE590B Lecture 4
More about P^1

Transforming Transformations

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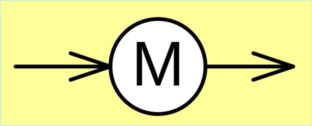
http://courses.cs.washington.edu/courses/cse590b/13au/

Previously On

CSE590b

Transformations

$$\mathbf{M} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

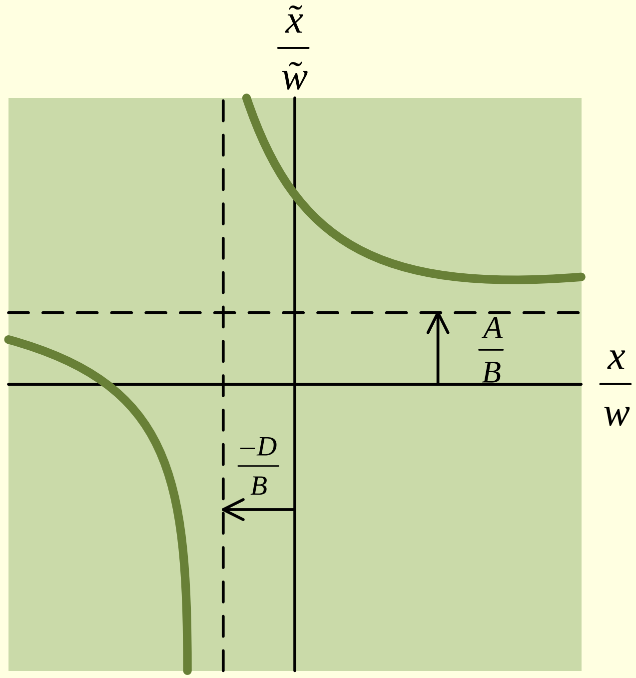


$$\begin{bmatrix} x & w \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \tilde{x} & \tilde{w} \end{bmatrix}$$

$$\frac{\tilde{x}}{\tilde{w}} = \frac{Ax + Cw}{Bx + Dw}$$

The Function

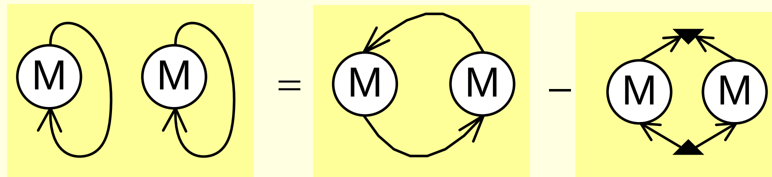
$$\frac{\tilde{x}}{\tilde{w}} = \frac{Ax + Cw}{Bx + Dw}$$

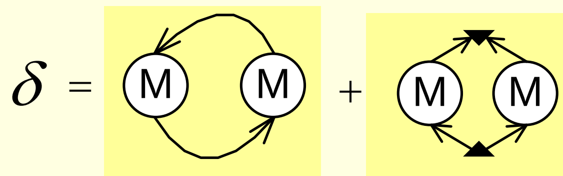


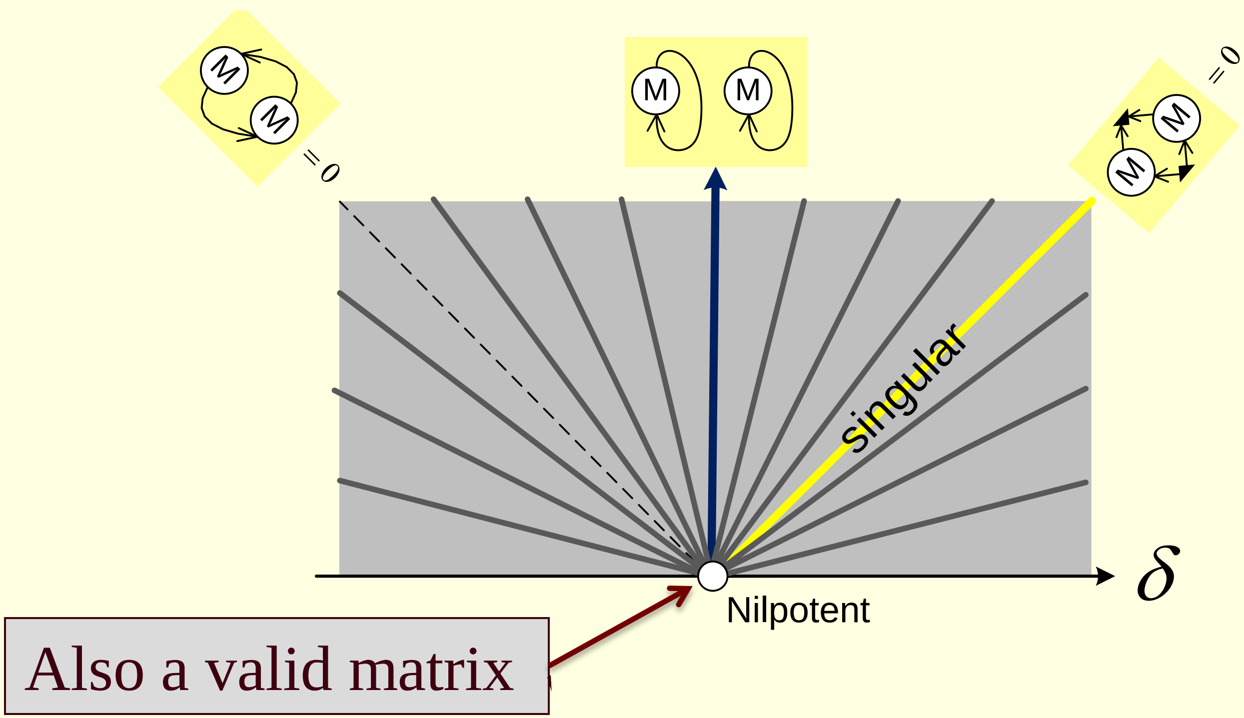
$$w = 0 \quad \Rightarrow \quad \frac{\tilde{x}}{\tilde{w}} = \frac{A}{B}$$

$$\tilde{w} = 0 \quad \Rightarrow \quad Bx + Dw = 0 \quad \Rightarrow \quad \frac{x}{w} = \frac{-D}{B}$$

The “Phase Space” of M







Possible numeric signatures

$$\frac{\delta}{t^2} = \chi$$

$$\tan^{-1}\left(\delta, t^2\right) = \phi$$

New Coordinate System

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} E+F & G+H \\ G-H & E-F \end{bmatrix}$$

$$\begin{aligned} \Delta &= AD - BC \\ &= (E+F)(E-F) - (G+H)(G-H) \\ &= (E^2 + H^2) - (F^2 + G^2) \end{aligned}$$

$$t = A + D = 2E$$

$$\begin{aligned} \delta &= (-A-D)^2 - 4(AD-BC) \\ &= 4(F^2 + G^2 - H^2) \end{aligned}$$

$$\Delta = E^2 + H^2 - F^2 - G^2$$

$$\tfrac{1}{4}t^2 = E^2$$

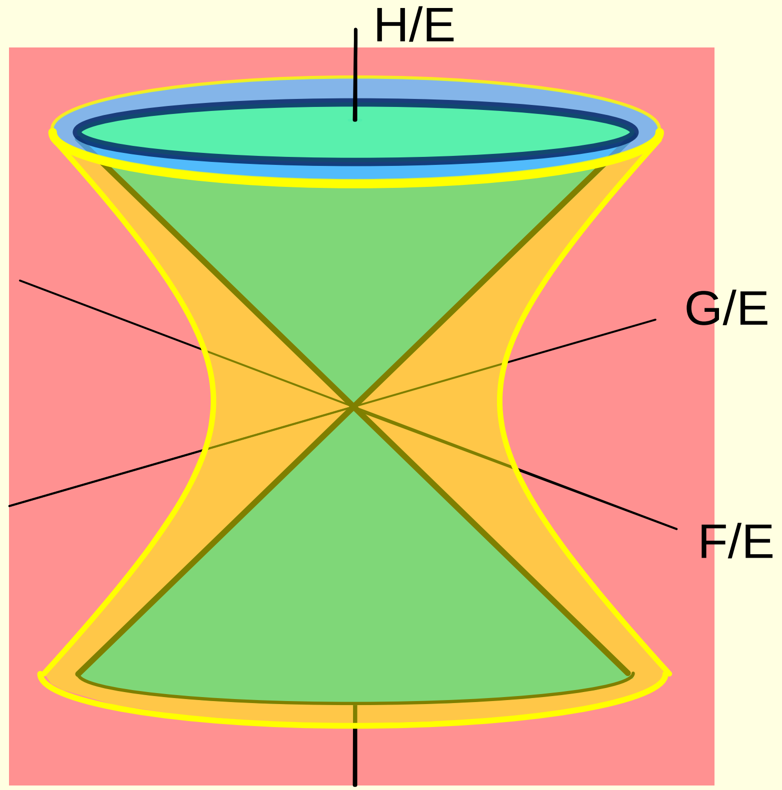
$$\tfrac{1}{4}\delta = \quad -H^2 + F^2 + G^2$$

Plot in EFGH space

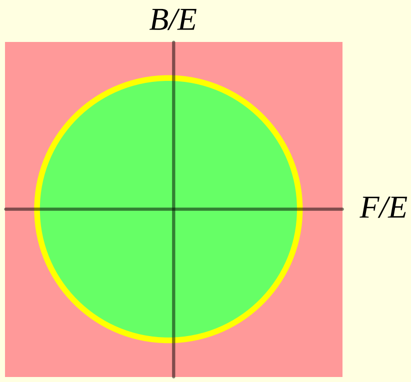
$$\Delta = 0 \quad \left(\frac{H}{E}\right)^2 = \left(\frac{F}{E}\right)^2 + \left(\frac{G}{E}\right)^2 - 1$$

$$\delta = 0 \quad \left(\frac{H}{E}\right)^2 = \left(\frac{F}{E}\right)^2 + \left(\frac{G}{E}\right)^2$$

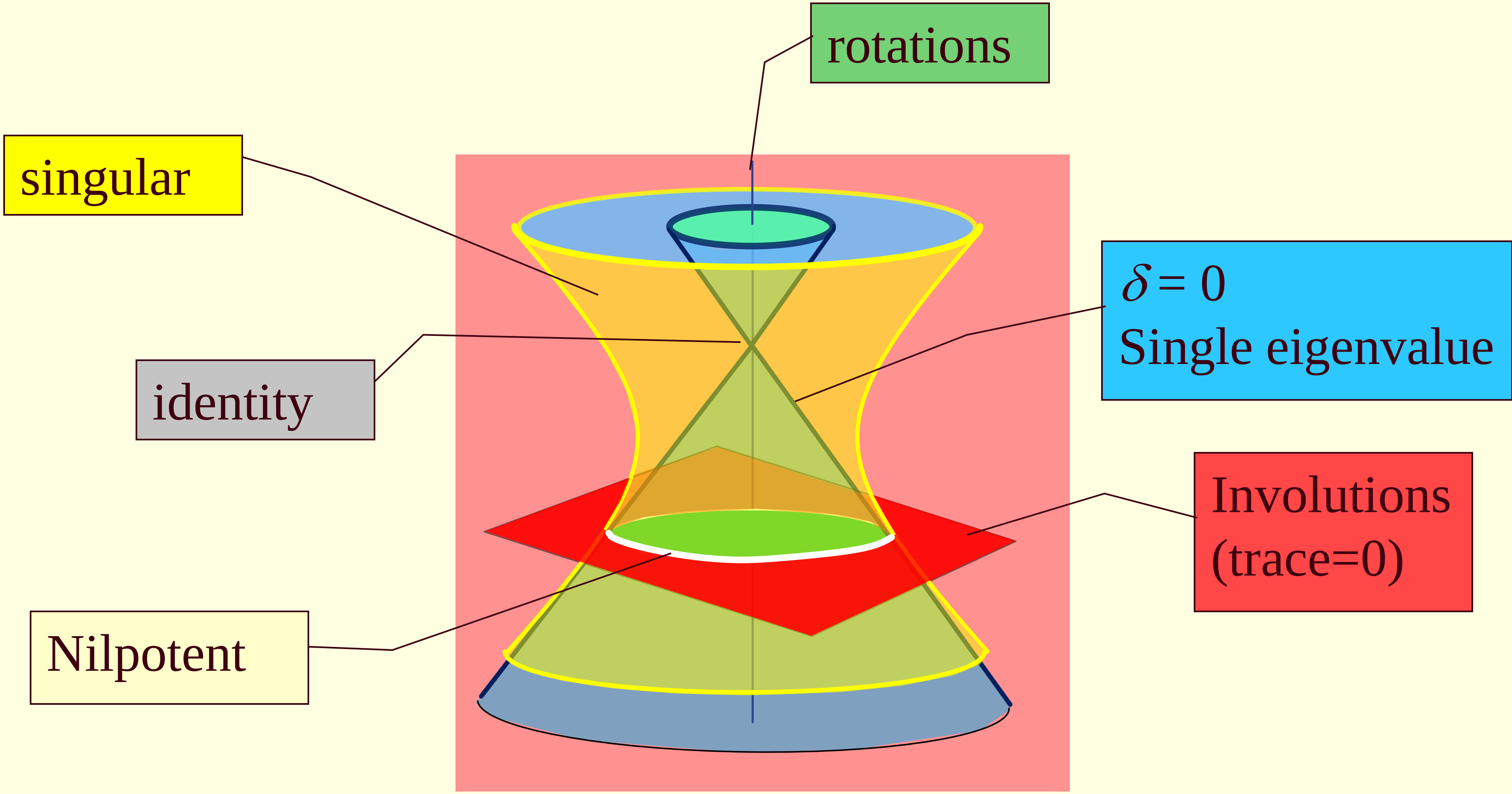
$t = 0$ plane at infinity



Compare with Q version



Roadmap of M



rotations
singular
$\delta = 0$
Single eigenvalue
identity
Involutions
(trace=0)
Nilpotent

And Now

Finding Real Eigenvalues

Find λ such that

$$\det\left(\mathbf{M}+\lambda\mathbf{I}\right)=0$$

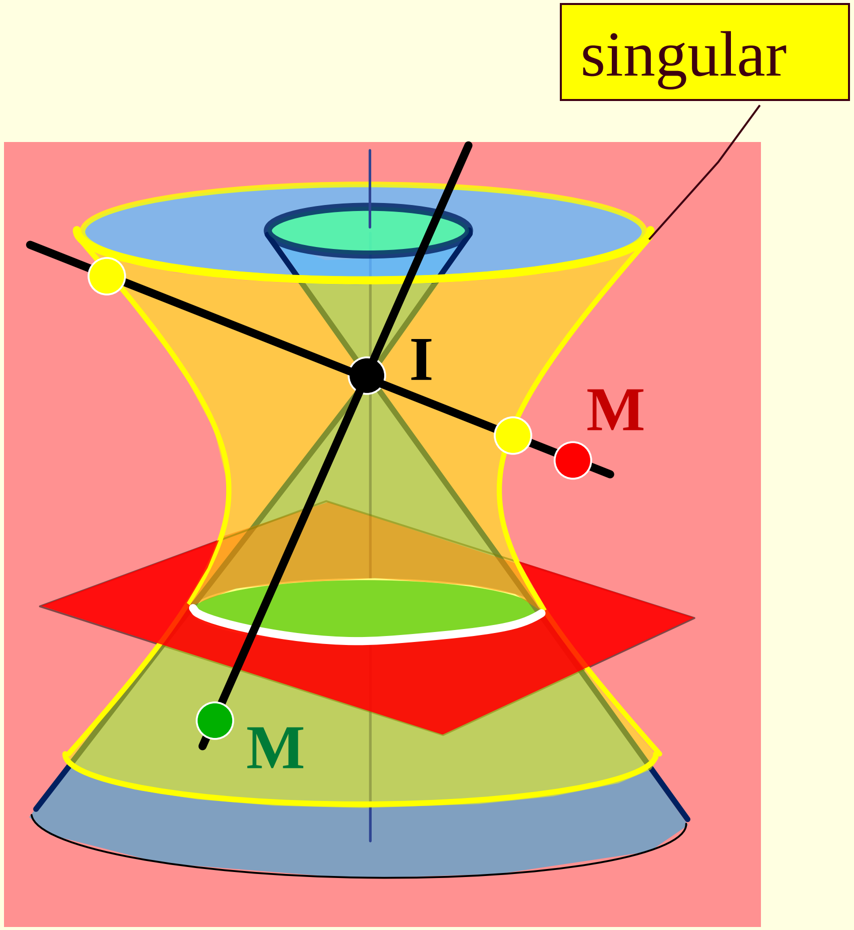
Linear combo of M and I is singular

Outside cone (red,blue):

Two intersections on line

Inside cone (green):

No intersections on line



Transformation of Transformations

$$\begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} = \mathbf{T}^* \begin{bmatrix} A & B \\ C & D \end{bmatrix} \mathbf{T}$$

$$\begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} = \begin{bmatrix} t & u \\ s & v \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} v & -u \\ -s & t \end{bmatrix}$$

$$\tilde{A} = tvA + uvC - stB - sqD$$

$$\tilde{B} = -utA - uuC + tB + tqD$$

$$\tilde{C} = svA + vvC - ssB - svD$$

$$\tilde{D} = -usA - uvC + tsB + tvD$$

$$\begin{bmatrix} \tilde{A} \\ \tilde{D} \\ \tilde{B} \\ \tilde{C} \end{bmatrix} = \begin{bmatrix} ts & -us & -ts & uv \\ -us & tv & ts & -uv \\ -tu & tu & tt & -uu \\ rv & -rv & -rr & vv \end{bmatrix} \begin{bmatrix} A \\ D \\ B \\ C \end{bmatrix}$$

Transformation of Transformations

$$\begin{bmatrix} \tilde{A} \\ \tilde{D} \\ \tilde{B} \\ \tilde{C} \end{bmatrix} = \begin{bmatrix} ts & -us & -ts & uv \\ -us & tv & ts & -uv \\ -tu & tu & tt & -uu \\ rv & -rv & -rr & vv \end{bmatrix} \begin{bmatrix} A \\ D \\ B \\ C \end{bmatrix}$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} E+F & G+H \\ G-H & E-F \end{bmatrix}$$

$$\begin{bmatrix} A \\ D \\ B \\ C \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} E \\ F \\ G \\ H \end{bmatrix}$$

$$\begin{bmatrix} \tilde{E} \\ \tilde{F} \\ \tilde{G} \\ \tilde{H} \end{bmatrix} = \begin{bmatrix} tv-us & 0 & 0 & 0 \\ 0 & tv+us & -ts+uv & -ts-uv \\ 0 & -tu+sv & \frac{1}{2}\left(tt-ss-uu+vv\right) & \frac{1}{2}\left(tt-ss+uu-vv\right) \\ 0 & -tu-sv & \frac{1}{2}\left(tt+ss-uu-vv\right) & \frac{1}{2}\left(tt+ss+uu+vv\right) \end{bmatrix} \begin{bmatrix} E \\ F \\ G \\ H \end{bmatrix}$$

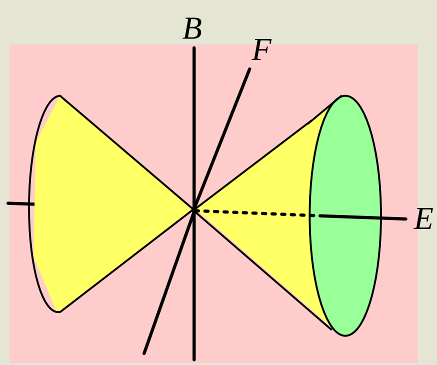
Distill with Rotation Transform

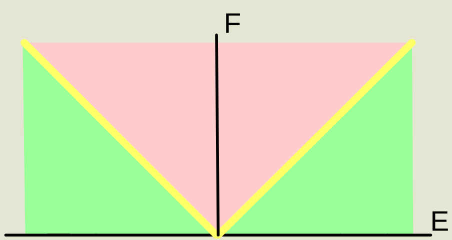
$$\begin{bmatrix} t & u \\ s & v \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} \tilde{E} \\ \tilde{F} \\ \tilde{G} \\ \tilde{H} \end{bmatrix} = \begin{bmatrix} tv-us & 0 & 0 & 0 \\ 0 & tv+us & -ts+uv & -ts-uv \\ 0 & -tu+sv & \frac{1}{2}(tt-ss-uu+vv) & \frac{1}{2}(tt-ss+uu-vv) \\ 0 & -tu-sv & \frac{1}{2}(tt+ss-uu-vv) & \frac{1}{2}(tt+ss+uu+vv) \end{bmatrix} \begin{bmatrix} E \\ F \\ G \\ H \end{bmatrix}$$

$$\begin{bmatrix} \tilde{E} \\ \tilde{F} \\ \tilde{G} \\ \tilde{H} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\theta & \sin 2\theta & 0 \\ 0 & -\sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} E \\ F \\ G \\ H \end{bmatrix}$$

Similar to what we did with Q:





Rotation Transform

Rotate to make G zero

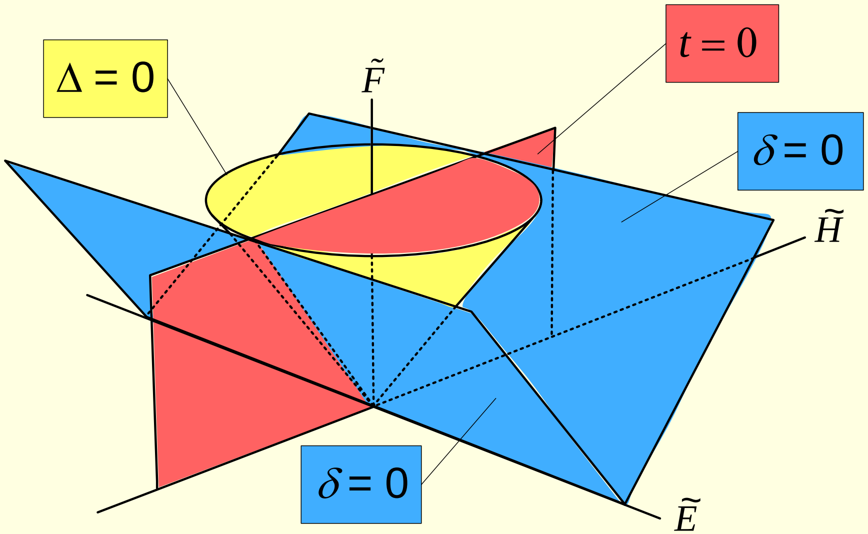
$$\begin{bmatrix} \tilde{E} \\ \tilde{F} \\ 0 \\ \tilde{H} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\theta & \sin 2\theta & 0 \\ 0 & -\sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} E \\ F \\ G \\ H \end{bmatrix}$$

Note: Not dividing by E yet

$$\Delta = \tilde{E}^2 + \tilde{H}^2 - \tilde{F}^2$$

$$\tfrac{1}{4}\delta = \left(\tilde{F}^2 - \tilde{H}^2\right) = \left(\tilde{F} + \tilde{H}\right)\left(\tilde{F} - \tilde{H}\right)$$

$$\tfrac{1}{2}t = \tilde{E}$$



Rotation Transform

Rotate to make G zero

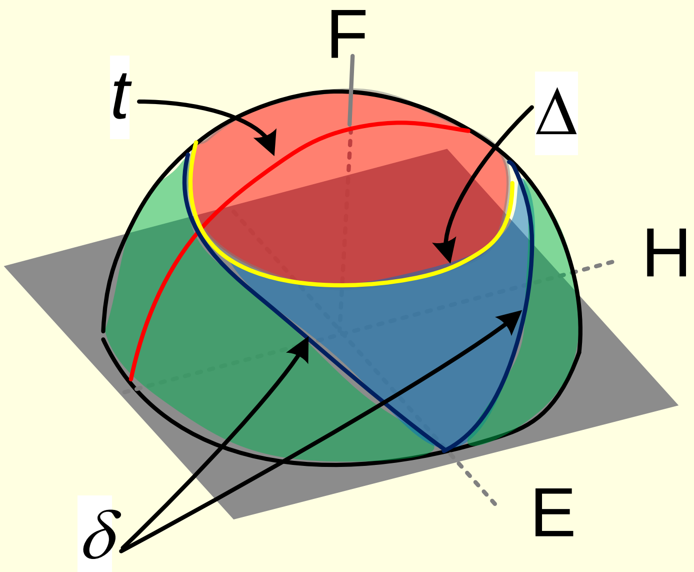
$$\begin{bmatrix} \tilde{E} \\ \tilde{F} \\ 0 \\ \tilde{H} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\theta & \sin 2\theta & 0 \\ 0 & -\sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} E \\ F \\ G \\ H \end{bmatrix}$$

$$\Delta = E^2 + H^2 - F^2$$

$$\tfrac{1}{4}\delta = \left(F^2 - H^2\right) = \left(F + H\right)\left(F - H\right)$$

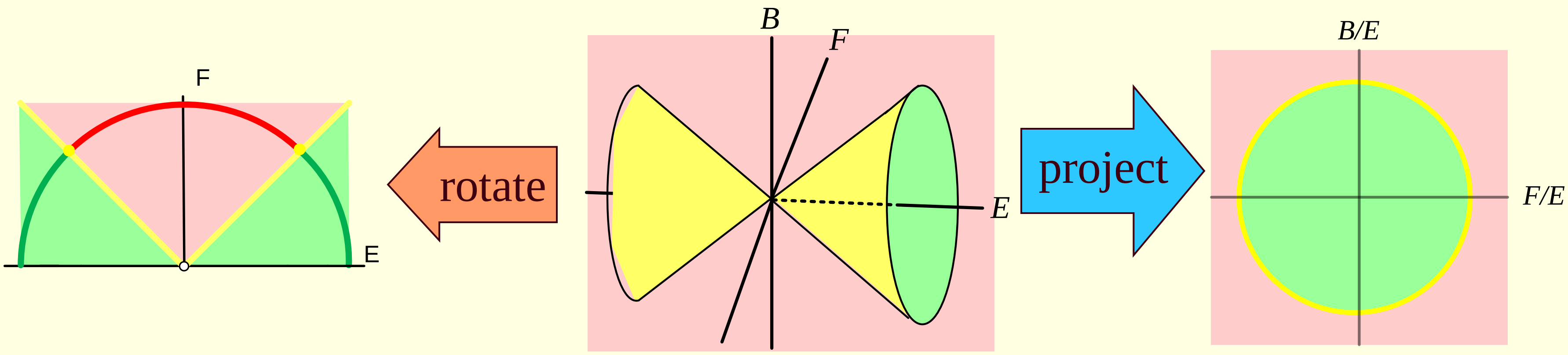
$$\tfrac{1}{2}t = E$$

Project onto unit sphere

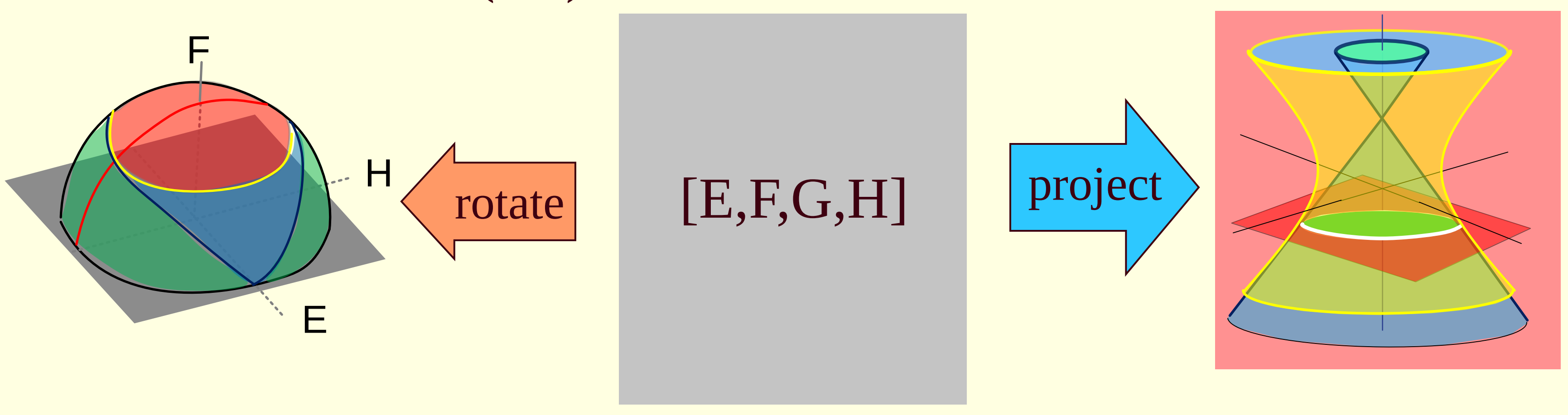


Mappings

Quadratic Polys (3D)



Transformation mtx (4D)



Sign Flips

Rotating 90 degrees flips (F,G) sign

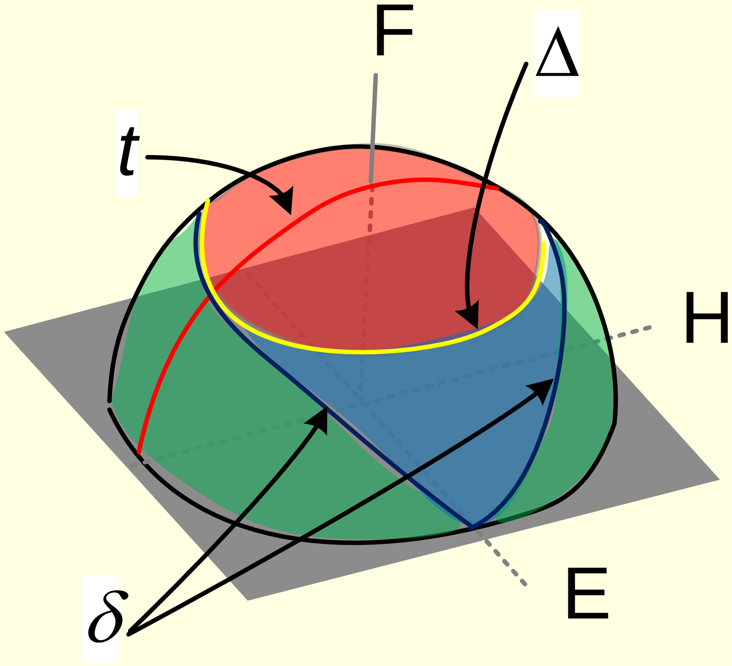
$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \tilde{E} \\ \tilde{F} \\ \tilde{G} \\ \tilde{H} \end{bmatrix} = \begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix} \begin{bmatrix} E \\ F \\ G \\ H \end{bmatrix}$$

Scaling by -1 in x flips (E,F) signs

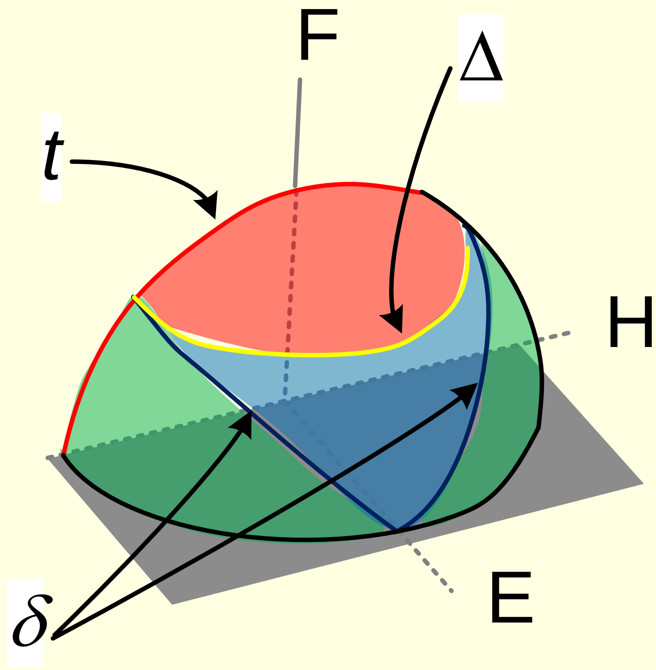
$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} \tilde{E} \\ \tilde{F} \\ \tilde{G} \\ \tilde{H} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} E \\ F \\ G \\ H \end{bmatrix}$$

Exchanging x,w flips (E,G) signs

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \tilde{E} \\ \tilde{F} \\ \tilde{G} \\ \tilde{H} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix} \begin{bmatrix} E \\ F \\ G \\ H \end{bmatrix}$$



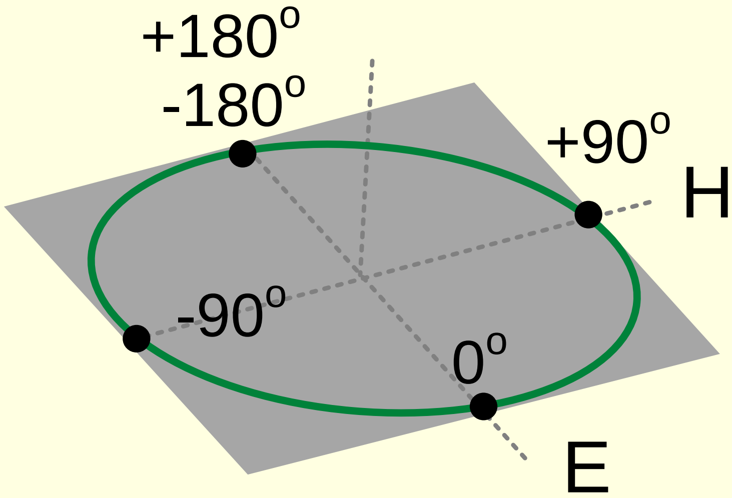
Sign Flips

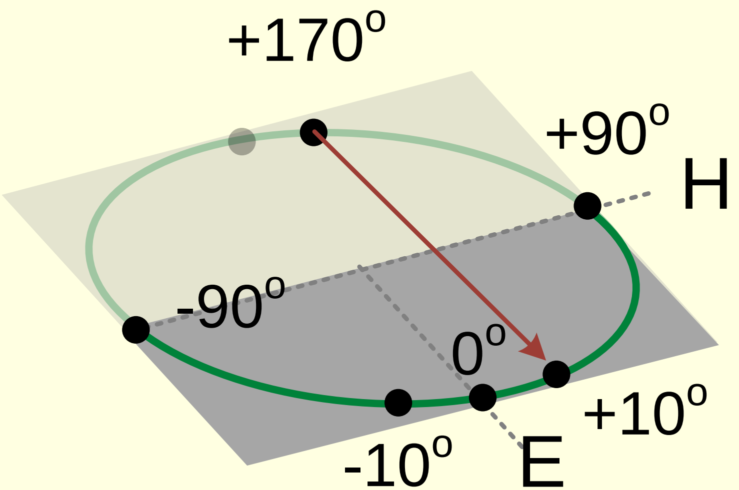


Exchanging x,w flips (E,G) signs

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \tilde{E} \\ \tilde{F} \\ \tilde{G} \\ \tilde{H} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix} \begin{bmatrix} E \\ F \\ G \\ H \end{bmatrix}$$

Sign Flip effect on angle range

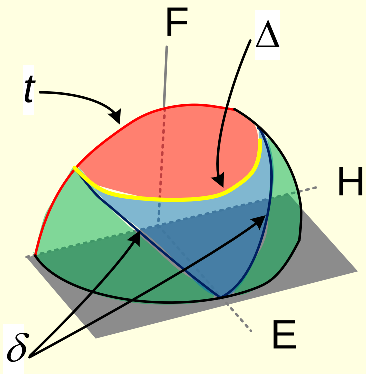




$$-10^\circ \not\cong +10^\circ$$

$$+170^\circ \cong +10^\circ$$

Further transform that keeps G=0



$$\begin{bmatrix} \tilde{E} \\ \tilde{F} \\ \tilde{G} \\ \tilde{H} \end{bmatrix} = \begin{bmatrix} tv-us & 0 & 0 \\ 0 & tv+us & -ts-uv \\ 0 & -tu+sv & \frac{1}{2}\left(tt-ss+uu-vv\right) \\ 0 & -tu-sv & \frac{1}{2}\left(tt+ss+uu+vv\right) \end{bmatrix} \begin{bmatrix} E \\ F \\ H \end{bmatrix}$$

$$-tu+sv=0$$

$$tt-ss+uu-vv=0$$

$$\begin{Bmatrix} s=u \\ v=t \end{Bmatrix} or \begin{Bmatrix} s=-u \\ v=-t \end{Bmatrix}$$

$$\begin{bmatrix} t & u \\ u & t \end{bmatrix}$$ = Diagonal scale

Effect of Diagonal Scale

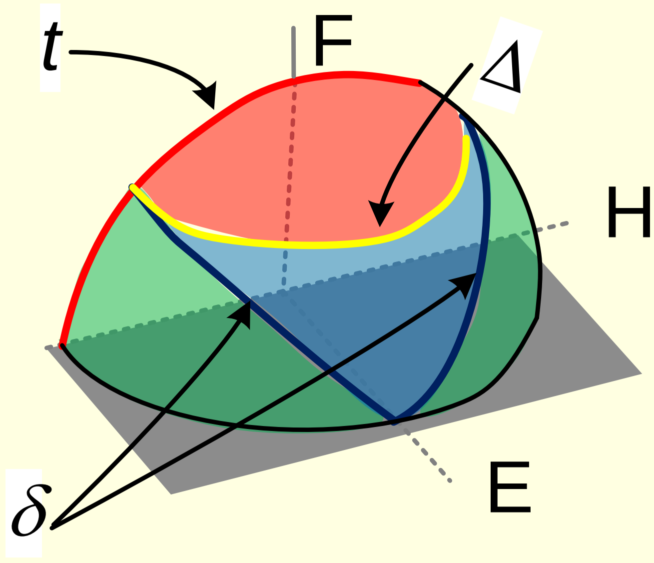
$$\begin{bmatrix} t & u \\ u & t \end{bmatrix} \Rightarrow \begin{bmatrix} \tilde{E} \\ \tilde{F} \\ \tilde{H} \end{bmatrix} = \begin{bmatrix} tt-uu & 0 & 0 \\ 0 & tt+uu & -2tu \\ 0 & -2tu & tt+uu \end{bmatrix} \begin{bmatrix} E \\ F \\ H \end{bmatrix}$$

$$\delta = 4\left(F^2 - H^2\right)$$

$$t^2 = 4E^2$$

$$\tilde{E}^2 = \left(tt-uu\right)^2 E^2$$

$$\tilde{F}^2 - \tilde{H}^2 = \left(tt-uu\right)^2 \left(\tilde{F}^2 - \tilde{H}^2\right)$$

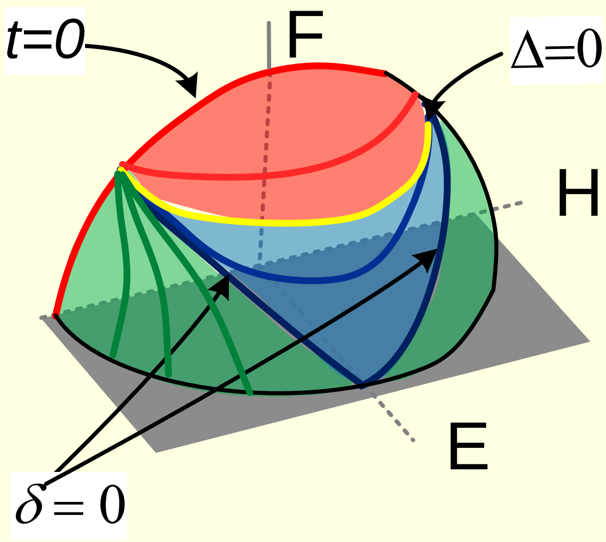


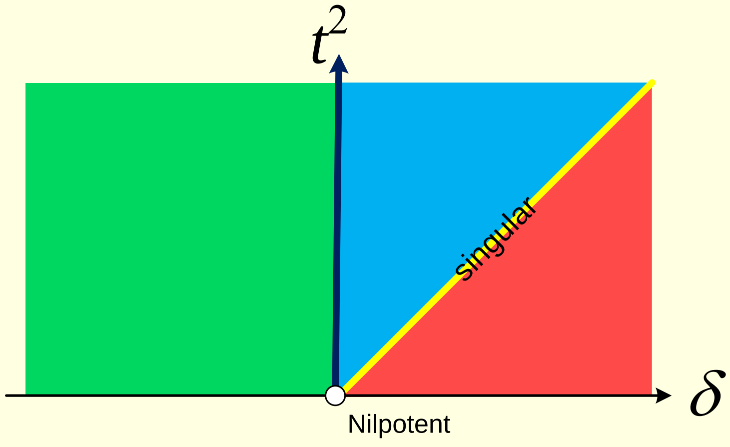
$$\frac{\tilde{E}^2}{\tilde{F}^2 - \tilde{H}^2} = \frac{E^2}{\left(F^2 - H^2\right)} = \frac{t^2}{\delta}$$

$$0 = \delta \tilde{E}^2 - t^2\left(\tilde{F}^2 - \tilde{H}^2\right)$$

Effect of Diagonal Scale

t^2	δ	$\delta\tilde{E}^2 + t^2\left(\tilde{H}^2 - \tilde{F}^2\right)$	
0	>0	$\delta\tilde{E}^2$	plane
>0	>0	$\delta\tilde{E}^2 + \left(t\tilde{H}\right)^2 - \left(t\tilde{F}\right)^2$	cone along F axis
>0	0	$\left(t\tilde{H}\right)^2 - \left(t\tilde{F}\right)^2$	intersecting planes
>0	<0	$\underset{neg}{\delta}\,\tilde{E}^2 + \left(t\tilde{H}\right)^2 - \left(t\tilde{F}\right)^2$	cone along H axis
0	<0	$\underset{neg}{\delta}\,\tilde{E}^2$	plane



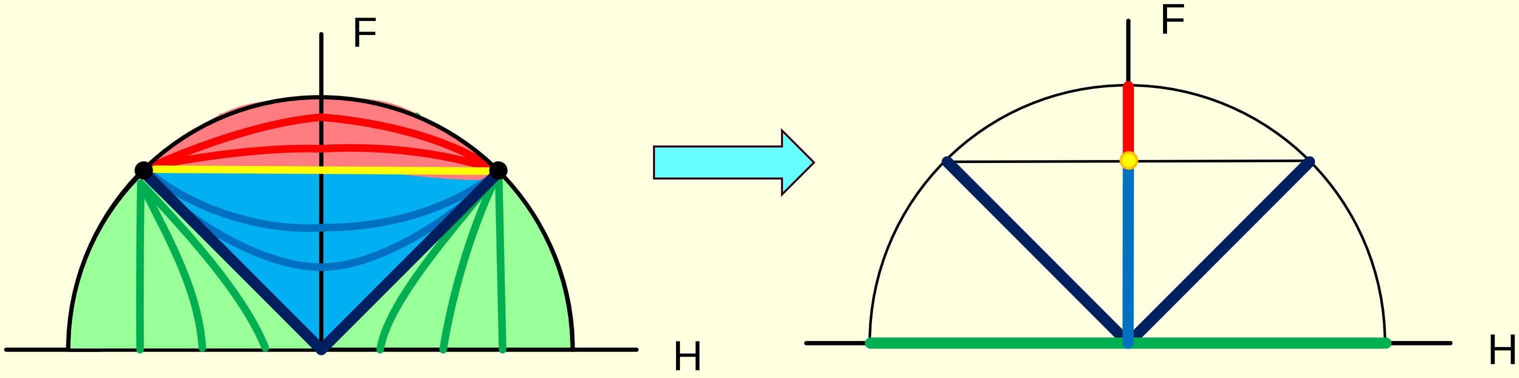


Diagonal Scale to get F or H zero

$$\begin{bmatrix} t & u \\ u & t \end{bmatrix} \Rightarrow \begin{bmatrix} \tilde{E} \\ \tilde{F} \\ \tilde{H} \end{bmatrix} = \begin{bmatrix} tt-uu & 0 & 0 \\ 0 & tt+uu & -2tu \\ 0 & -2tu & tt+uu \end{bmatrix} \begin{bmatrix} E \\ F \\ H \end{bmatrix}$$

$\tilde{F} = ttF - 2tuH + uuF, \quad \text{discrim} = H^2 - F^2$ If positive can make F=0

$\tilde{H} = ttH - 2tuF + uuH, \quad \text{discrim} = F^2 - H^2$ If positive can make H=0



δ=0 case

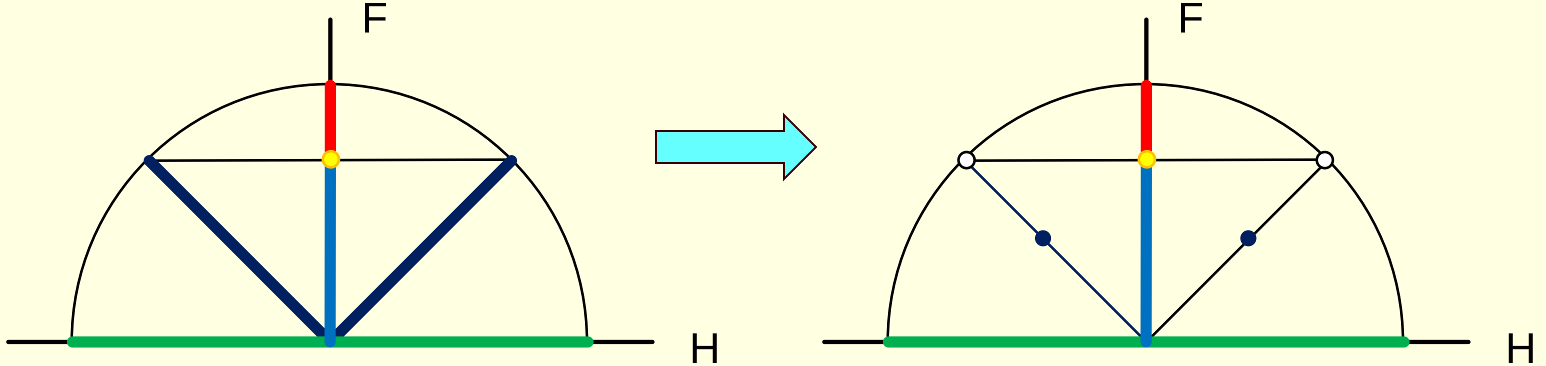
$$\begin{bmatrix}\tilde{E}\\ \tilde{F}\\ \tilde{H}\end{bmatrix}=\begin{bmatrix}tt-uu & 0 & 0\\ 0 & tt+uu & -2tu\\ 0 & -2tu & tt+uu\end{bmatrix}\begin{bmatrix}E\\ F\\ H\end{bmatrix}$$

$$H=-F$$

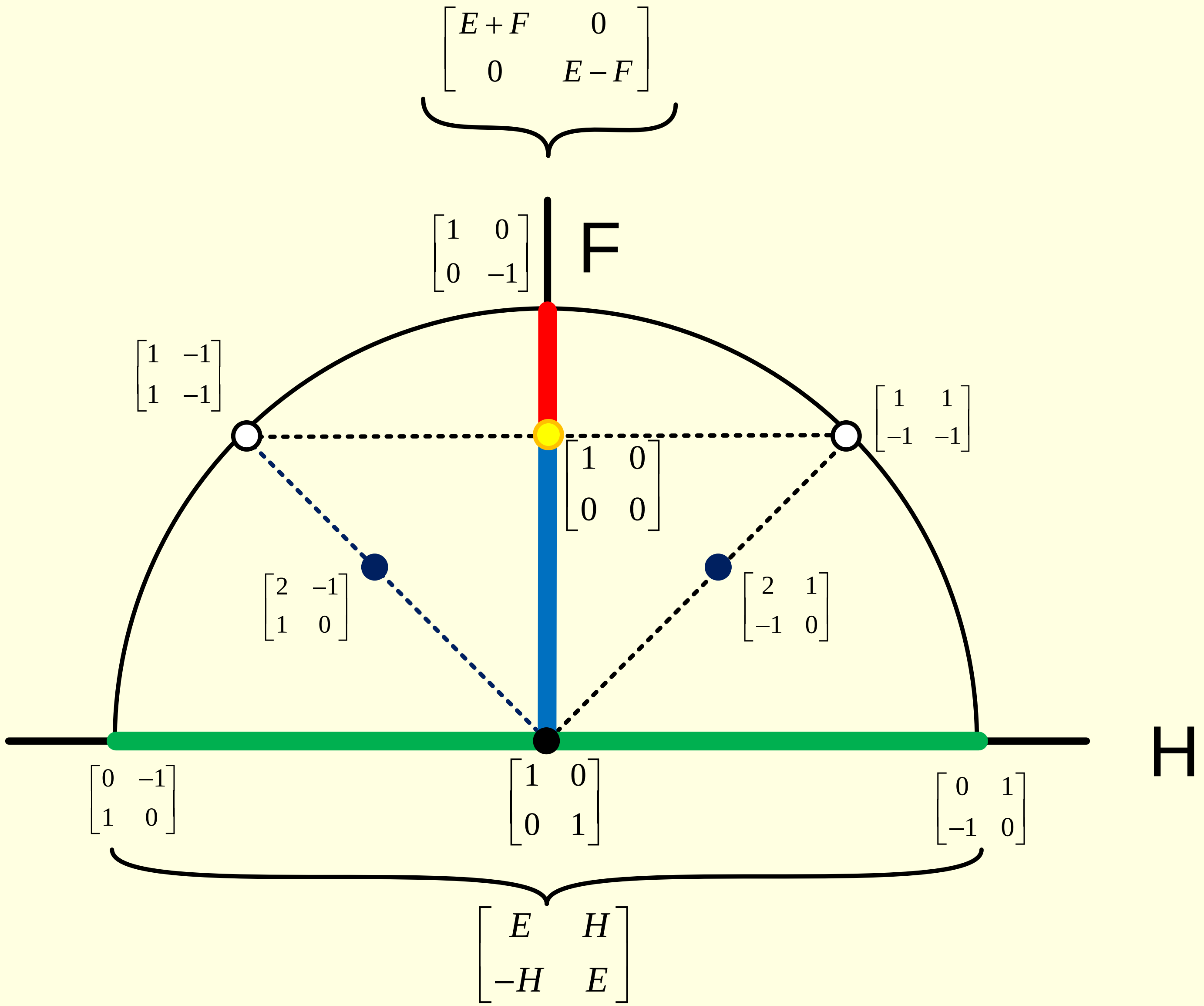
$$\begin{bmatrix}\tilde{E}\\ \tilde{F}\end{bmatrix}=\begin{bmatrix}tt-uu & 0\\ 0 & tt+2tu+uu\end{bmatrix}\begin{bmatrix}E\\ F\end{bmatrix}=(t+u)\begin{bmatrix}t-u & 0\\ 0 & t+u\end{bmatrix}\begin{bmatrix}E\\ F\end{bmatrix}$$

$$H=+F$$

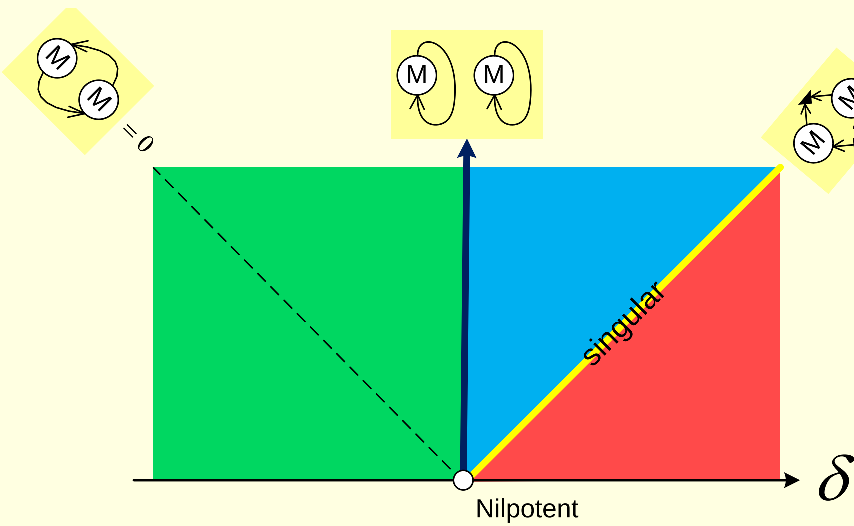
$$\begin{bmatrix}\tilde{E}\\ \tilde{F}\end{bmatrix}=\begin{bmatrix}tt-uu & 0\\ 0 & tt-2tu+uu\end{bmatrix}\begin{bmatrix}E\\ F\end{bmatrix}=(t-u)\begin{bmatrix}t+u & 0\\ 0 & t-u\end{bmatrix}\begin{bmatrix}E\\ F\end{bmatrix}$$

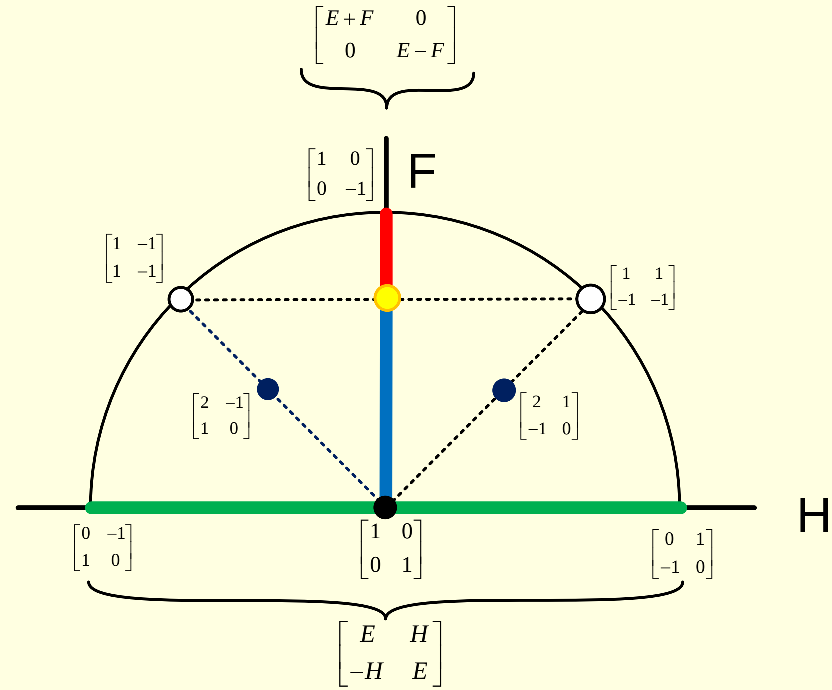


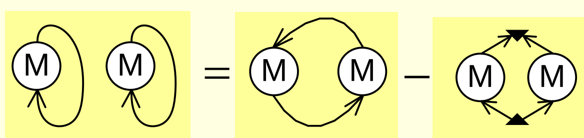
Distilled EFGH Space

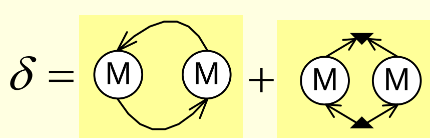


Phase Dgm and Distilled

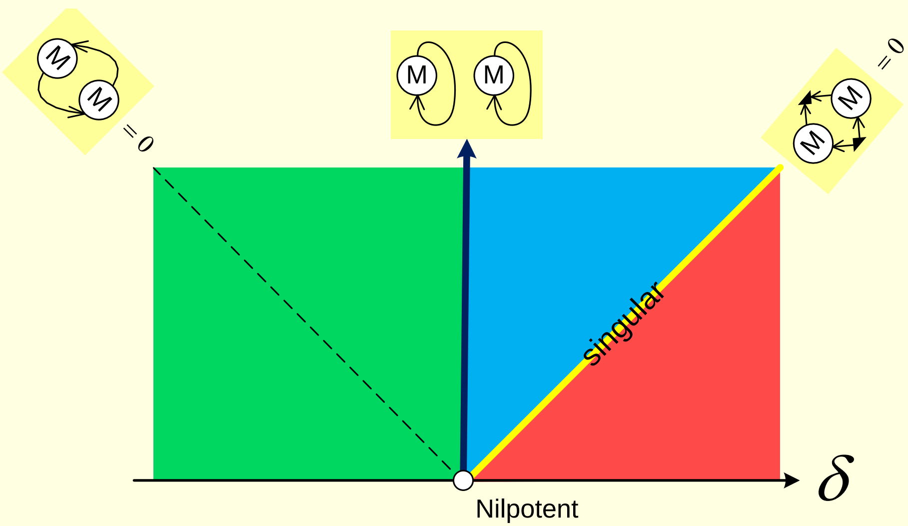


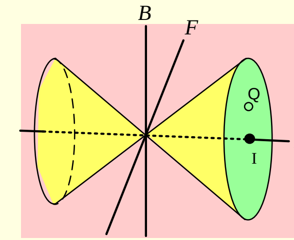


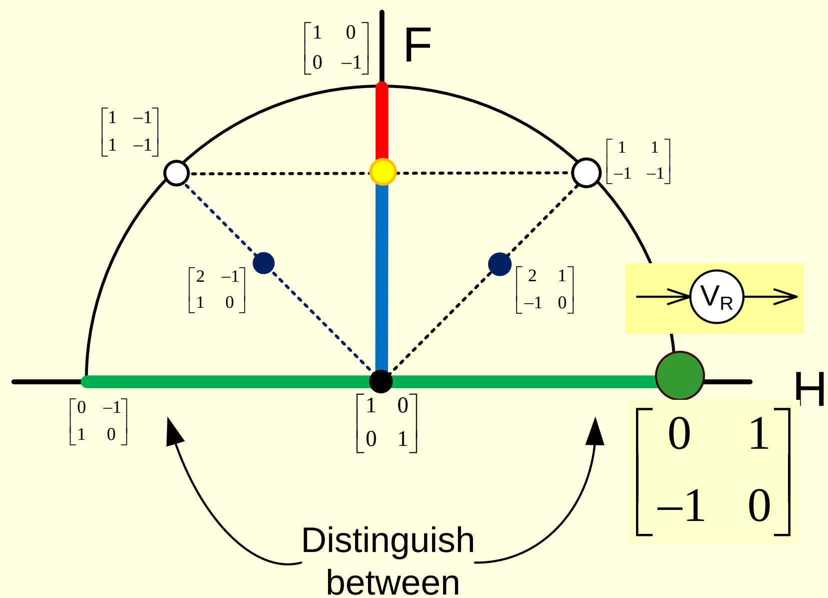


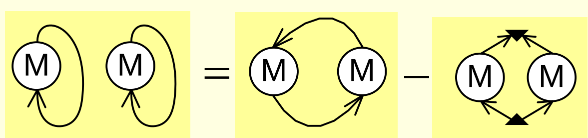


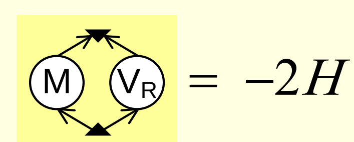
Phase Dgm and Distilled

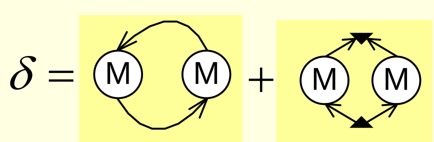


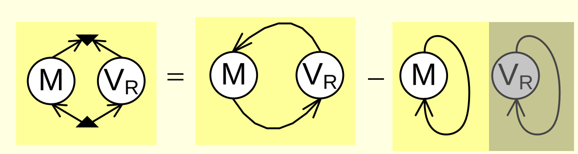




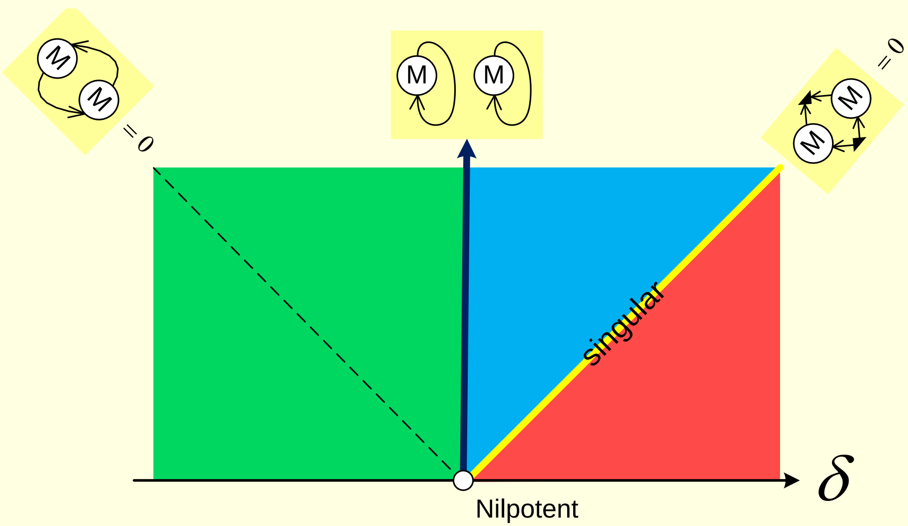


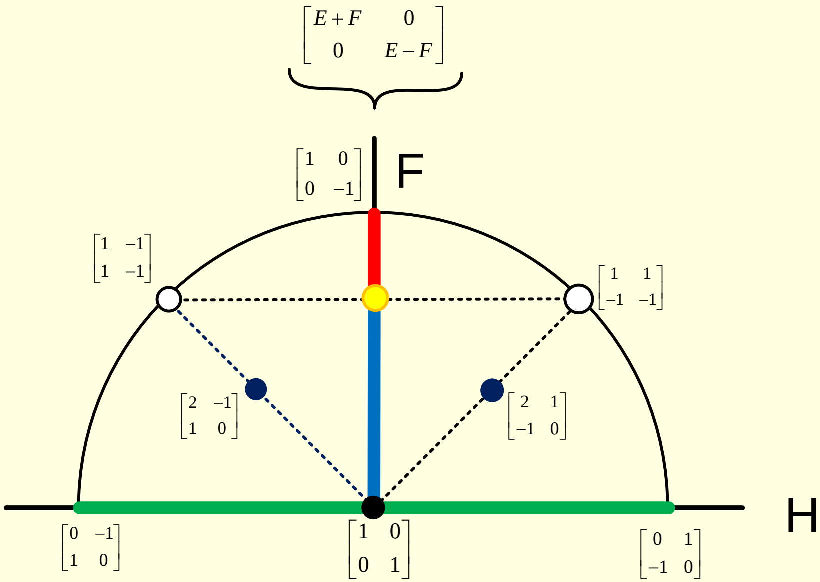


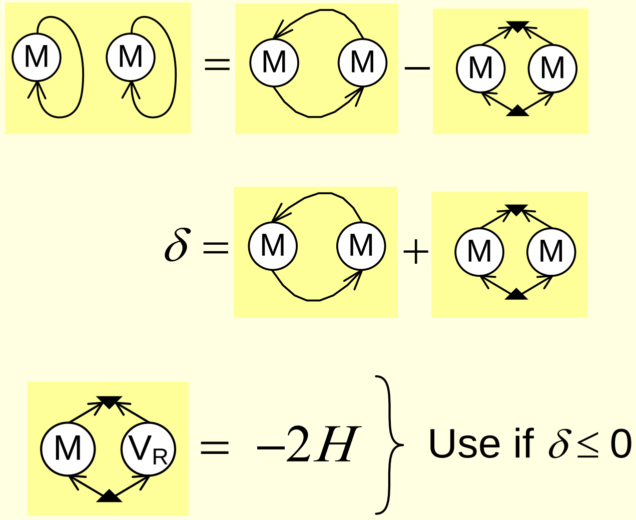




Phase Dgm and Distilled

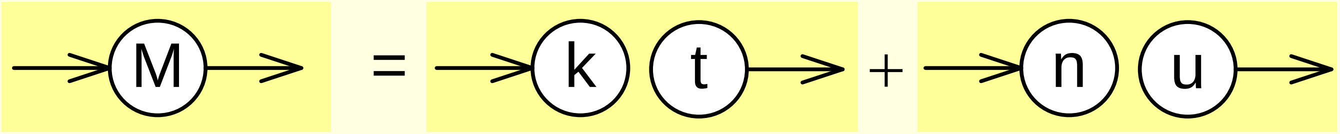






Internal Structure

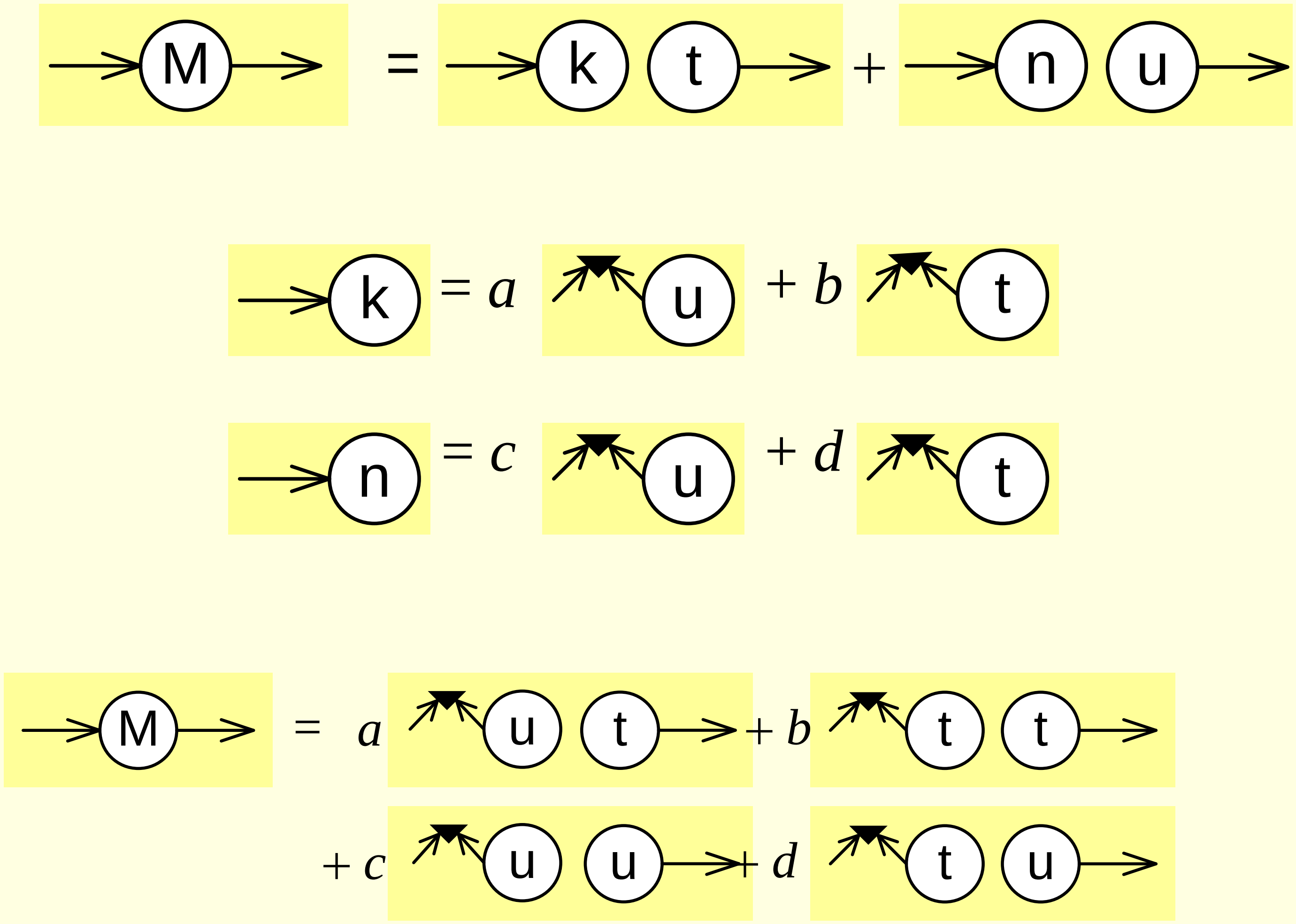
Using outer products to make M



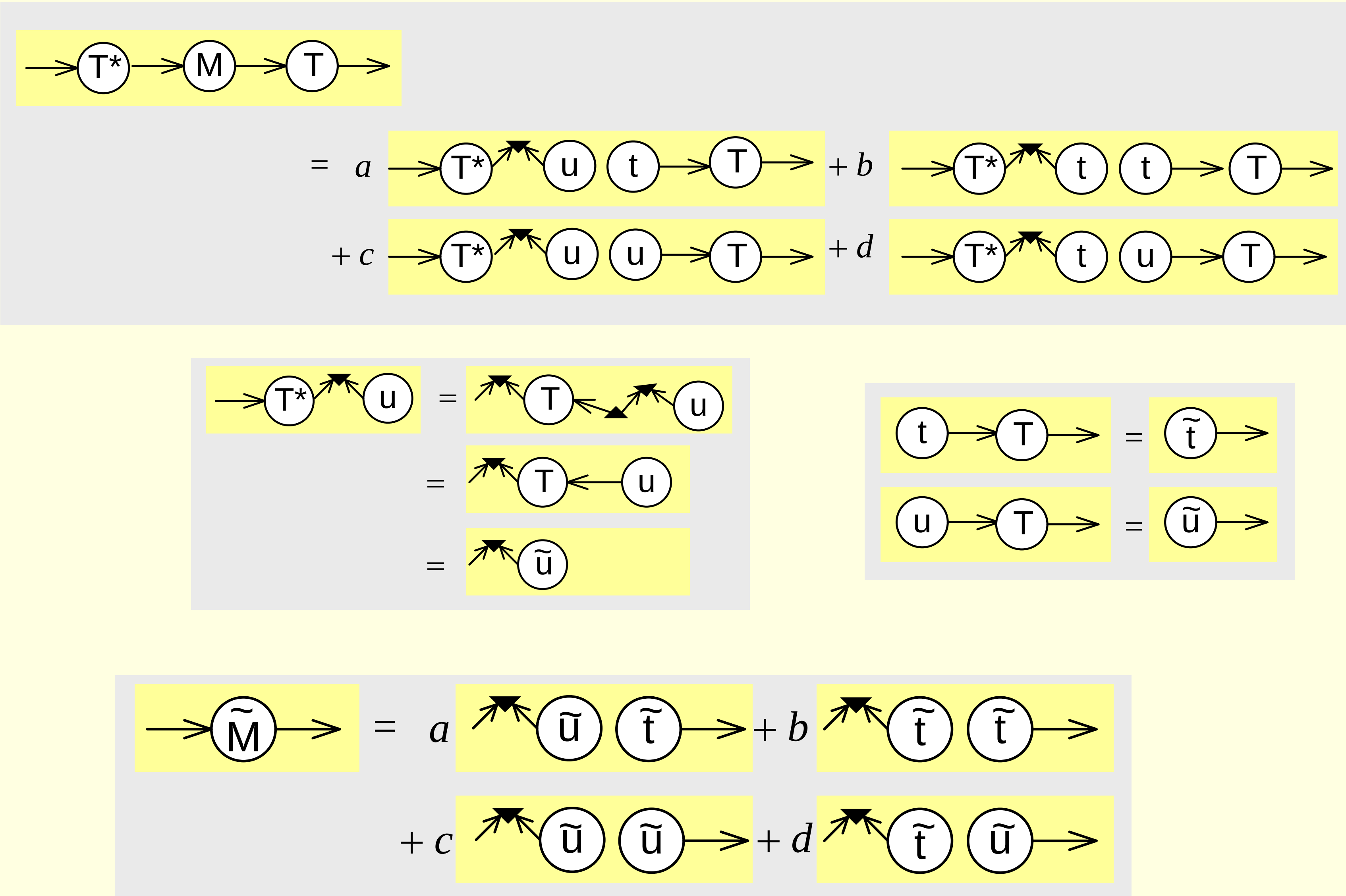
$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A \\ C \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} + \begin{bmatrix} B \\ D \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} A & B \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix}$$

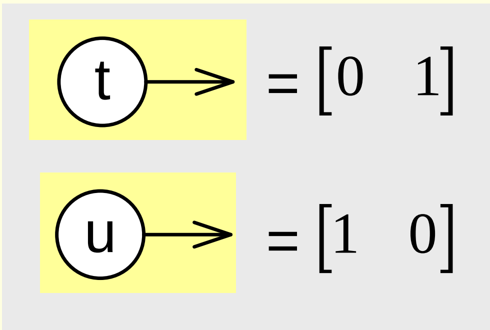
Write k,n in terms of t,u

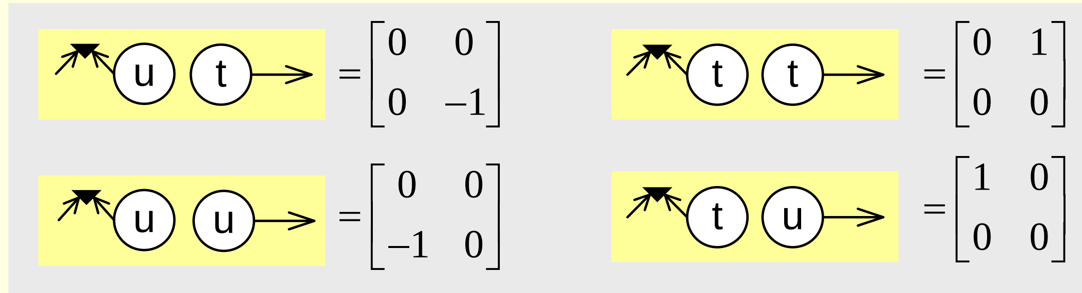


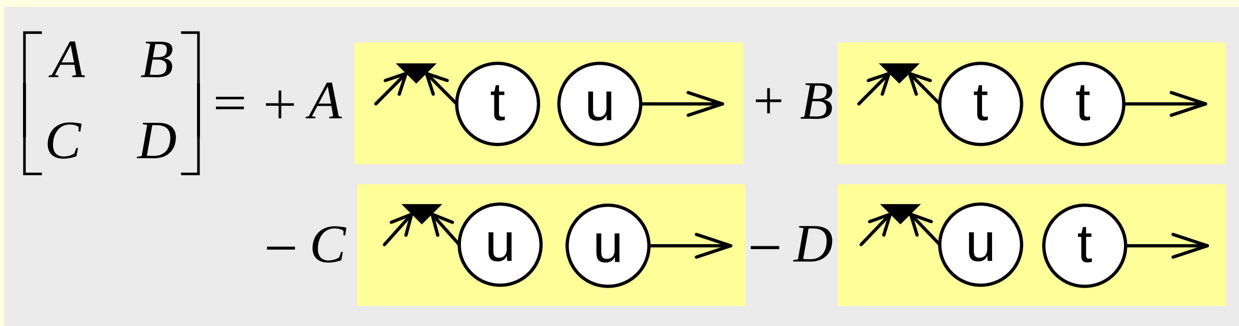
Transformation

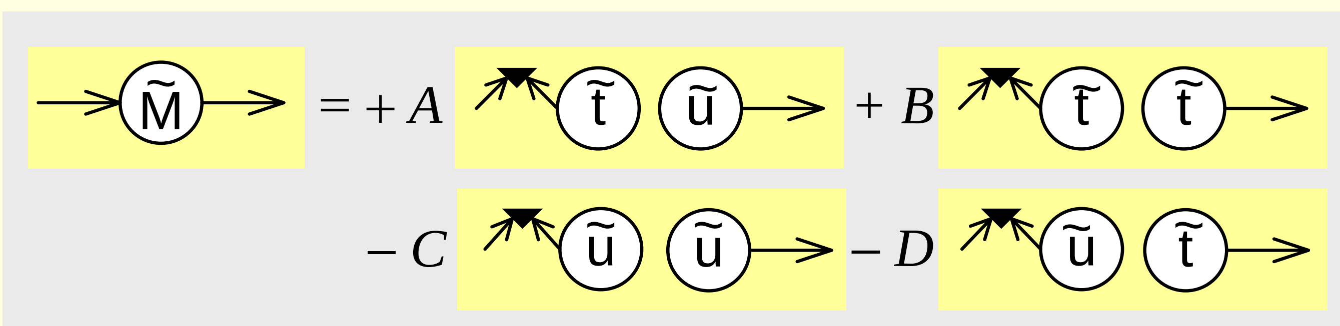


Pick nice t,u

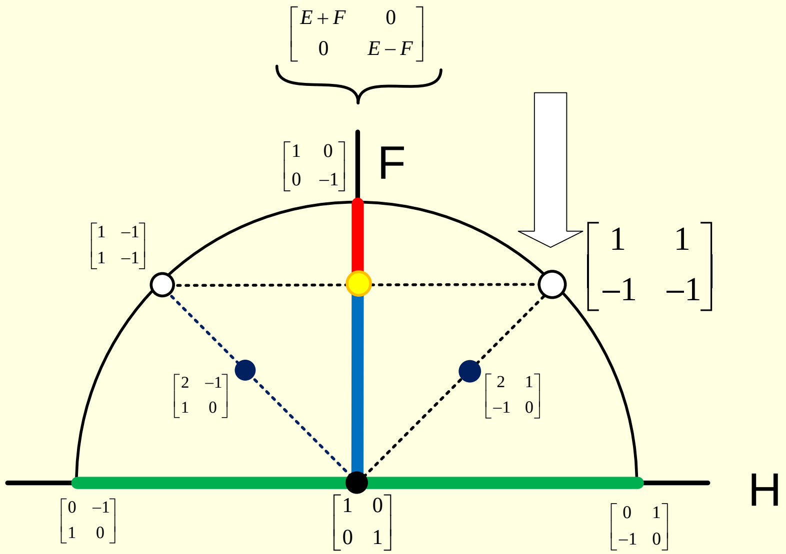

t = [0 1]
u = [1 0]


u t = [0 0; 0 –1]
t t = [0 1; 0 0]
u u = [0 0; –1 0]
t u = [1 0; 0 0]


[A B; C D] = + A t u + B t t
– C u u – D u t


M̃ = + A t̃ ũ + B t̃ t̃
– C ũ ũ – D ũ t̃

Nilpotent



Distilled:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} E+F & G+H \\ G-H & E-F \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} E \\ F \\ G \\ H \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

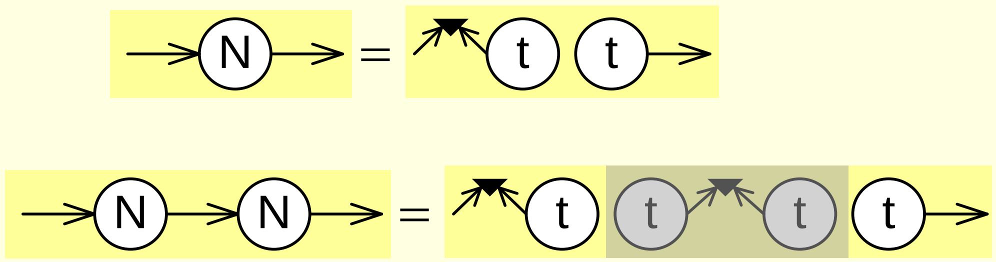
Can Transform to:

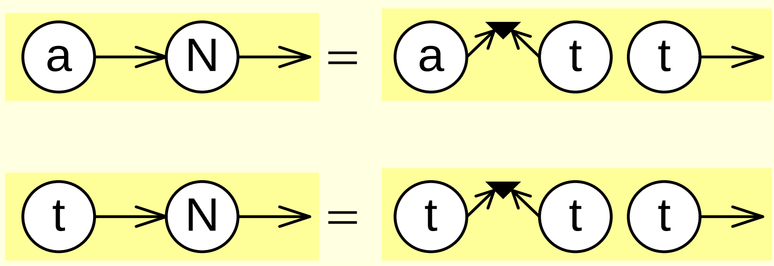
$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} E+F & G+H \\ G-H & E-F \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

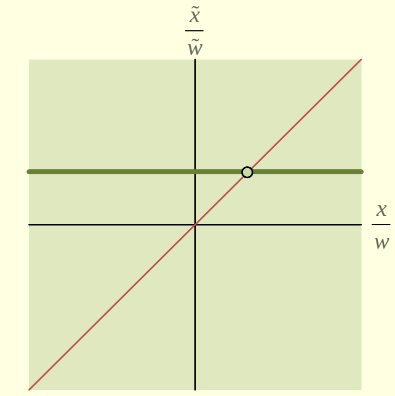
$$\begin{bmatrix} E \\ F \\ G \\ H \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

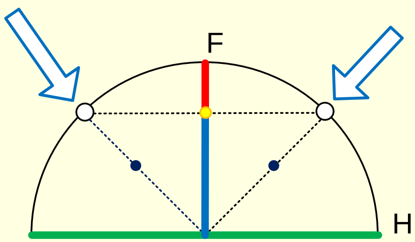
Nilpotent

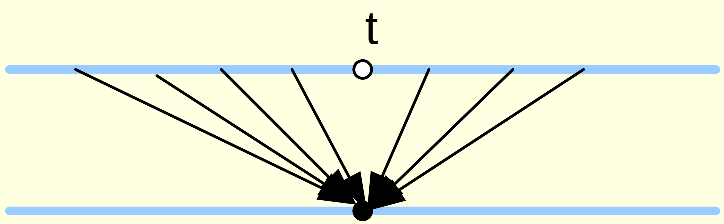
$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

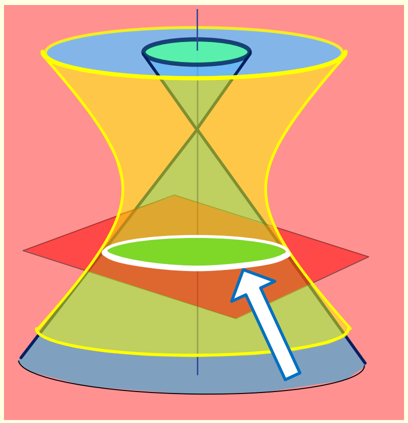


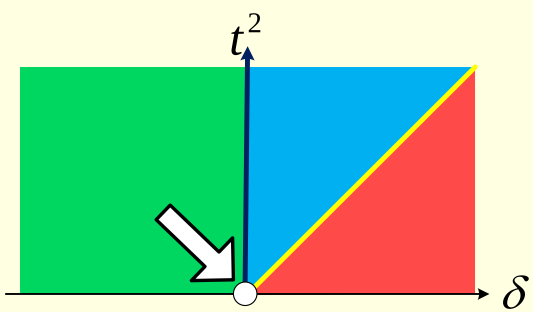






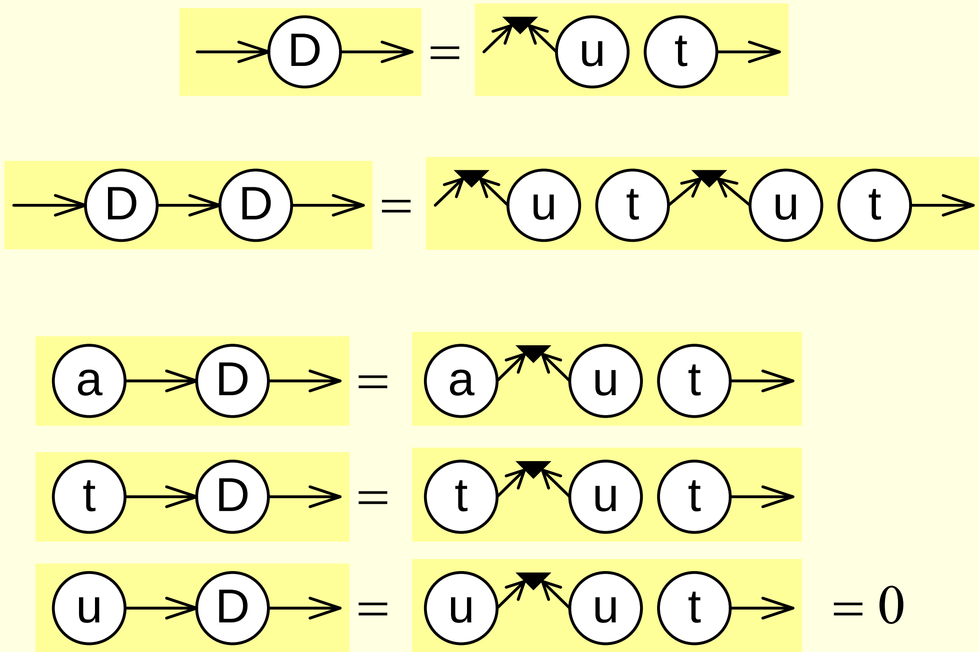


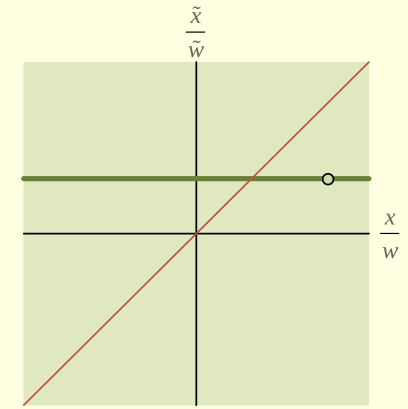


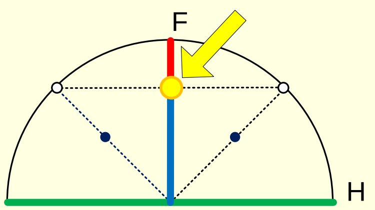


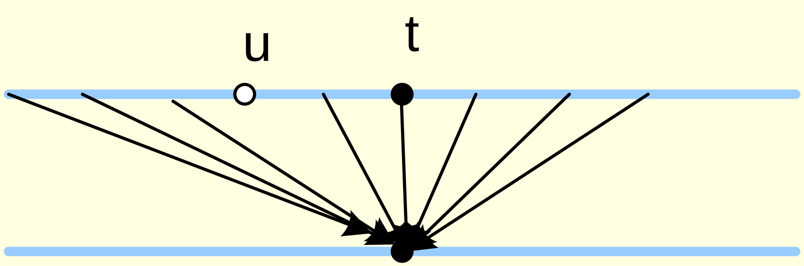
Idempotent

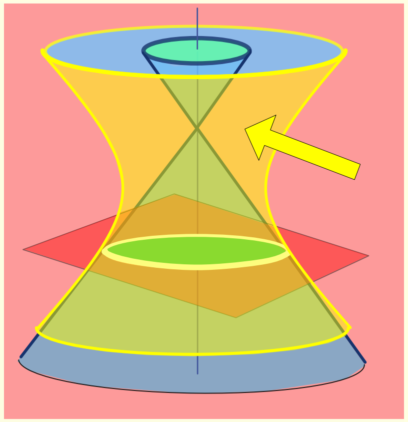
$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

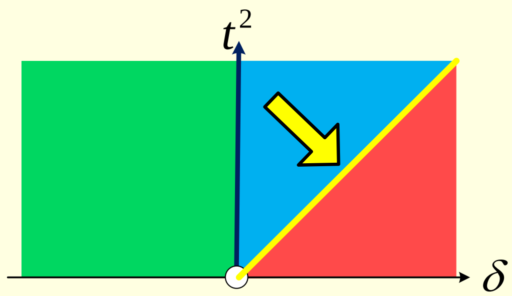




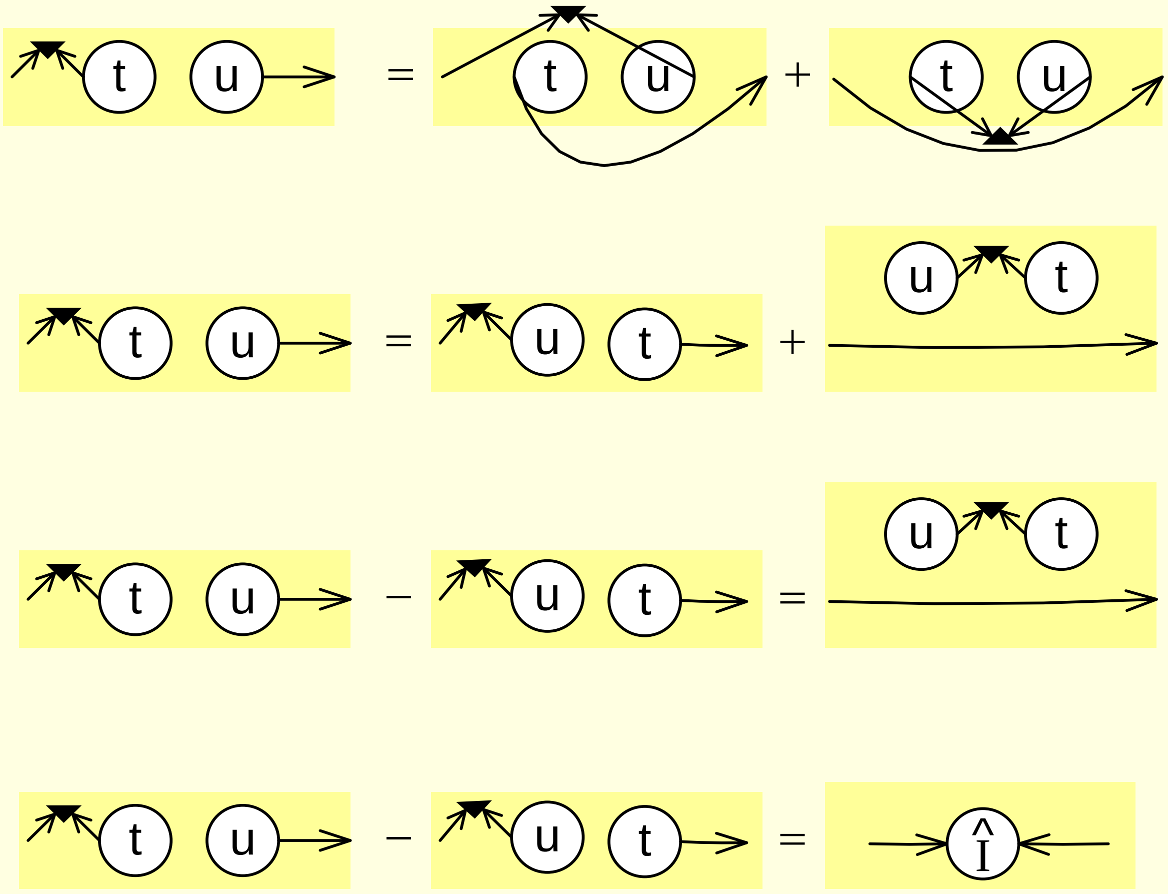


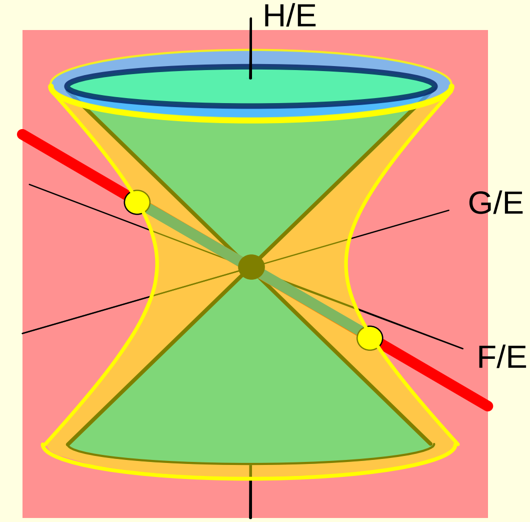






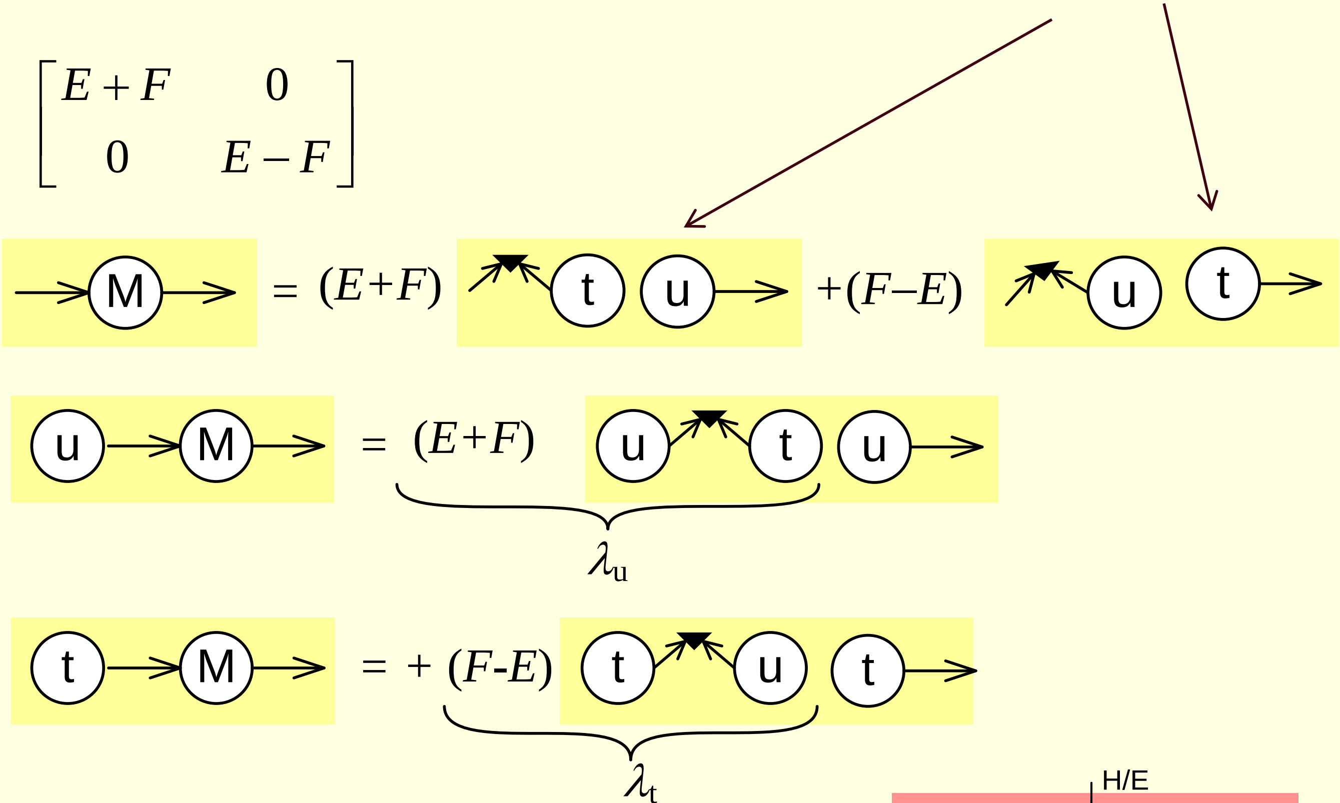
An Identity

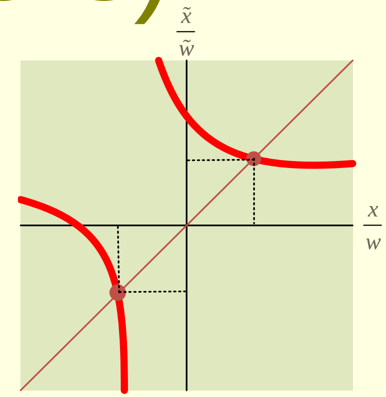


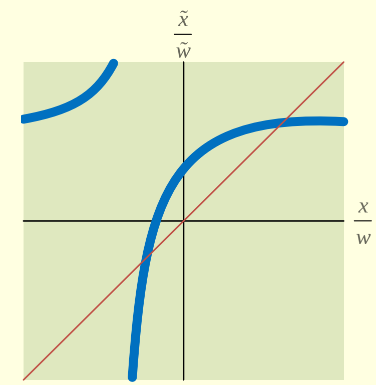


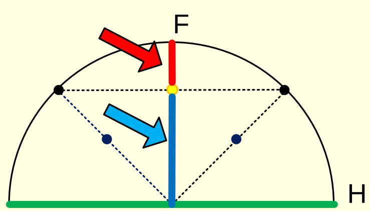
General Scales (Eigenvectors)

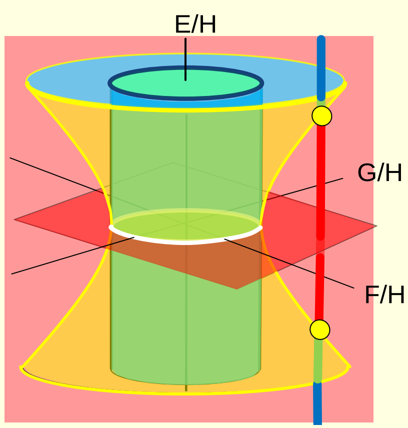
$$\begin{bmatrix} E+F & 0 \\ 0 & E-F \end{bmatrix}$$

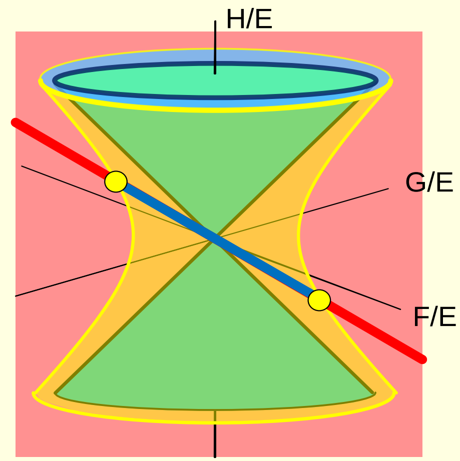


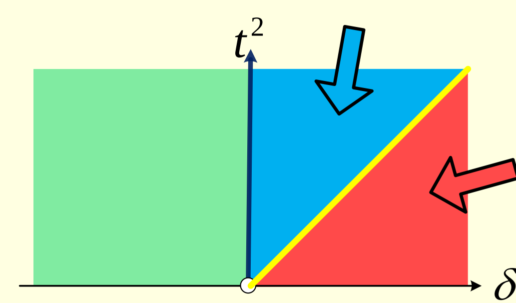






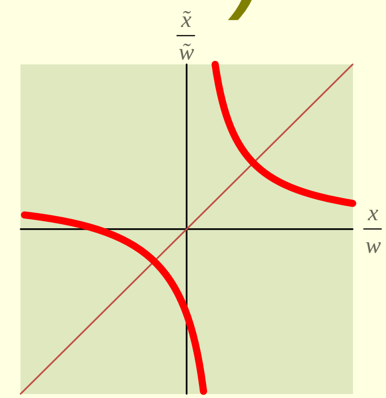


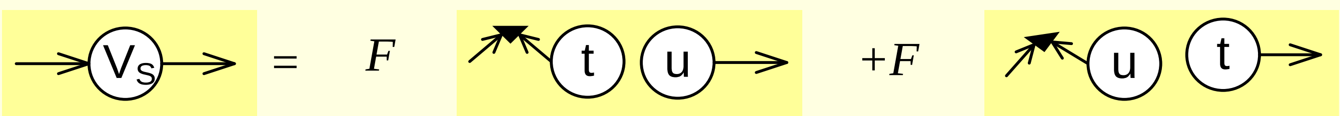


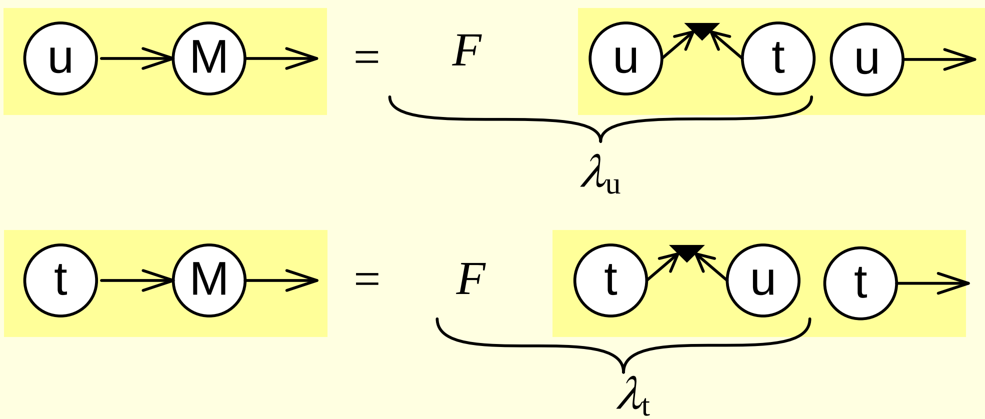


Scale Involution (Eigenvectors)

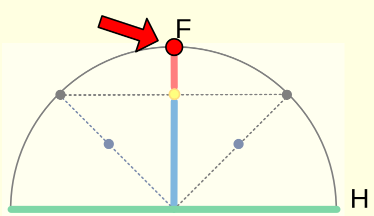
$$\begin{bmatrix} E+F & 0 \\ 0 & E-F \end{bmatrix}, E=0$$

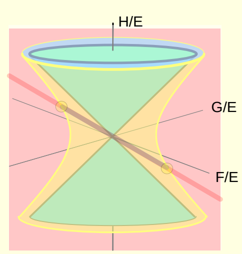


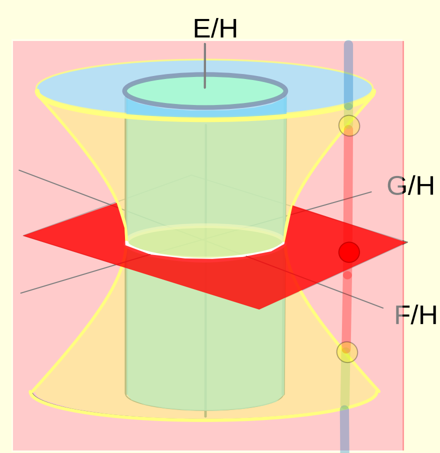


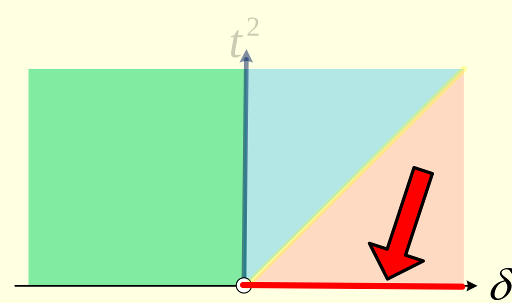


$$\lambda_u = -\lambda_t$$



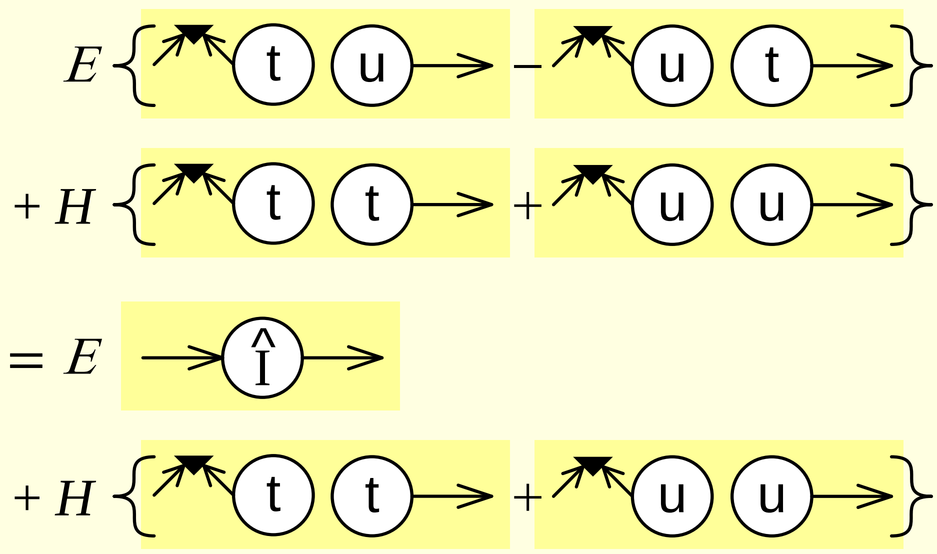


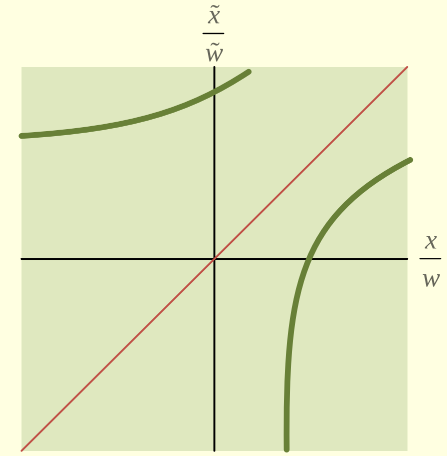


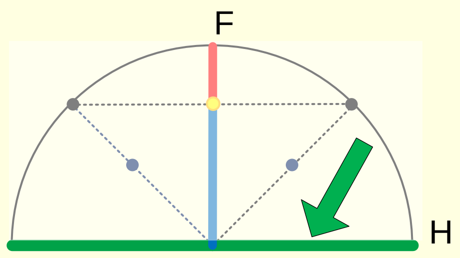


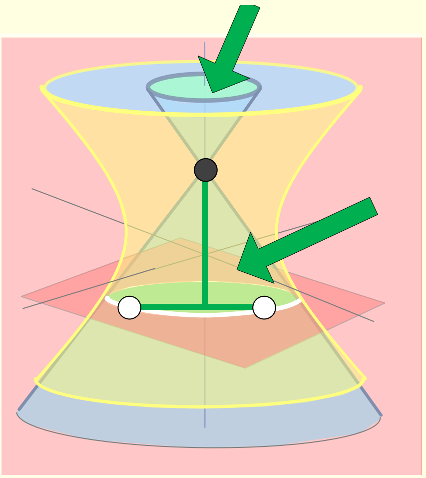
Rotation

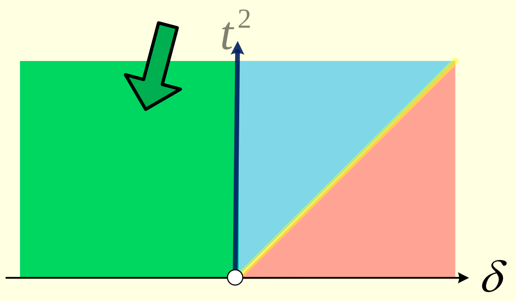
$$\begin{bmatrix} E & H \\ -H & E \end{bmatrix}$$





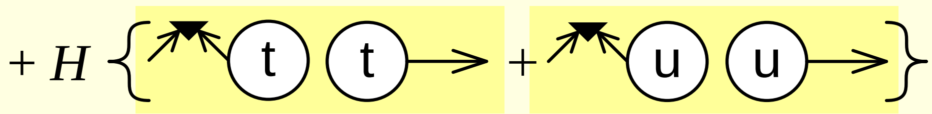


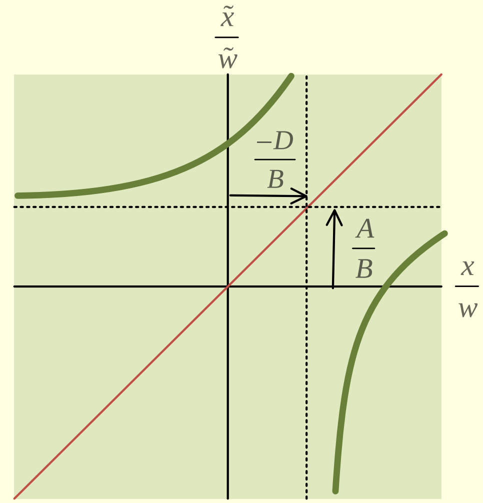


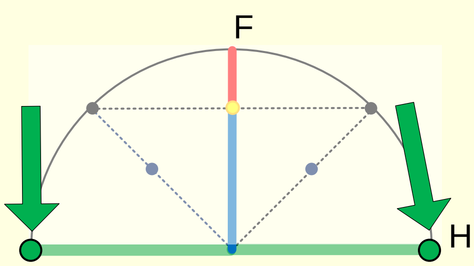


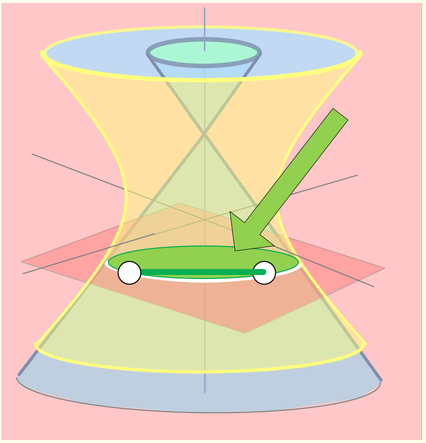
Rotation Involution

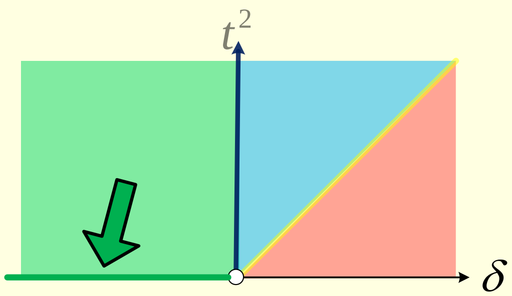
$$\begin{bmatrix} E & H \\ -H & E \end{bmatrix}, E = 0$$





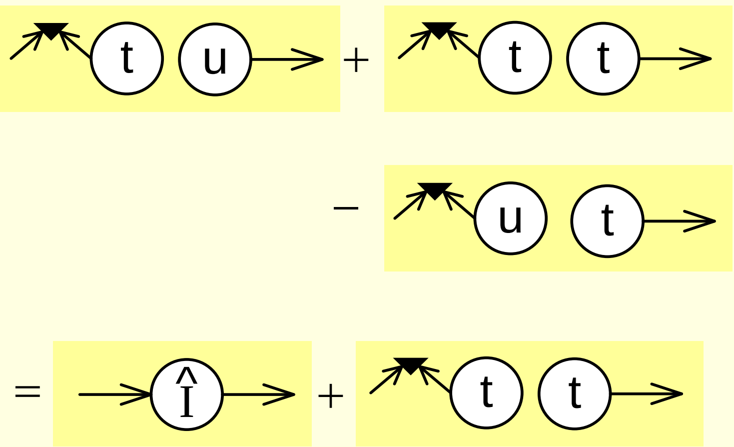


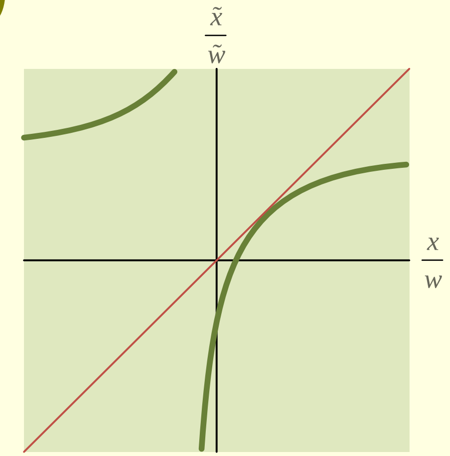


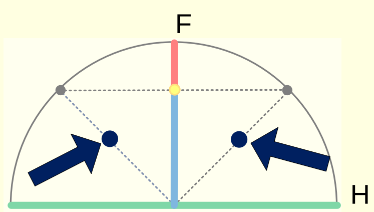


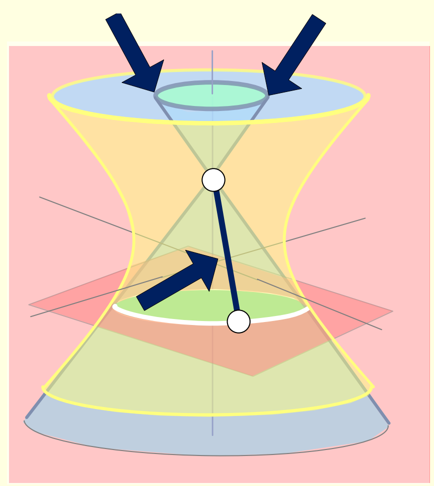
Single eigenvalue $\delta = 0$

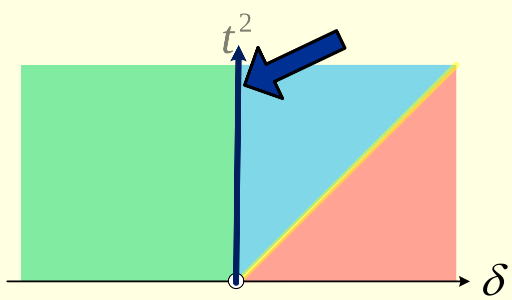
$$\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$$



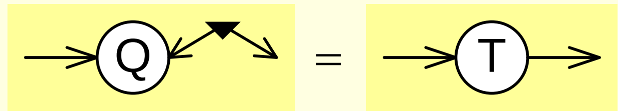


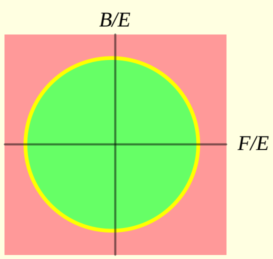


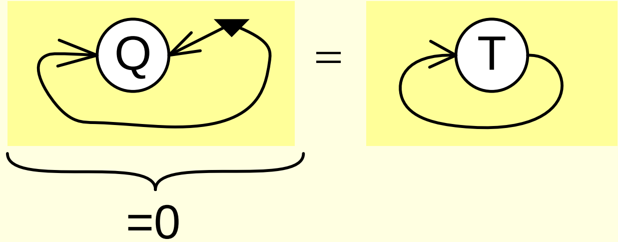


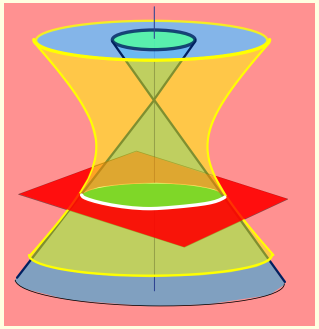


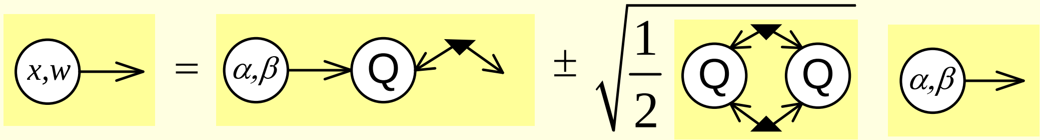
Pure & Mixed Tensors - A Relation

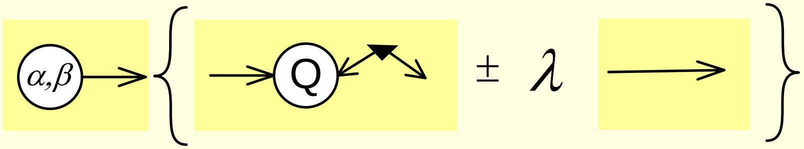


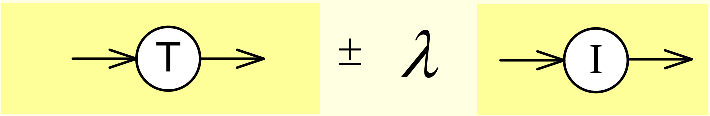












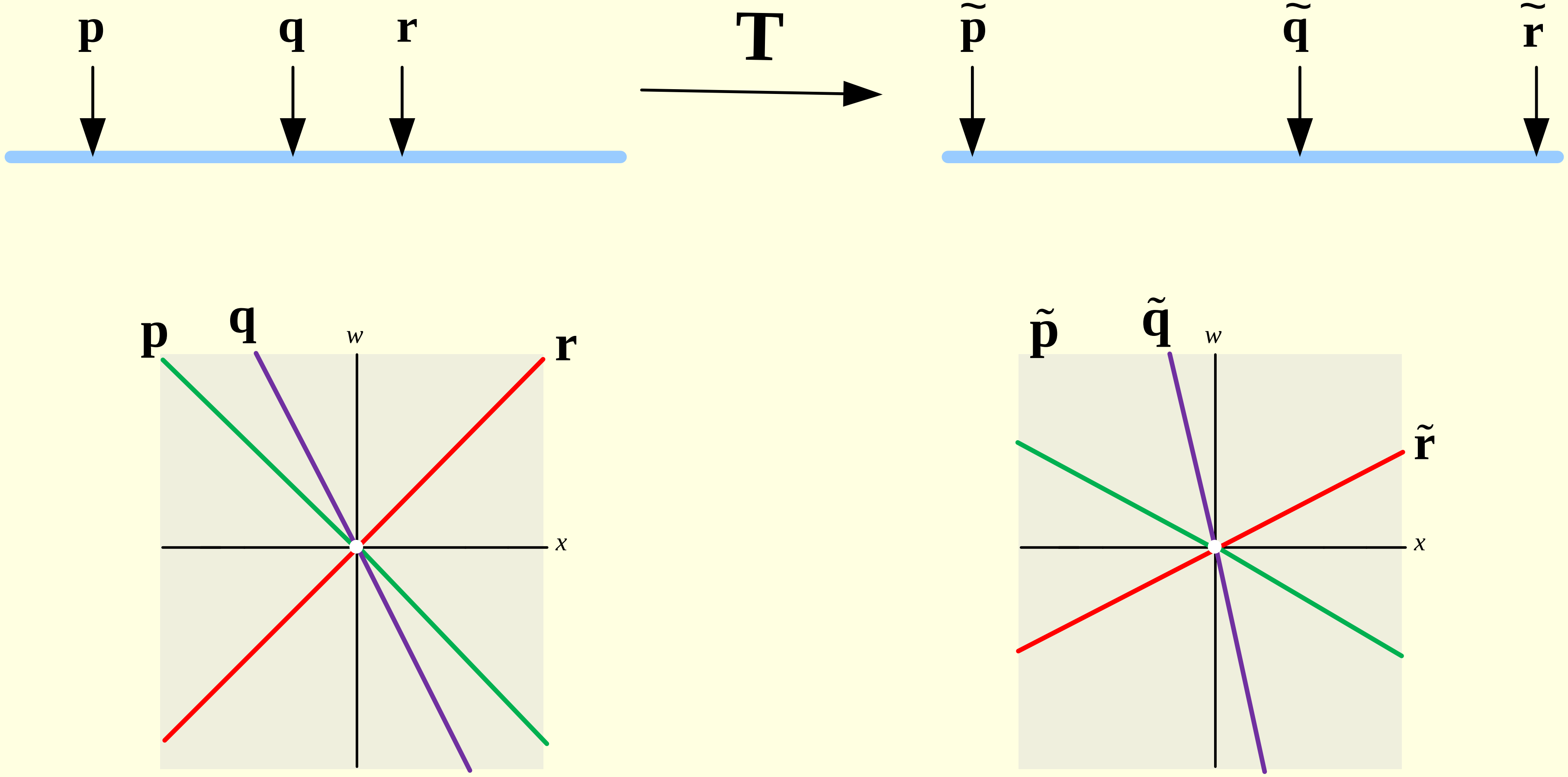
Roots of **Q**

are

Eigenvalues of **T**

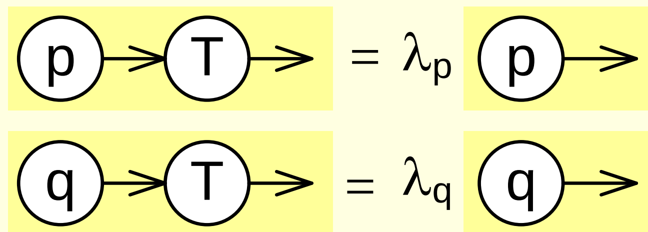
Exemplary Transformations

Exemplary Transformation

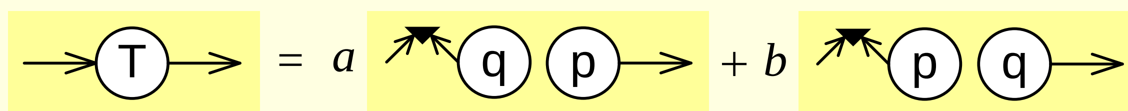


Construct T given Eigenvectors

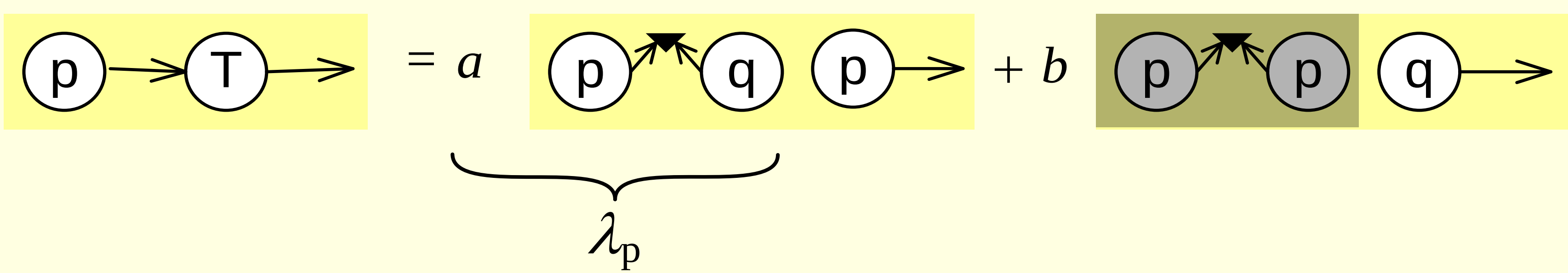
Want



Basic Answer

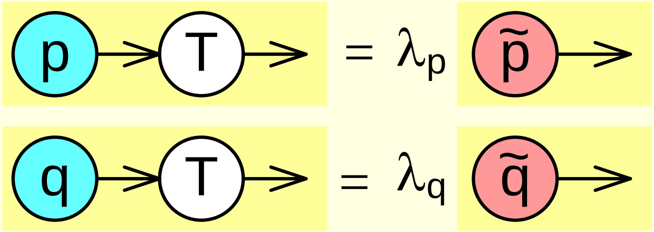


How it works for p

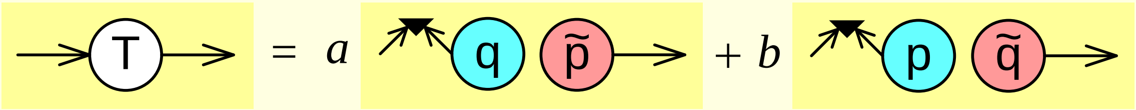


Construct T given two different output points

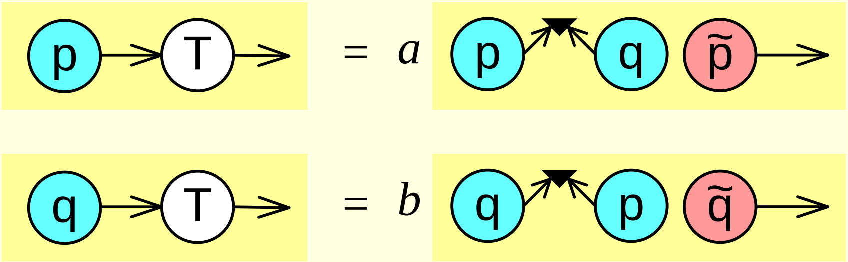
Want



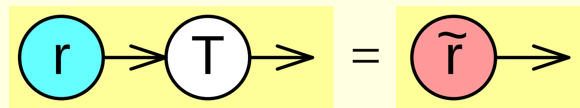
Basic Answer

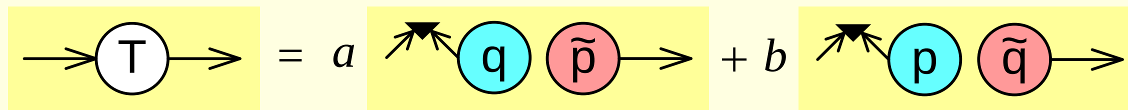


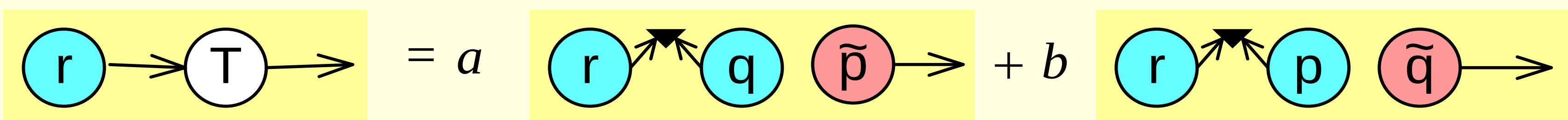
Works for p and q



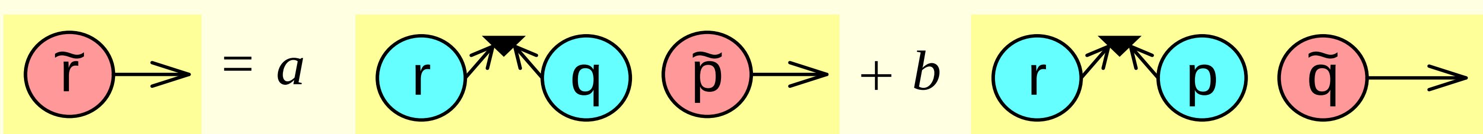
Third point

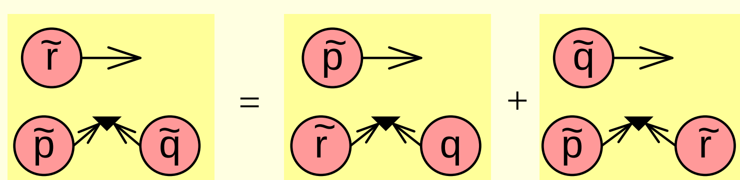




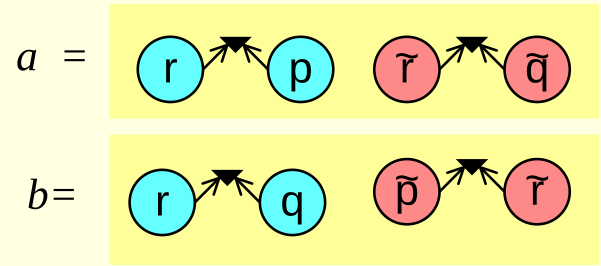


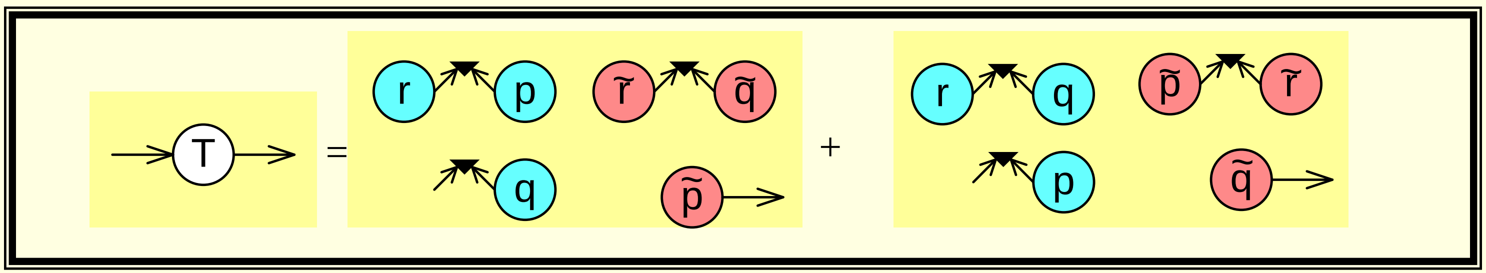
Pick a and b to make



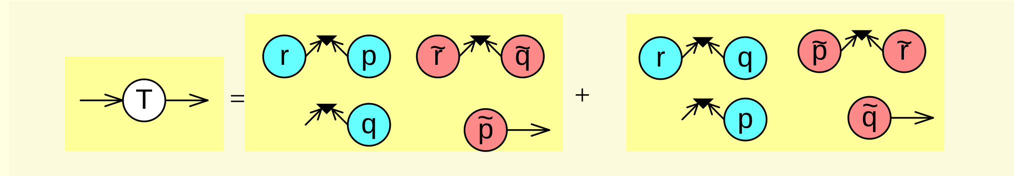


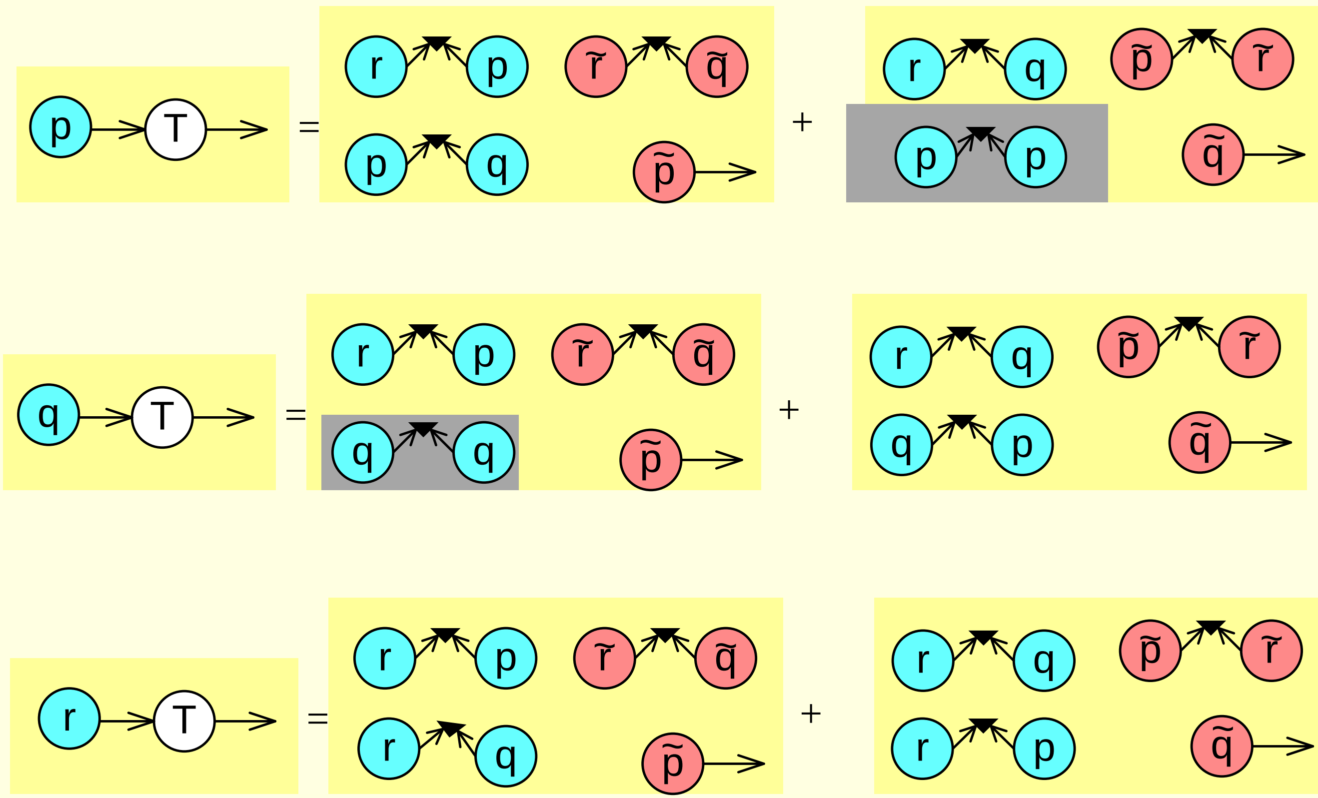
The answer



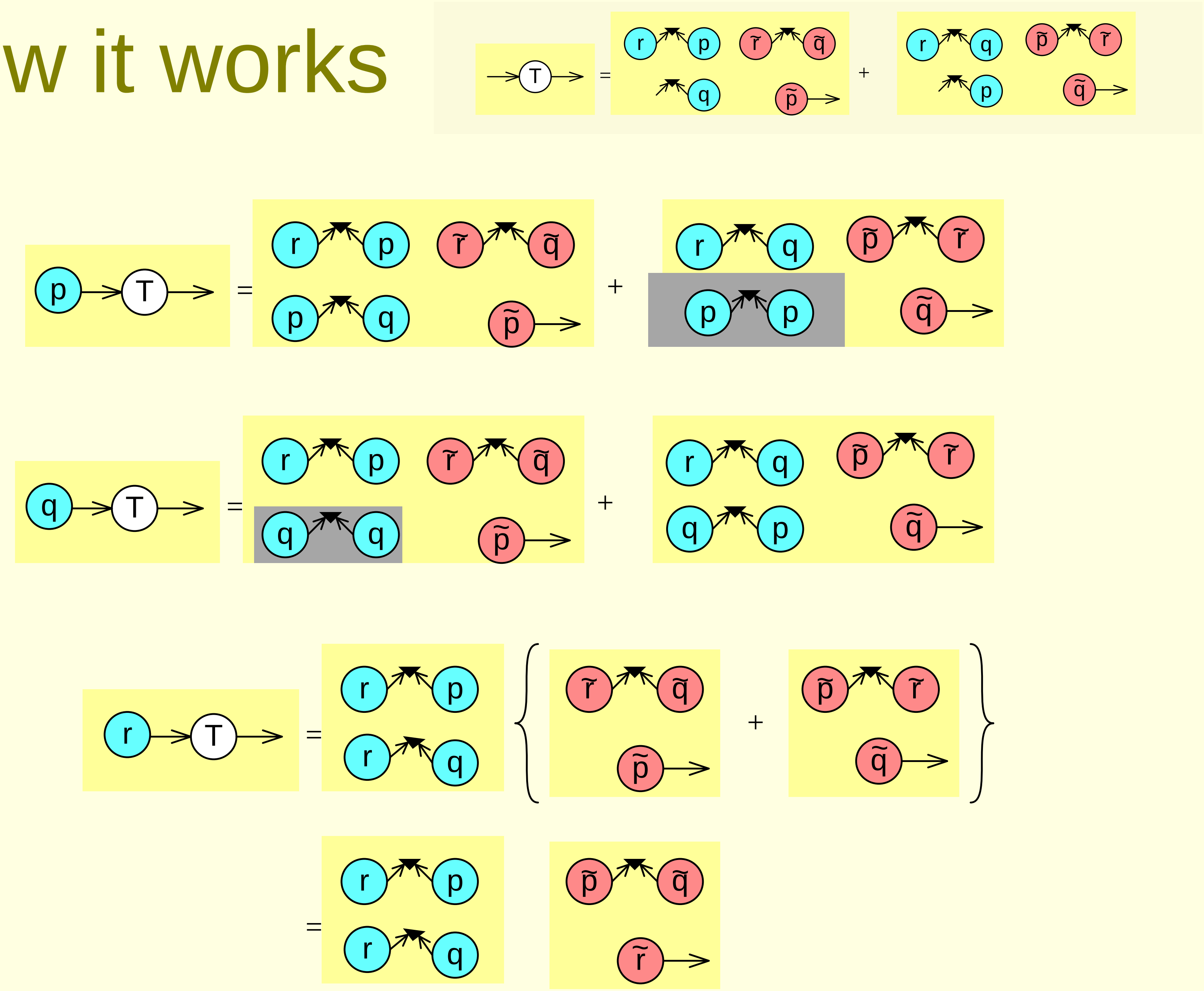


How it works

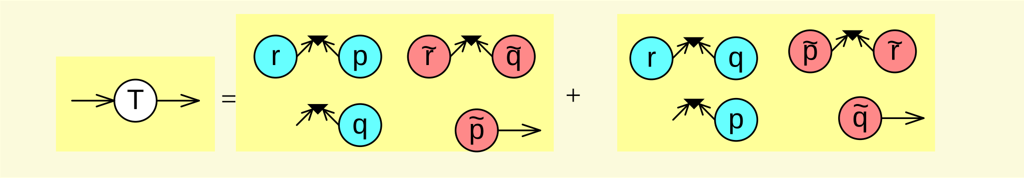


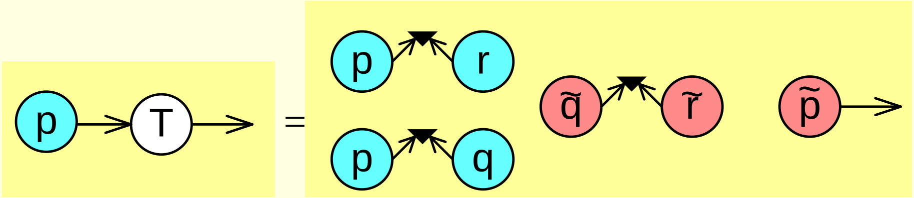


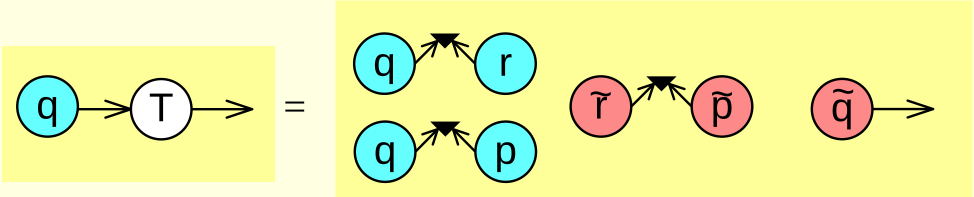
How it works

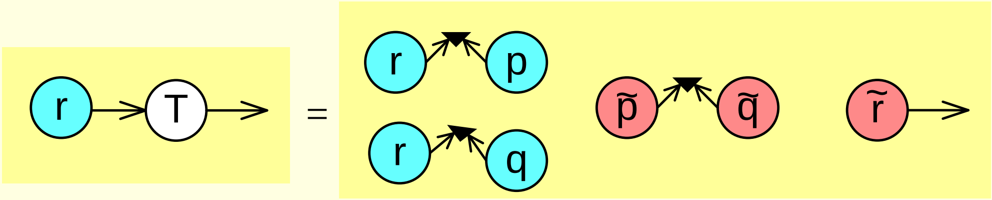


Net

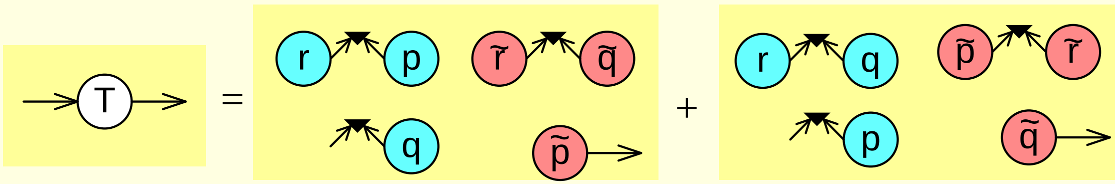


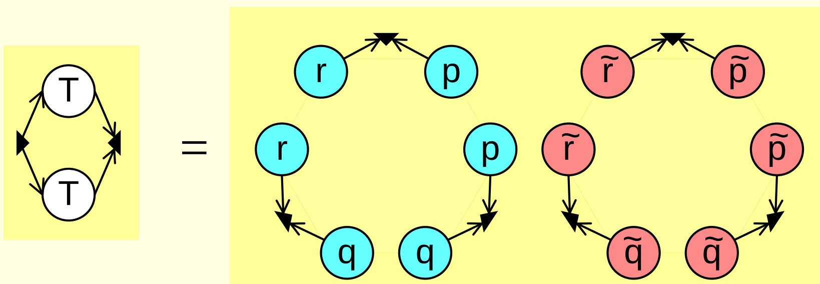






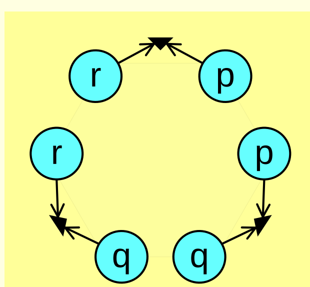
Determinant



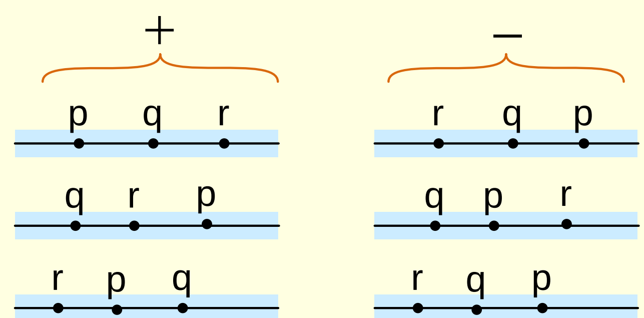


Determinant is negative exactly when necessary for order reversal

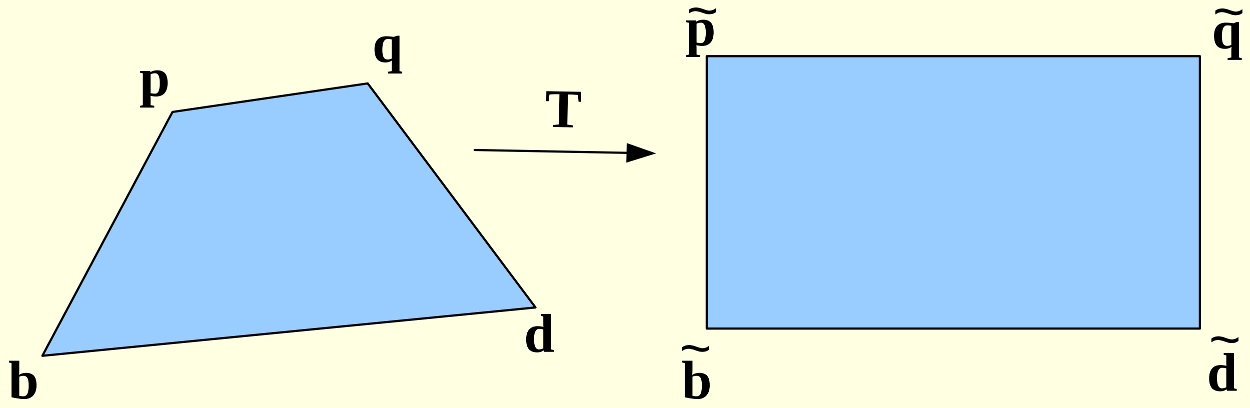
An invariant of the points p,q,r



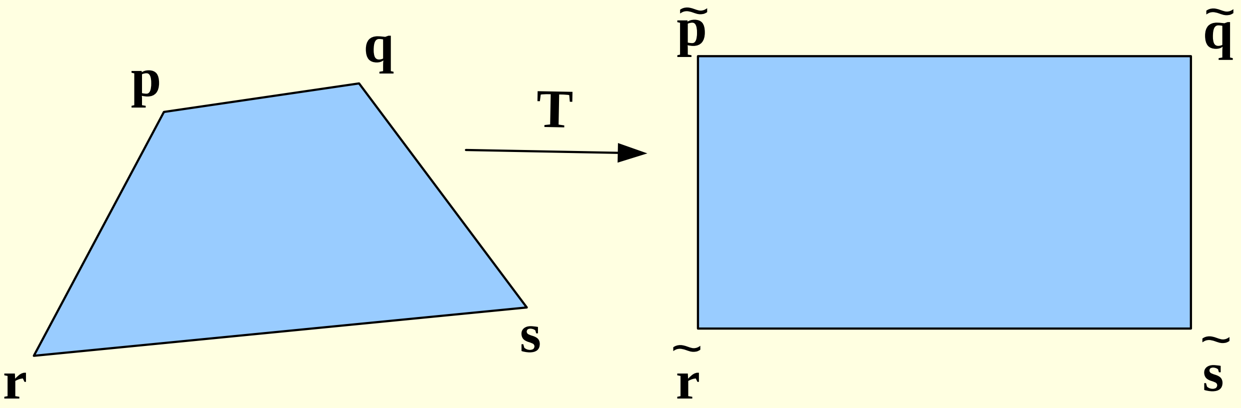
Geometric meaning of sign

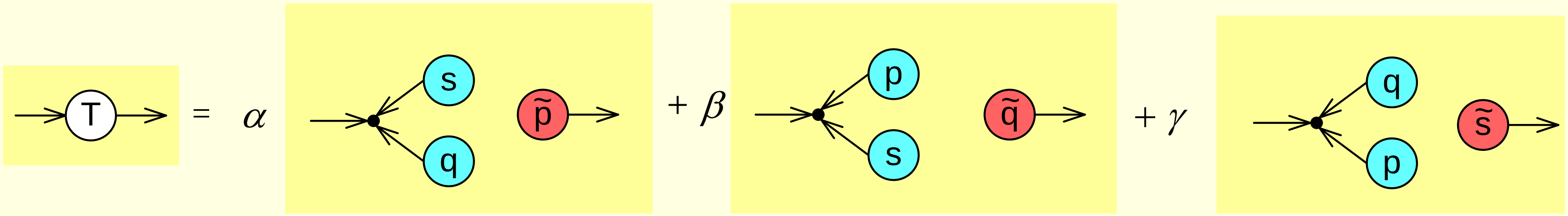


Higher Dimensions



Higher Dimensions

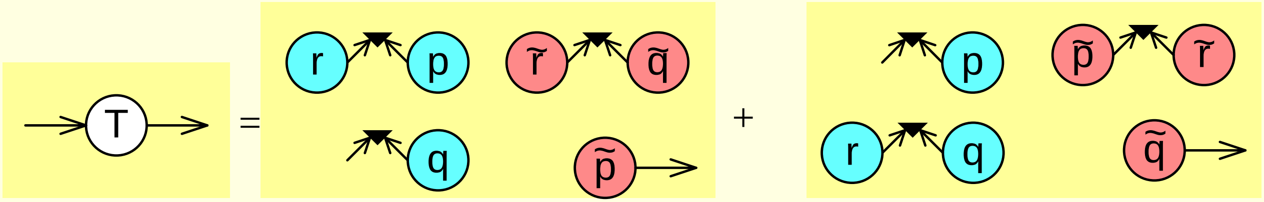


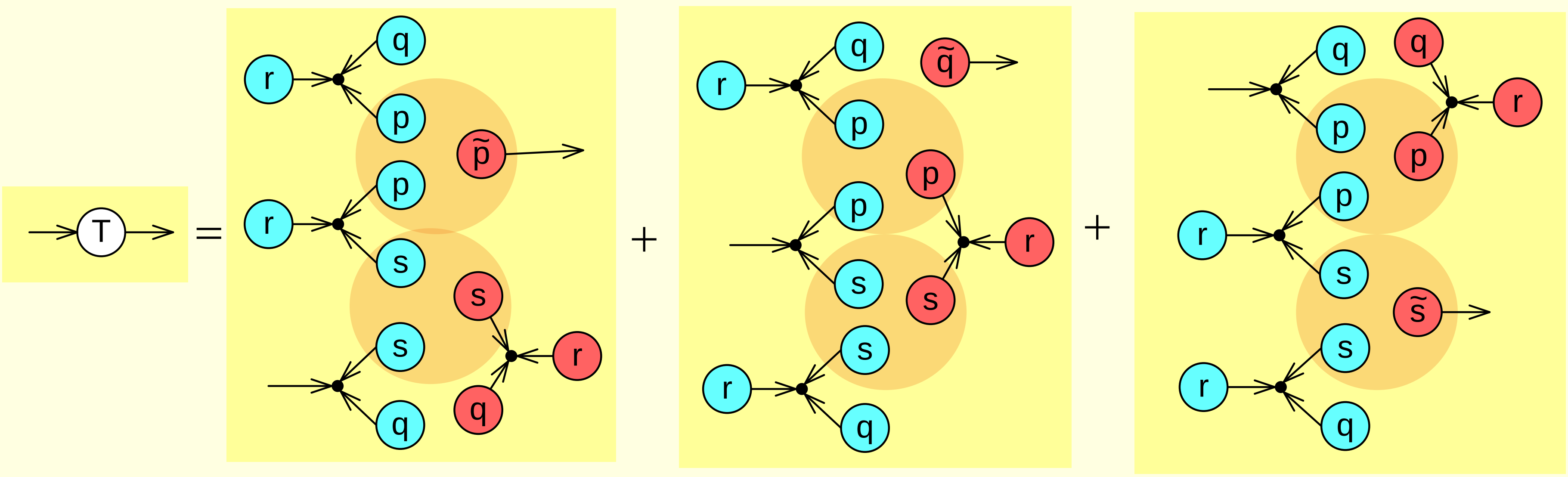


Works for **p,q,s**

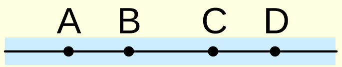
Now find α,β,γ to make it work for **r**

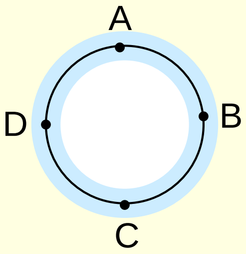
Higher Dimensions

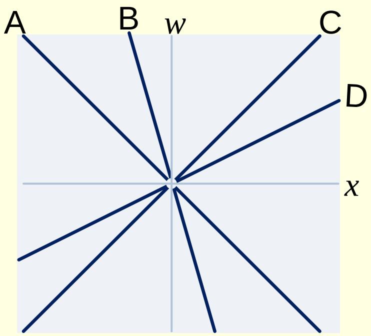




Four Points in P^1

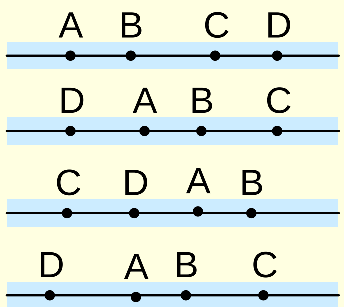


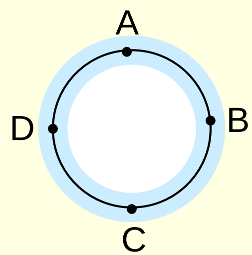


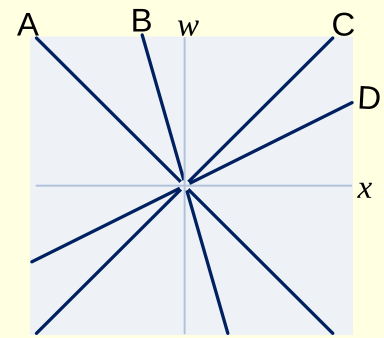


Interleaving of Four Points

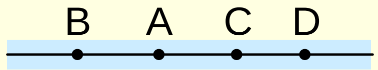
Same interleaving

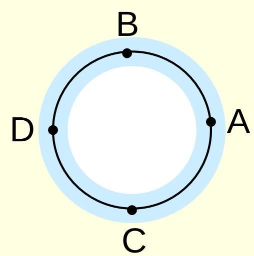


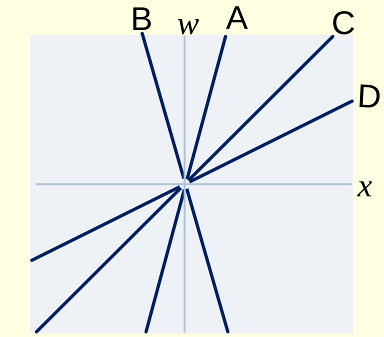




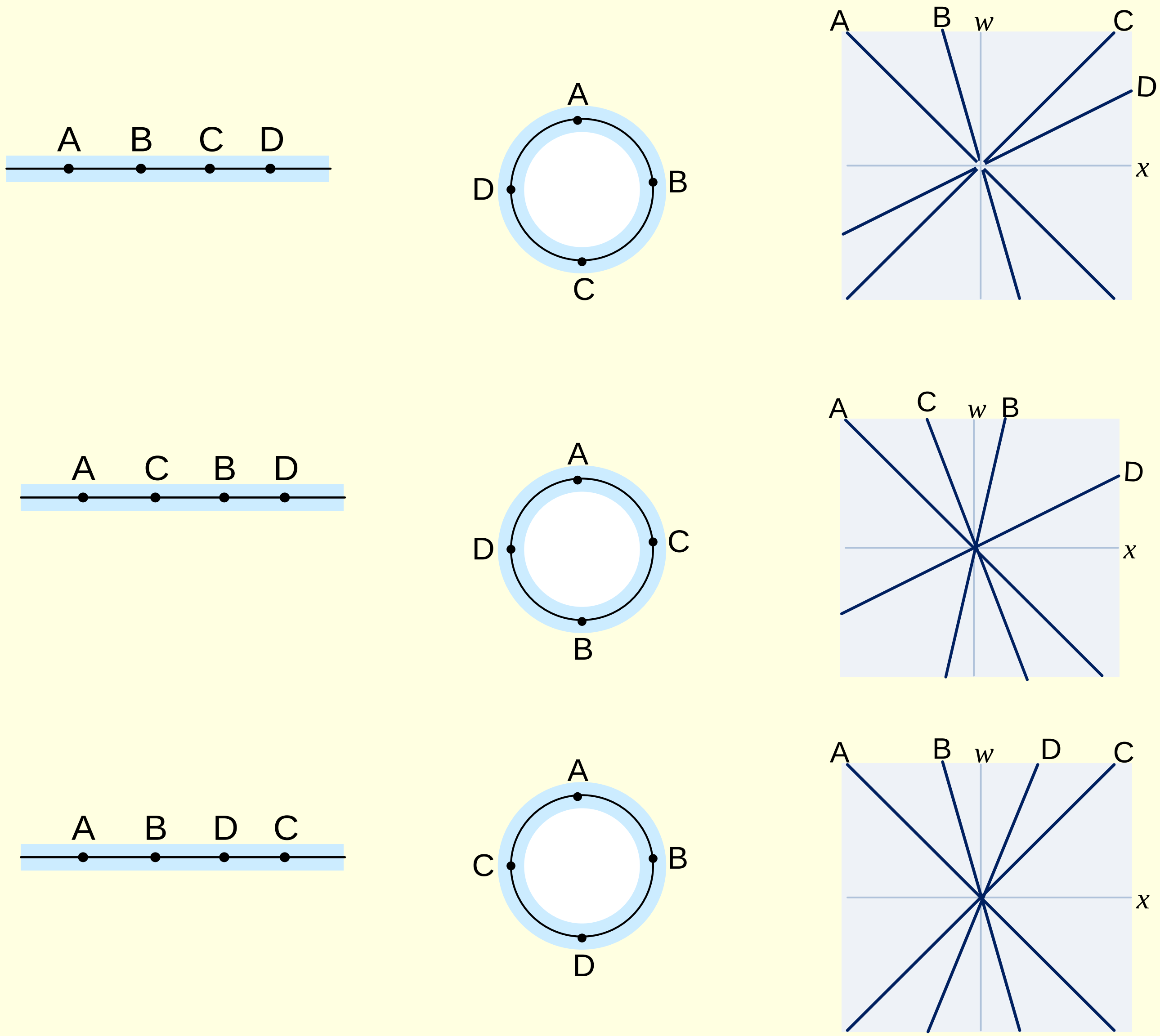
Different interleaving



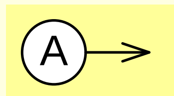


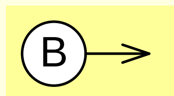


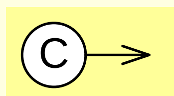
Three possible interleavings

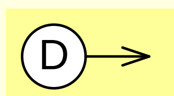


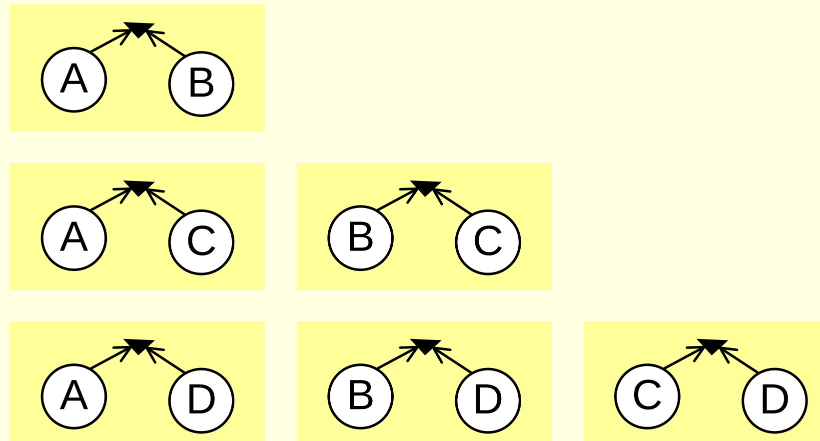
Diagrams for Four Points

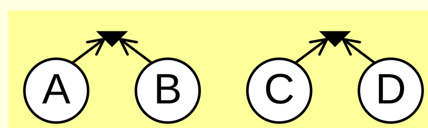
$\mathbf{A} = \begin{bmatrix} x_A & w_A \end{bmatrix} =$ 

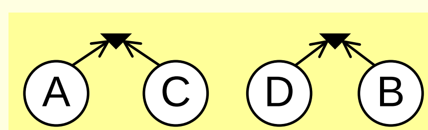
$\mathbf{B} = \begin{bmatrix} x_B & w_B \end{bmatrix} =$ 

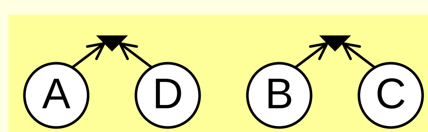
$\mathbf{C} = \begin{bmatrix} x_C & w_C \end{bmatrix} =$ 

$\mathbf{D} = \begin{bmatrix} x_D & w_D \end{bmatrix} =$ 

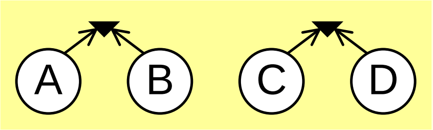


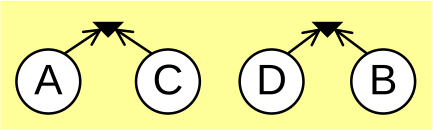
$V_1 =$ 

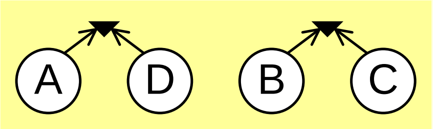
$V_2 =$ 

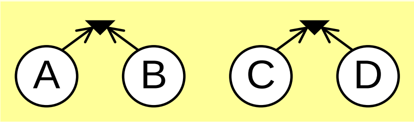
$V_3 =$ 

Diagrams for Four Points

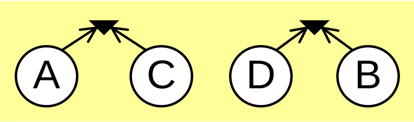
$V_1 =$ 

$V_2 =$ 

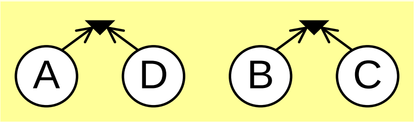
$V_3 =$ 



$+$



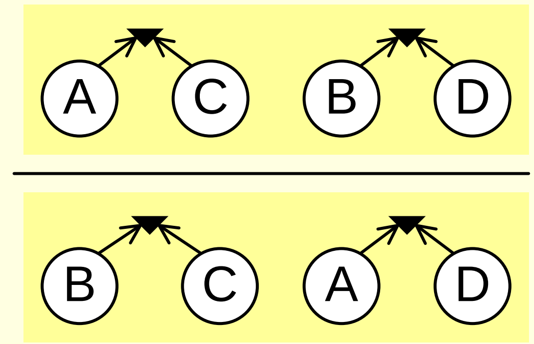
$+$



$= 0$

$$V_1 + V_2 + V_3 = 0$$

Cross Ratio



$$= \frac{-V_2}{V_3}$$

"Absolute Invariant"

Could pick any of $-\frac{V_1}{V_2},\ -\frac{V_2}{V_3},\ -\frac{V_3}{V_1}$ as "cross ratio"

Relationship between

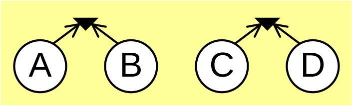
$$-\frac{V_1}{V_2} = \chi$$

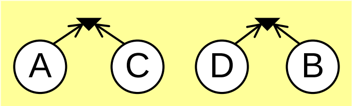
$$-\frac{V_2}{V_3} = \frac{-V_2}{-V_1 - V_2} = \frac{1}{\chi + 1}$$

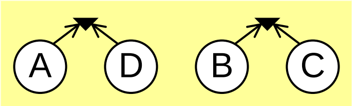
$$-\frac{V_3}{V_1} = \frac{V_1 + V_2}{V_1} = 1 + \frac{1}{\chi}$$

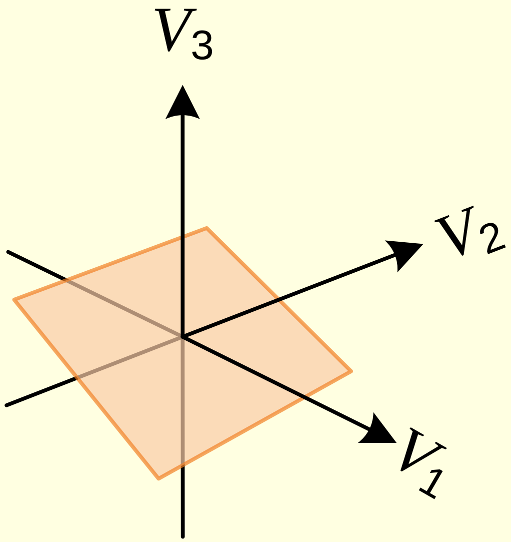
3D view of invariant space

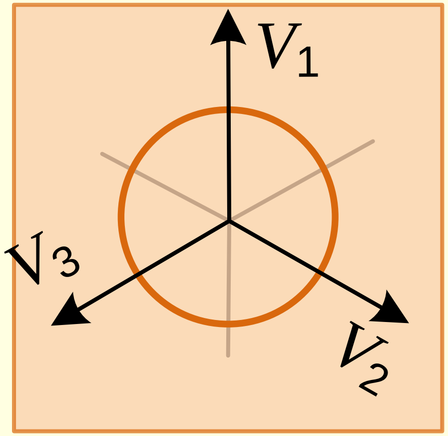
$$V_1 + V_2 + V_3 = 0$$

$V_1 =$ 

$V_2 =$ 

$V_3 =$ 





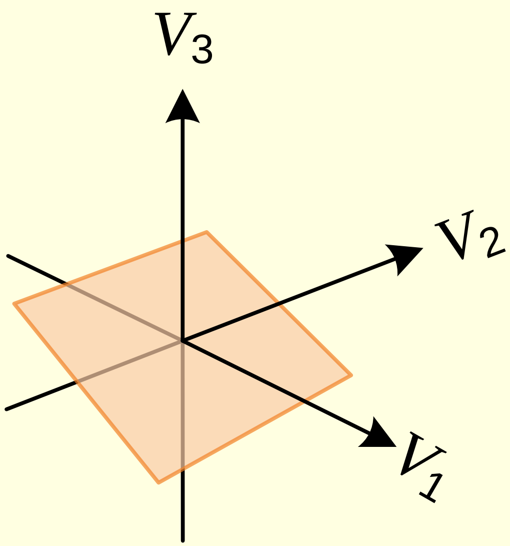
Homogeneously scale to normalize onto unit circle

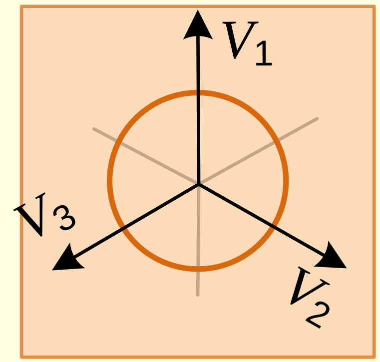
$$V_1^{\;2} + V_2^{\;2} + V_3^{\;2} = 1$$

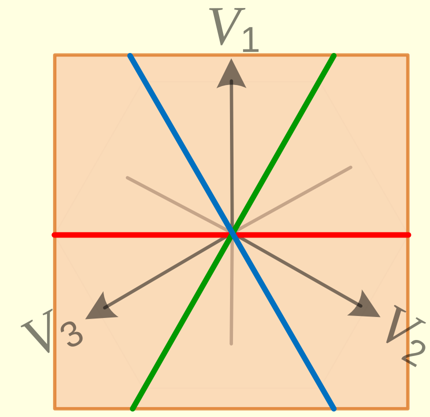
$$\phi = \cos^{-1}\left(V_1\right)$$

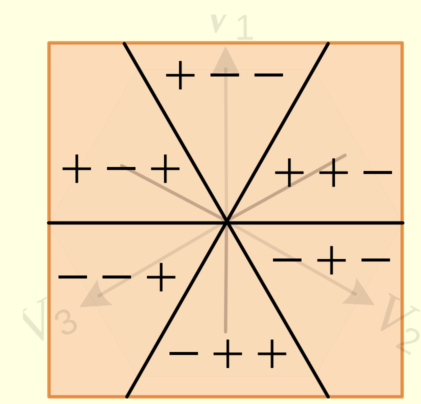
3D view of invariant space

$$V_1 + V_2 + V_3 = 0$$









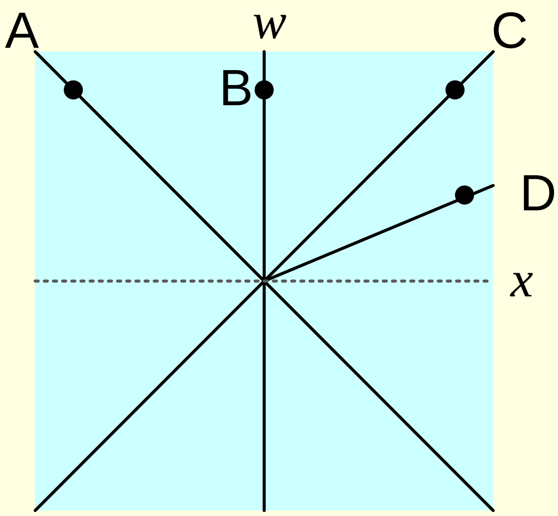
Determine which sign indicates which interleaving

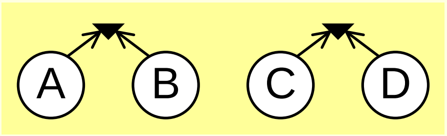
$$\mathbf{A} = \begin{bmatrix} -1 & 1 \end{bmatrix}$$

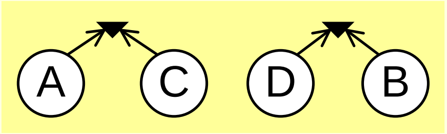
$$\mathbf{B} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

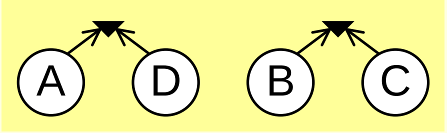
$$\mathbf{C} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} x & w \end{bmatrix}$$



$V_1 =$  $= -w + x$

$V_2 =$  $= \quad -2x$

$V_3 =$  $= +w + x$

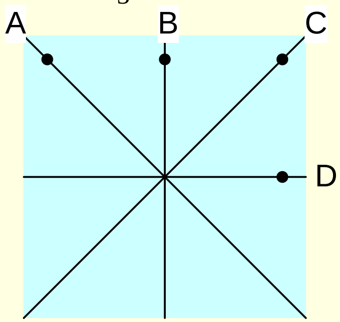
Follow in V space

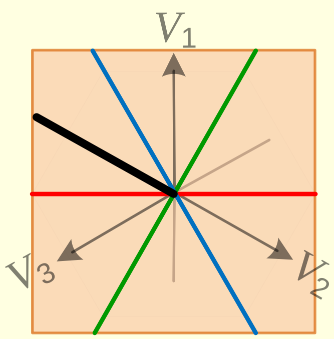
$$\mathbf{D} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$V_1 = +1$

$V_2 = -2$

$V_3 = +1$



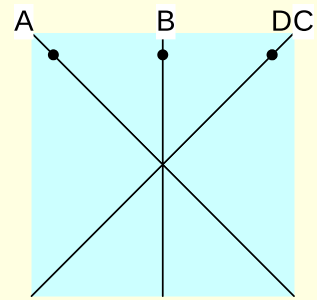


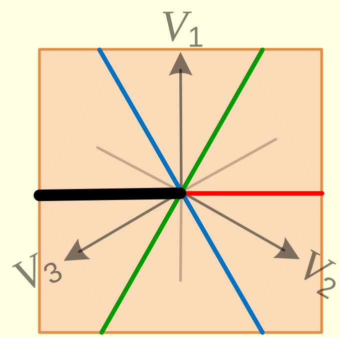
$$\mathbf{D} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$V_1 = 0$

$V_2 = -2$

$V_3 = +2$



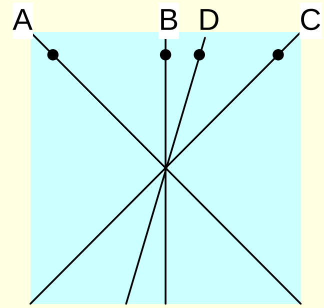


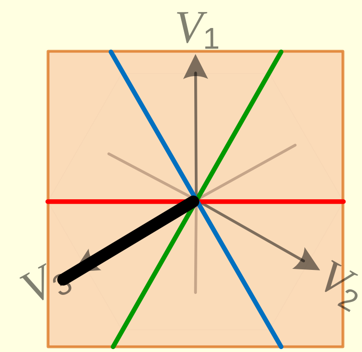
$$\mathbf{D} = \begin{bmatrix} \frac{1}{2} & \frac{3}{2} \end{bmatrix}$$

$V_1 = -1$

$V_2 = -1$

$V_3 = +2$





Follow in V space

$\mathbf{D} = [1 \quad 0]$
$V_1 = +1$
$V_2 = -2$
$V_3 = +1$

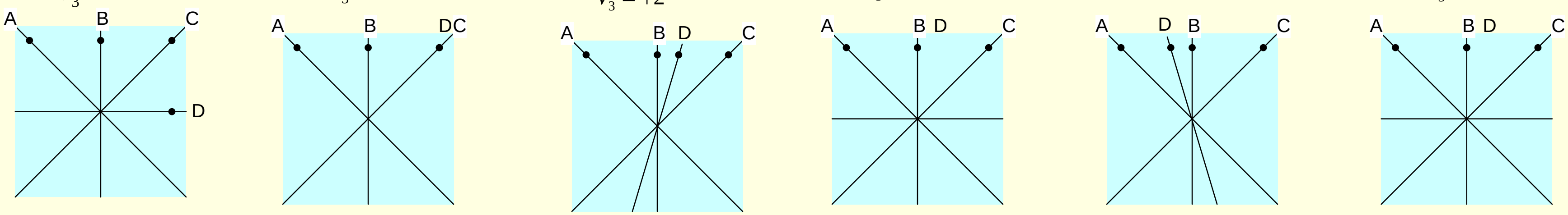
$\mathbf{D} = [1 \quad 1]$
$V_1 = 0$
$V_2 = -2$
$V_3 = +2$

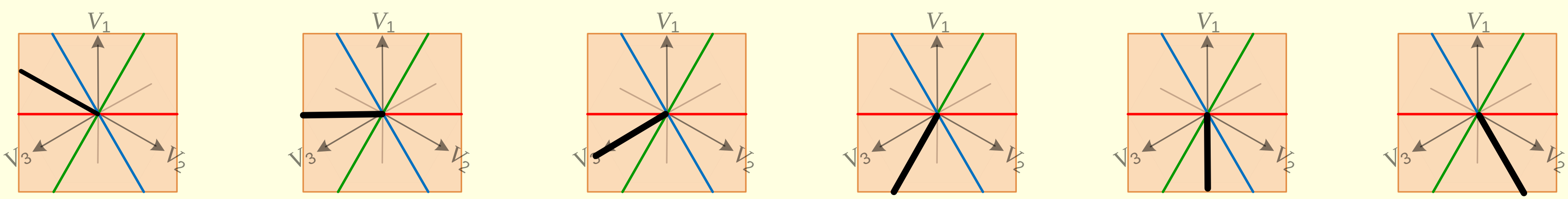
$\mathbf{D} = \left[\frac{1}{2} \quad \frac{3}{2}\right]$
$V_1 = -1$
$V_2 = -1$
$V_3 = +2$

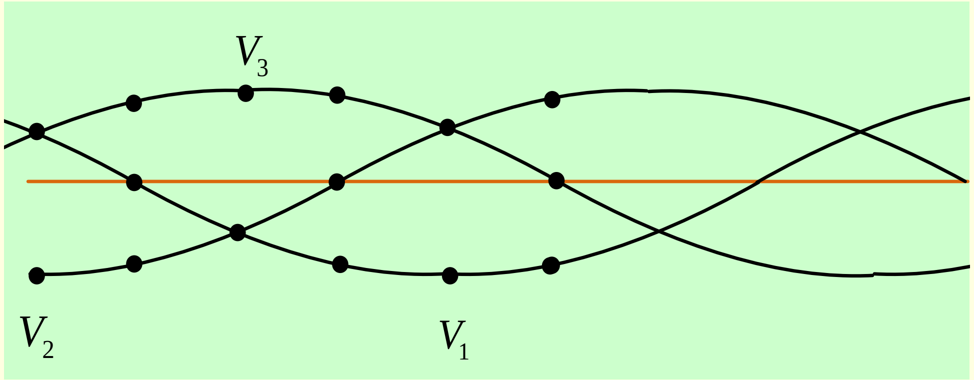
$\mathbf{D} = [0 \quad 1]$
$V_1 = -1$
$V_2 = 0$
$V_3 = +1$

$\mathbf{D} = \left[-\frac{1}{2} \quad \frac{3}{2}\right]$
$V_1 = -2$
$V_2 = +1$
$V_3 = +1$

$\mathbf{D} = [-1 \quad 1]$
$V_1 = -1$
$V_2 = 0$
$V_3 = +1$







Follow in V space

$\mathbf{D} = [1 \quad 0]$, $V_1 = +1$, $V_2 = -2$, $V_3 = +1$

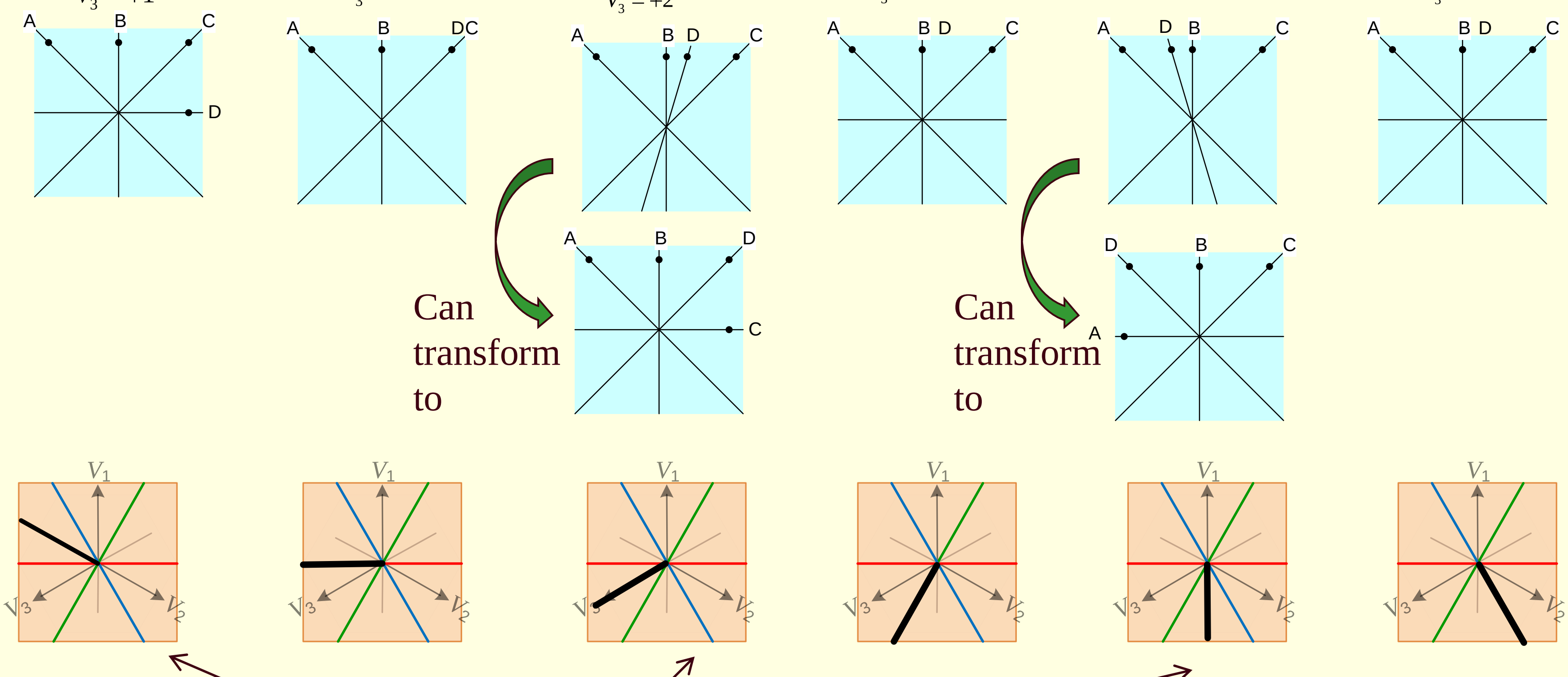
$\mathbf{D} = [1 \quad 1]$, $V_1 = 0$, $V_2 = -2$, $V_3 = +2$

$\mathbf{D} = \left[\frac{1}{2} \quad \frac{3}{2}\right]$, $V_1 = -1$, $V_2 = -1$, $V_3 = +2$

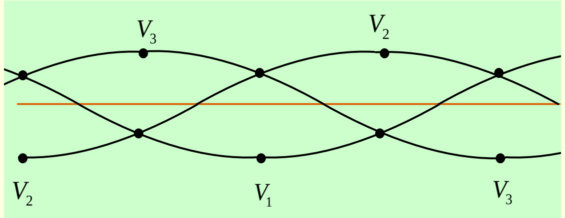
$\mathbf{D} = [0 \quad 1]$, $V_1 = -1$, $V_2 = 0$, $V_3 = +1$

$\mathbf{D} = \left[-\frac{1}{2} \quad \frac{3}{2}\right]$, $V_1 = -2$, $V_2 = +1$, $V_3 = +1$

$\mathbf{D} = [-1 \quad 1]$, $V_1 = -1$, $V_2 = 0$, $V_3 = +1$



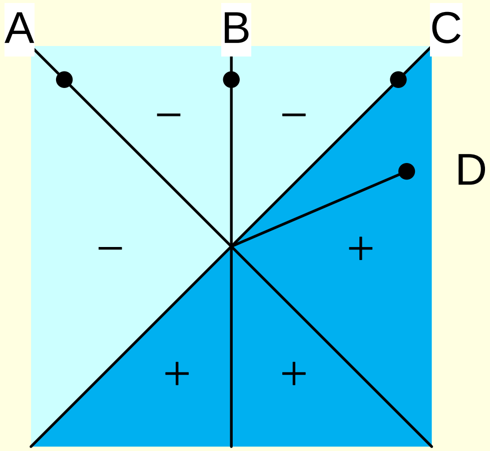
Harmonic Set



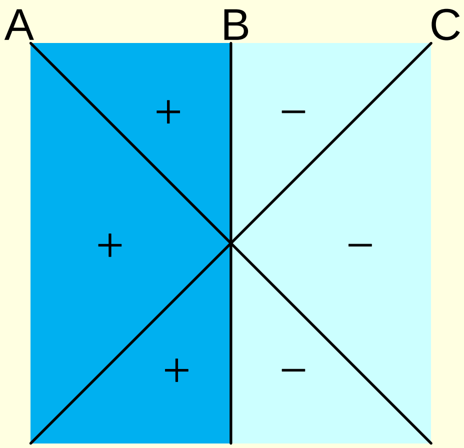
Determine which sign indicates which interleaving

$$\mathbf{A} = [-1 \quad 1]$$
$$\mathbf{B} = [\;0 \quad 1]$$
$$\mathbf{C} = [\;1 \quad 1]$$
$$\mathbf{D} = [x \quad w]$$

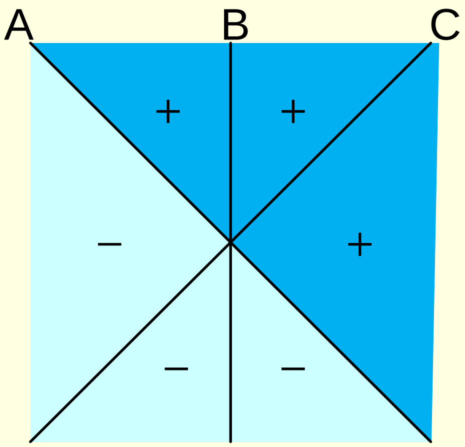
$$V_1 = +(\mathbf{A} \wedge \mathbf{B})(\mathbf{C} \wedge \mathbf{D}) = -w + x$$
$$V_2 = -(\mathbf{A} \wedge \mathbf{C})(\mathbf{B} \wedge \mathbf{D}) = \quad -2x$$
$$V_3 = +(\mathbf{A} \wedge \mathbf{D})(\mathbf{B} \wedge \mathbf{C}) = +w + x$$



$V_1 = -w + x$

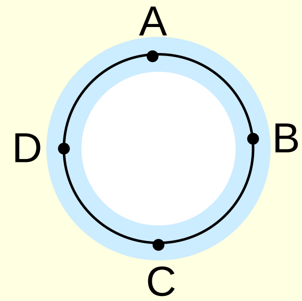
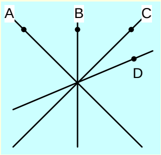
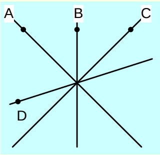
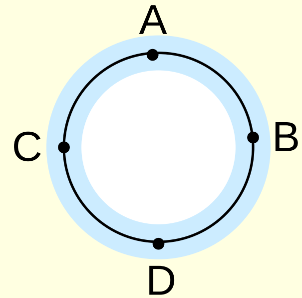
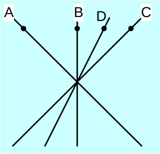
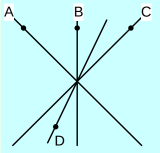
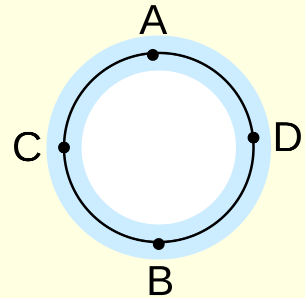
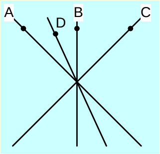
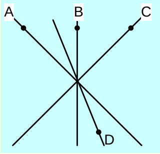


$V_2 = -2x$



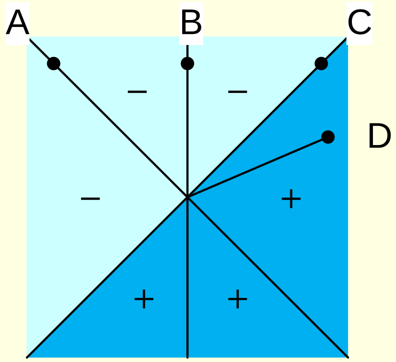
$V_3 = w + x$

Signs and interleaving

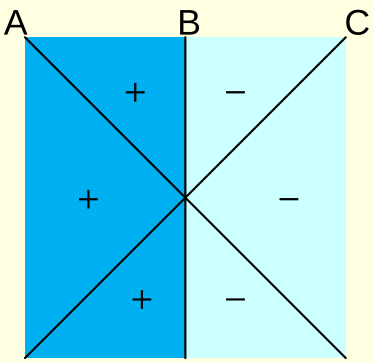
Sign(V_1,V_2,V_3)	Ordering of points on P^1	View in (x,w) plane
$+-+$		
$-+-$		
$--+$		
$++-$		
$-++$		
$+--$		

Better Interleaving Test

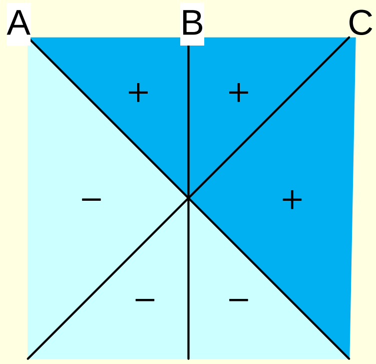
Old way



$V_1 = -w + x$

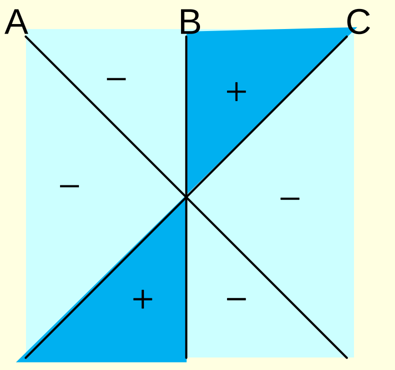


$V_2 = -2x$

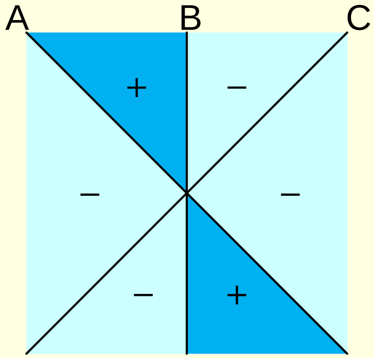


$V_3 = w + x$

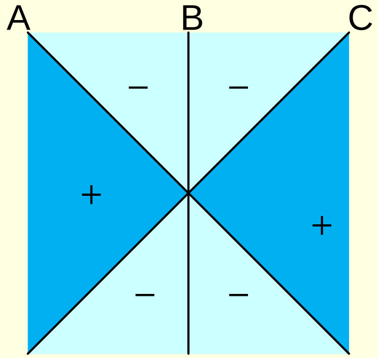
Better way



$V_1V_2 = (-w + x)(-2x) = 2x(w - x)$

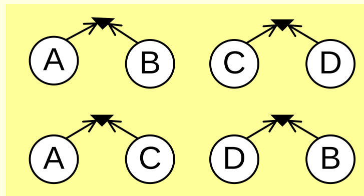


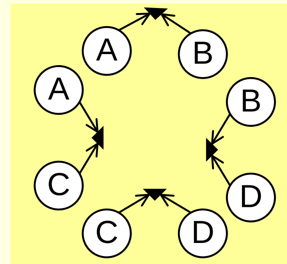
$V_2V_3 = -2x(w + x)$

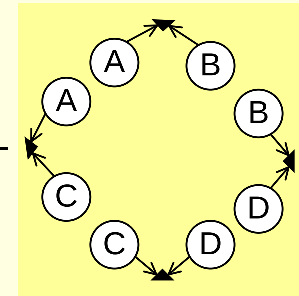


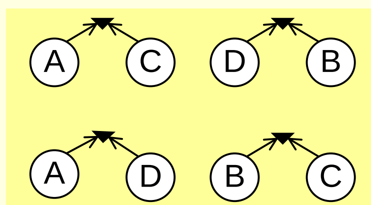
$V_3V_1 = (w + x)(-w + x) = -w^2 + x^2$

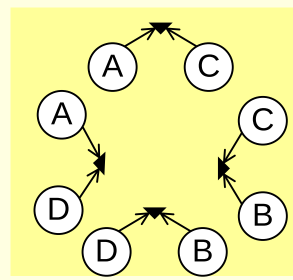
Diagrams

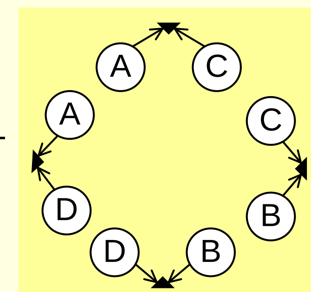
$V_1 V_2 =$ 

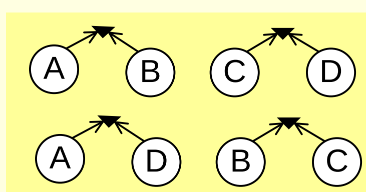
$=$ 

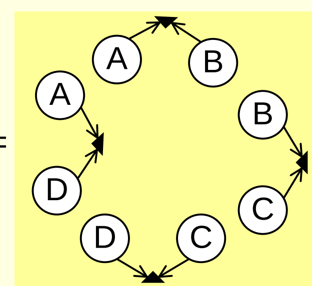
$= -$ 

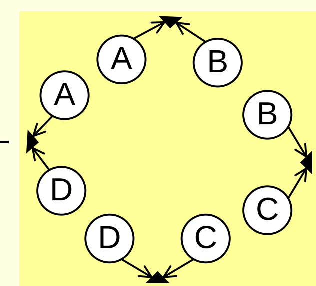
$V_2 V_3 =$ 

$=$ 

$= -$ 

$V_3 V_1 =$ 

$=$ 

$= -$ 

Best Interleaving Test