CSE590B Lecture 5
Cubic Bezier Curves

Moving up in degree,

Moving up in dimension

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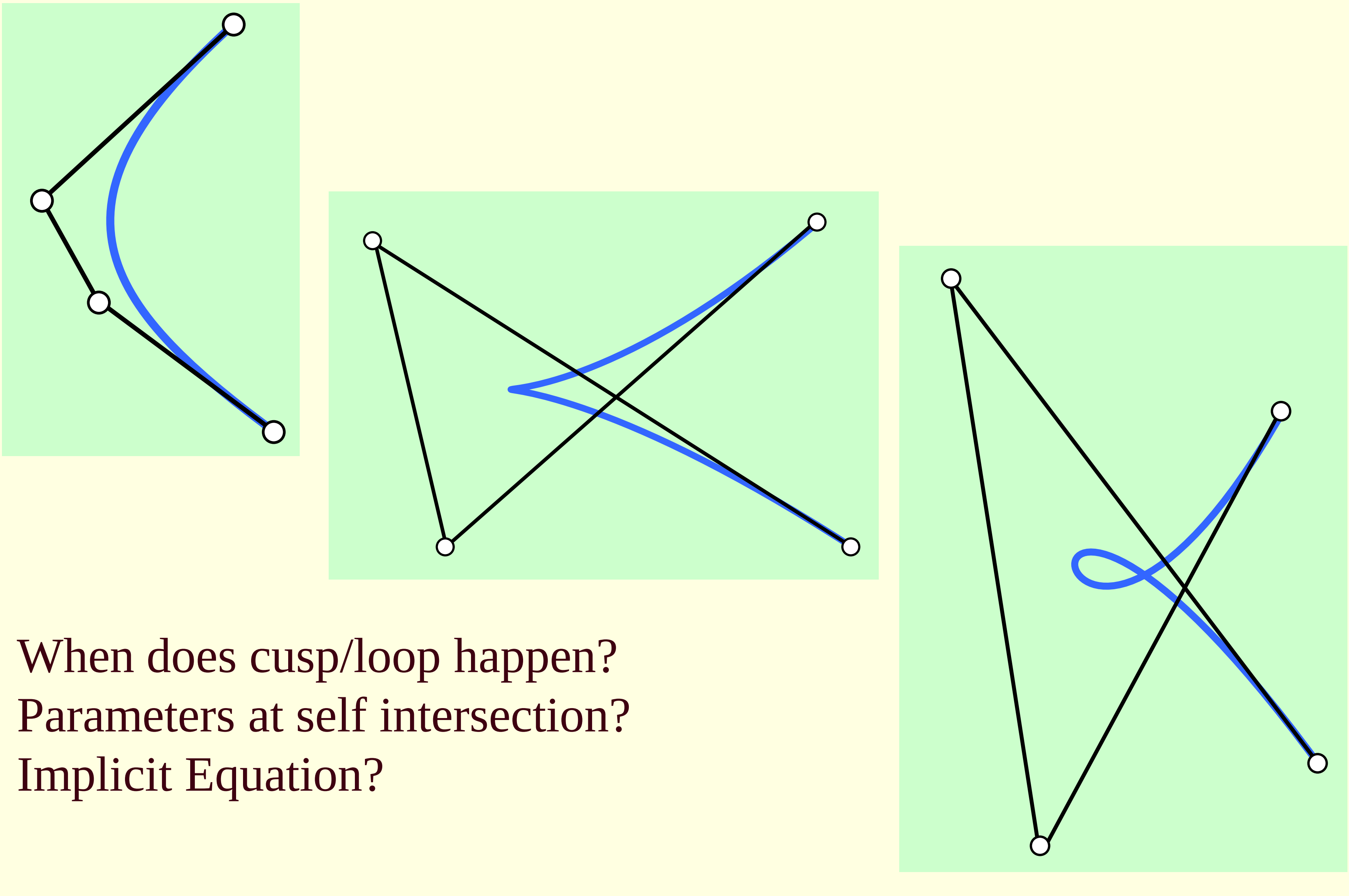
JimBlinn.Com

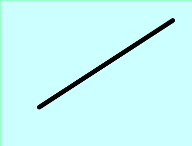
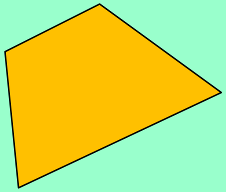
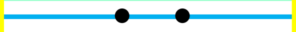
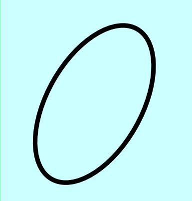
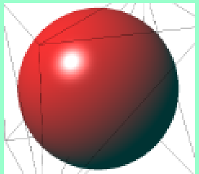
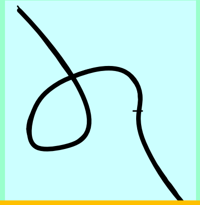
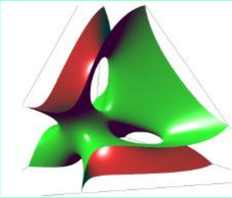
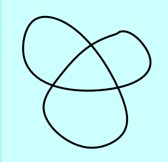
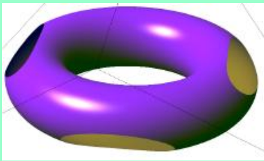
http://courses.cs.washington.edu/courses/cse590b/13au/

Cubic Bezier Curves

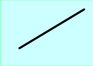
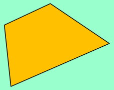
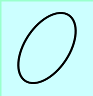
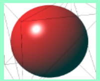
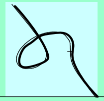
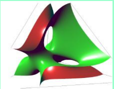
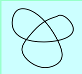
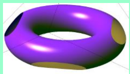


When does cusp/loop happen?
Parameters at self intersection?
Implicit Equation?

The Grid

	$2D=P^1$ Point sets on line	$3D=P^2$ Curves in plane	4D=P3 Surfaces in space
Linear			
Quadratic			
Cubic			
Quartic			

Other Dimensions

	2D=P1 Point sets on line	3D=P2 Curves in plane	4D=P3 Surfaces in space
Linear			
Quadratic			
Cubic			
Quartic			

2D algebra

Points in P^1

$$\begin{bmatrix} x & w \end{bmatrix}$$

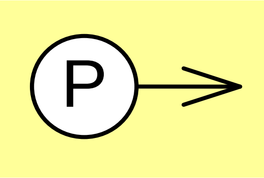
$$X = \frac{x}{w}$$

3D algebra

Points in P^2

$$\begin{bmatrix} x & y & w \end{bmatrix}$$

$$\begin{bmatrix} X & W \end{bmatrix} = \begin{bmatrix} \frac{x}{w} & \frac{y}{w} \end{bmatrix}$$



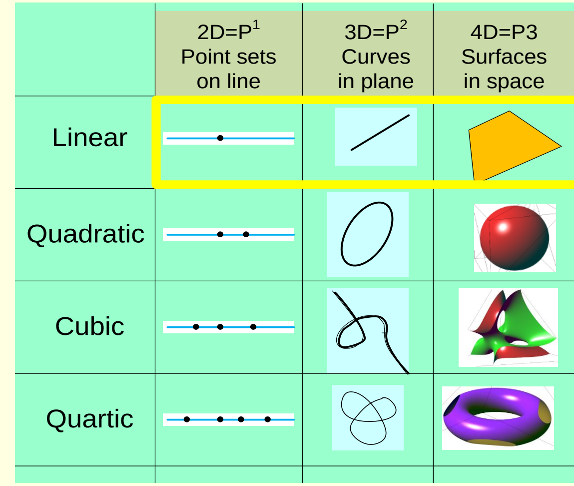
4D algebra

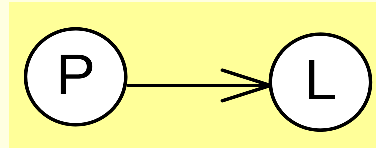
Points in P^3

$$\begin{bmatrix} x & y & z & w \end{bmatrix}$$

$$\begin{bmatrix} X & Y & Z \end{bmatrix} = \begin{bmatrix} \frac{x}{w} & \frac{y}{w} & \frac{z}{w} \end{bmatrix}$$

Other Dimensions





Point in P^1

2D algebra

$$f(x,w) = Ax + Bw = \begin{bmatrix} x & w \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix}$$

Lines in P^2

3D algebra

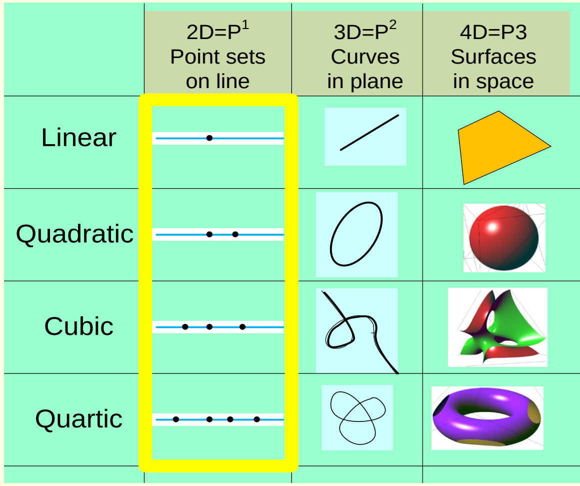
$$f(x,y,w) = Ax + By + Cw = \begin{bmatrix} x & y & w \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix}$$

Planes in P^3

4D algebra

$$f(x,y,z,w) = Ax + By + Cz + Dw = \begin{bmatrix} x & y & z & w \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}$$

Other Orders



$$\mathbf{L}(x,w) = Ax + Bw$$

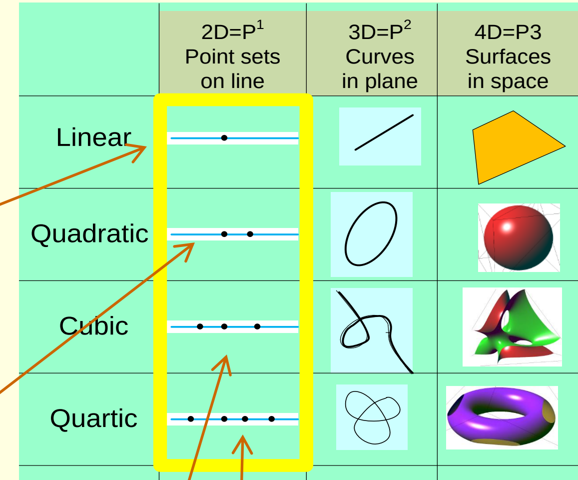
$$\mathbf{Q}(x,w) = Ax^2 + 2Bxw + Cw^2$$

$$\mathbf{C}(x,w) = Ax^3 + 3Bx^2w + 3Cxw^2 + Dw^3$$

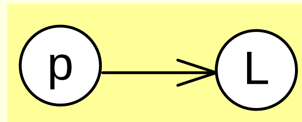
$$\mathbf{F}(x,w) = Ax^4 + 4Bx^3w + 6Cx^2w^2 + 4Dxw^3 + Ew^4$$

$$\mathbf{N}(x,w) = Ax^5 + 5Bx^4w + 10Cx^3w^2 + 10Dx^2w^3 + 5Exw^4 + Fw^5$$

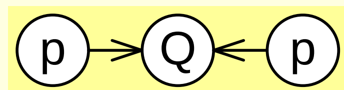
Other Orders



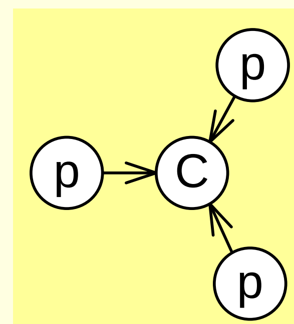
$$\mathbf{L}(x,w)=\begin{bmatrix} x & w \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} =$$



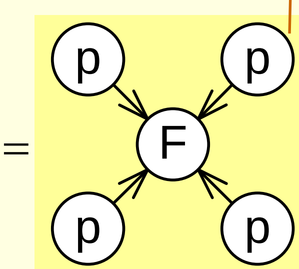
$$\mathbf{Q}(x,w)=\begin{bmatrix} x & w \end{bmatrix}\begin{bmatrix} A & B \\ B & C \end{bmatrix}\begin{bmatrix} x \\ w \end{bmatrix} =$$



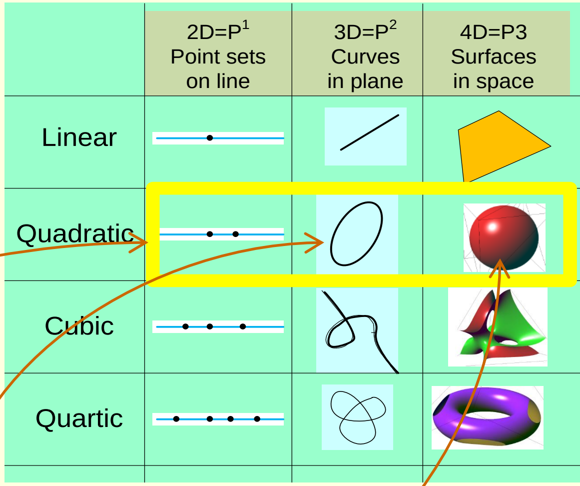
$$\mathbf{C}(x,w)=\begin{bmatrix} x & w \end{bmatrix}\left\{\begin{bmatrix}\begin{bmatrix} A & B \\ B & C \end{bmatrix} & \begin{bmatrix} B & C \\ C & D \end{bmatrix}\end{bmatrix}\begin{bmatrix} x \\ w \end{bmatrix}\right\}\begin{bmatrix} x \\ w \end{bmatrix} =$$



$$\mathbf{F}(x,w)=\begin{bmatrix} x & w \end{bmatrix}\left\{\begin{bmatrix} x & w \end{bmatrix}\begin{bmatrix}\begin{bmatrix} A & B \\ B & C \end{bmatrix} & \begin{bmatrix} B & C \\ C & D \end{bmatrix} \\ \begin{bmatrix} B & C \\ C & D \end{bmatrix} & \begin{bmatrix} C & D \\ D & E \end{bmatrix}\end{bmatrix}\begin{bmatrix} x \\ w \end{bmatrix}\right\}\begin{bmatrix} x \\ w \end{bmatrix} =$$



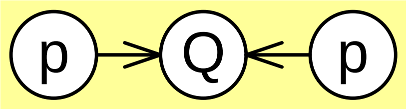
Quadratic, Different dimensions



$$\mathbf{Q}(x,w)=\begin{bmatrix}x & w\end{bmatrix}\begin{bmatrix}A & B\\ B & C\end{bmatrix}\begin{bmatrix}x\\ w\end{bmatrix}$$

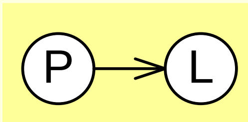
$$\mathbf{Q}(x,y,w)=\begin{bmatrix}x & y & w\end{bmatrix}\begin{bmatrix}A & B & C\\ B & D & E\\ C & E & F\end{bmatrix}\begin{bmatrix}x\\ y\\ w\end{bmatrix}$$

=

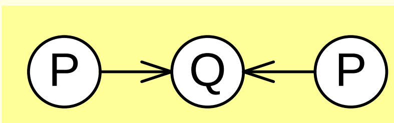


$$\mathbf{Q}(x,y,z,w)=\begin{bmatrix}x & y & z & w\end{bmatrix}\begin{bmatrix}A & B & C & D\\ B & E & F & G\\ C & F & H & J\\ D & G & J & K\end{bmatrix}\begin{bmatrix}x\\ y\\ z\\ w\end{bmatrix}$$

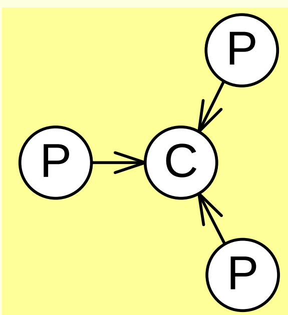
Diagram Same Across Dimensions



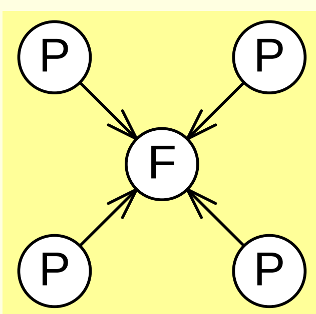
$= 0$



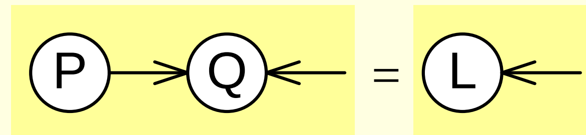
$= 0$

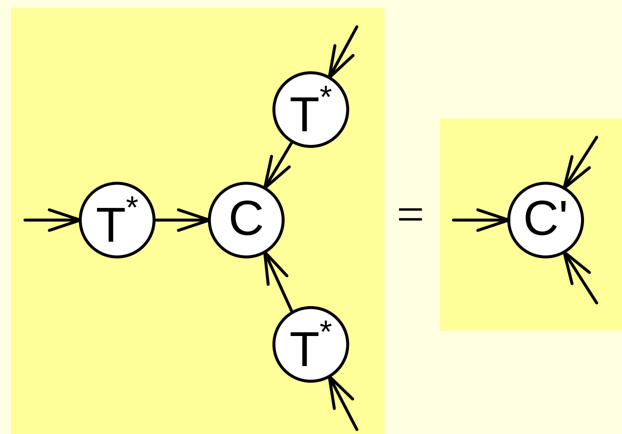


$= 0$

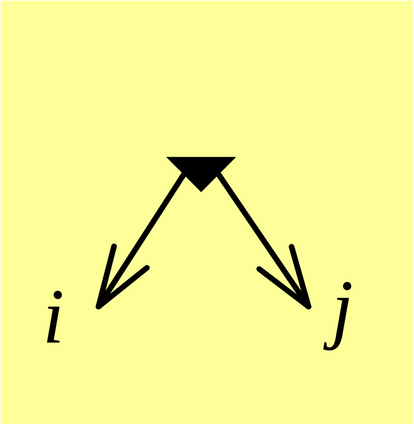


$= 0$

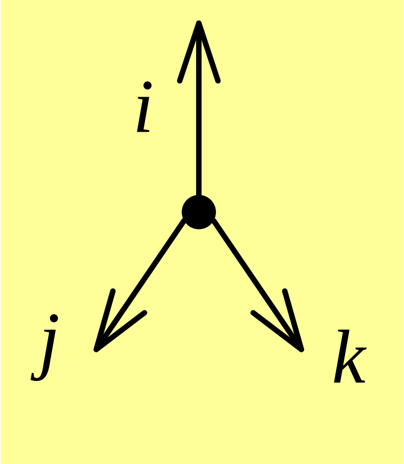




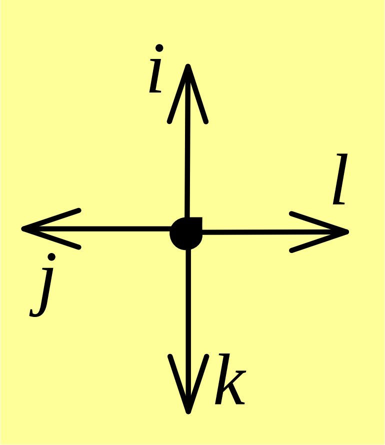
Dimensionality and Epsilon



ε^{ij}



ε^{ijk}



ε^{ijkl}

2D algebra

1D geometry

3D algebra

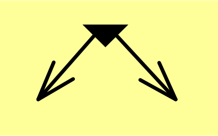
2D geometry

4D algebra

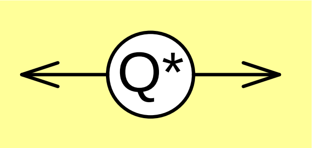
3D geometry

2D Epsilon

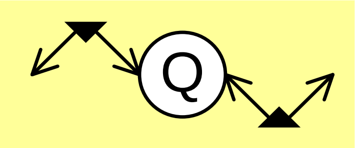
$$\varepsilon = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} =$$

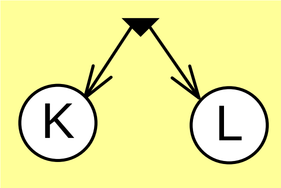


Adjugate of Matrix

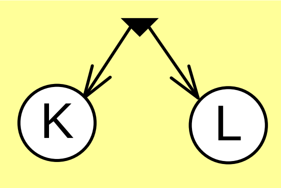


=



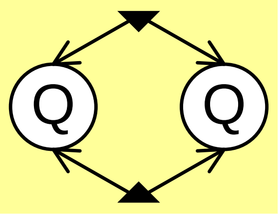


$= -$



Determinant of Matrix

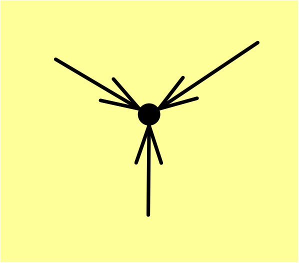
-2detQ =

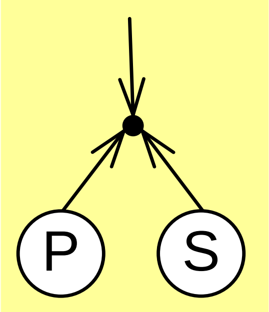


3D Epsilon

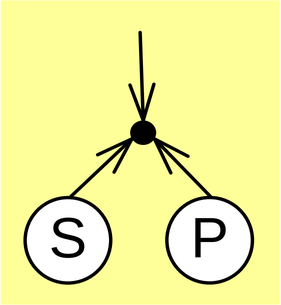
$$\varepsilon_{ijk} = +1 \quad \text{if } ijk \text{ is an even permutation of 012}$$
$$\varepsilon_{ijk} = -1 \quad \text{if } ijk \text{ is an odd permutation of 012}$$
$$\varepsilon_{ijk} = 0 \quad \text{otherwise}$$

$$\varepsilon = \begin{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{bmatrix} =$$



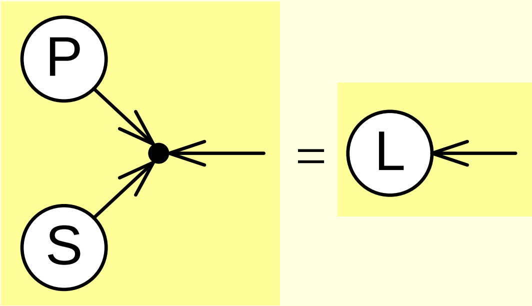


$= -$

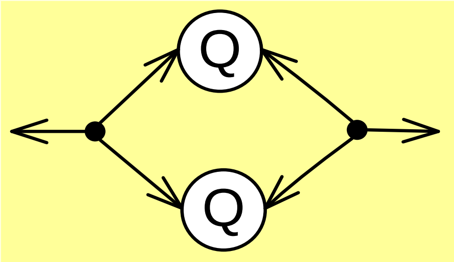


3D Epsilon Usage

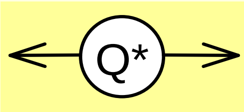
Line thru two points



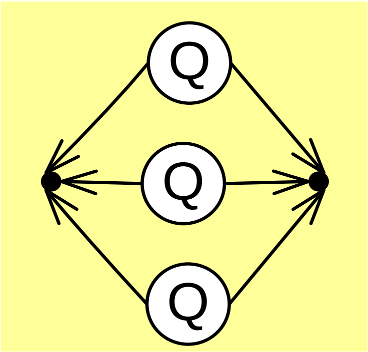
Adjugate of Matrix



$= -2$



Determinant of Matrix



$= -6 \det \mathbf{Q}$

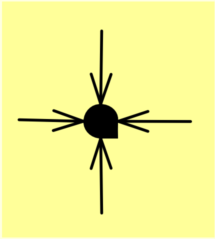
4D Epsilon

$$\varepsilon_{ijkl} = +1 \quad \text{if } ijkl \text{ is an even permutation of 0123}$$

$$\varepsilon_{ijkl} = -1 \quad \text{if } ijkl \text{ is an odd permutation of 0123}$$

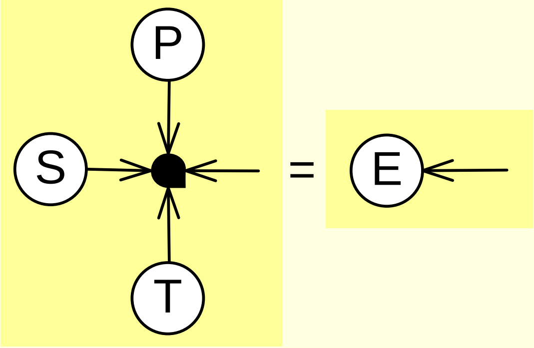
$$\varepsilon_{ijkl} = \ \ 0 \quad \text{otherwise}$$

$$\varepsilon = \begin{bmatrix}
\begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix} &
\begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&-1&0 \end{bmatrix} &
\begin{bmatrix} 0&0&0&0\\0&0&0&-1\\0&0&0&0\\0&1&0&0 \end{bmatrix} &
\begin{bmatrix} 0&0&0&0\\0&0&1&0\\0&-1&0&0\\0&0&0&0 \end{bmatrix} \\
\begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&-1\\0&0&1&0 \end{bmatrix} &
\begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix} &
\begin{bmatrix} 0&0&0&1\\0&0&0&0\\0&0&0&0\\-1&0&0&0 \end{bmatrix} &
\begin{bmatrix} 0&0&-1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0 \end{bmatrix} \\
\begin{bmatrix} 0&0&0&0\\0&0&0&1\\0&0&0&0\\0&-1&0&0 \end{bmatrix} &
\begin{bmatrix} 0&0&0&-1\\0&0&0&0\\0&0&0&0\\1&0&0&0 \end{bmatrix} &
\begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix} &
\begin{bmatrix} 0&1&0&0\\-1&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix} \\
\begin{bmatrix} 0&0&0&0\\0&0&-1&0\\0&1&0&0\\0&0&0&0 \end{bmatrix} &
\begin{bmatrix} 0&0&1&0\\0&0&0&0\\-1&0&0&0\\0&0&0&0 \end{bmatrix} &
\begin{bmatrix} 0&-1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix} &
\begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}
\end{bmatrix} =$$

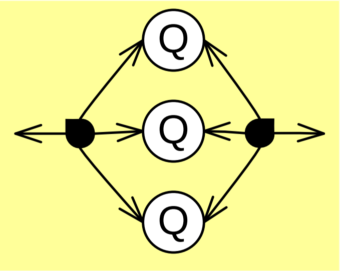


4D Epsilon Usage

3 Points = A Plane

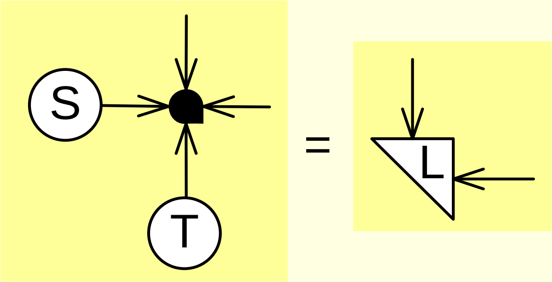


Adjugate of Matrix

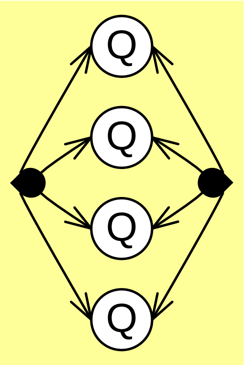


= 6 adj **Q**

2 Points = A Line

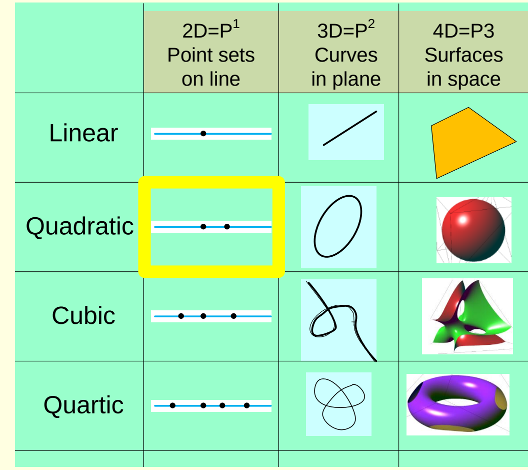


Determinant of Matrix

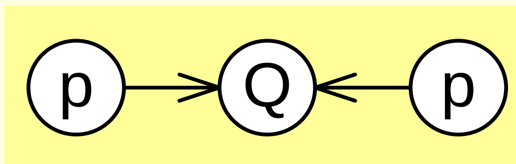


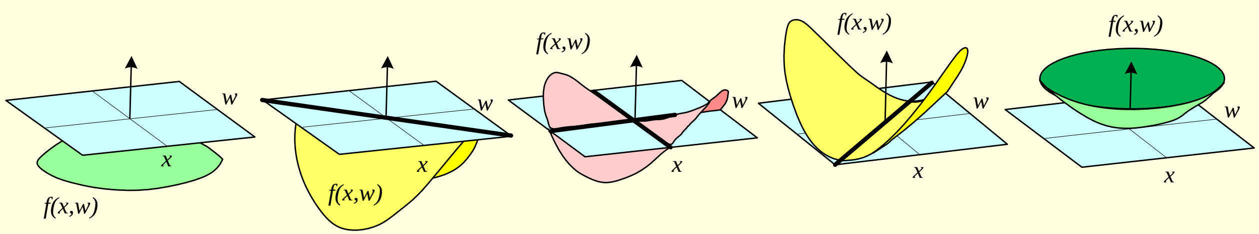
= 24 det **Q**

P^1 Quadratic Polynomial

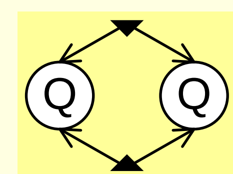


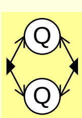
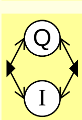
$$\mathbf{Q}(x,w) = Ax^2 + 2Bxw + Cw^2$$

$=$ 



discriminant



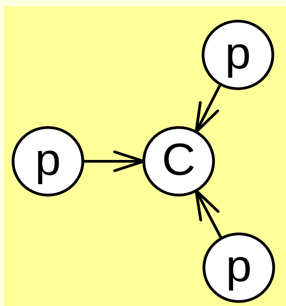
	$\frac{1}{1}-$	2-	11	2+	$\frac{1}{1}+$
	$-$	0	$+$	0	$-$
	$+$	$+$		$-$	$-$

P^1 Cubic Polynomial

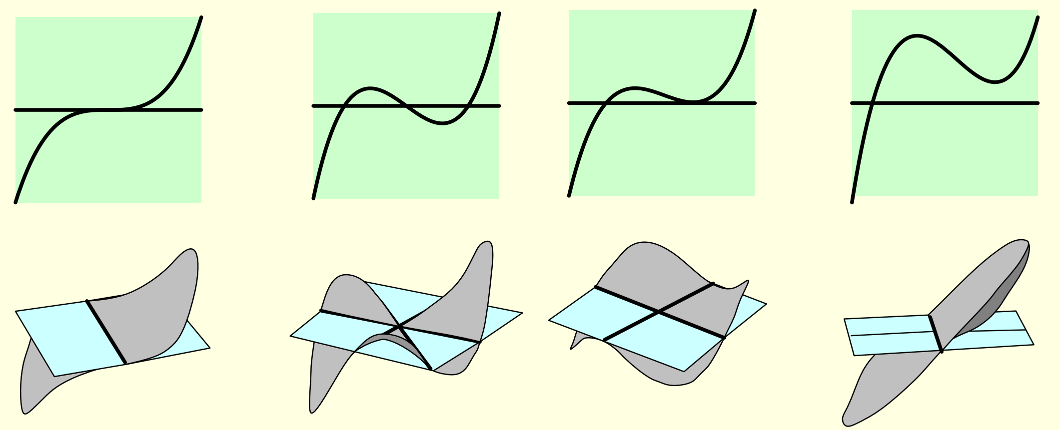
$$\mathbf{C}(x,w) = Ax^3 + 3Bx^2w + 3Cxw^2 + Dw^3$$

$$= \begin{bmatrix} x & w \end{bmatrix} \left\{ \left[\begin{bmatrix} A & B \\ B & C \end{bmatrix} \quad \begin{bmatrix} B & C \\ C & D \end{bmatrix} \right] \begin{bmatrix} x \\ w \end{bmatrix} \right\} \begin{bmatrix} x \\ w \end{bmatrix}$$

=



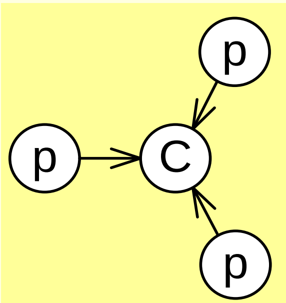
	$2D=P^1$ Point sets on	$3D=P^2$ Curves in plane	4D=P3 Surfaces in space
Linear			
Quadratic			
Cubic			
Quartic			

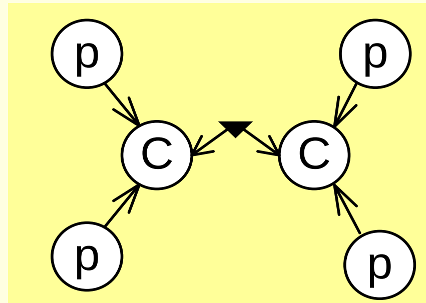


Type 3 Type 111 Type 12 Type $1\frac{1}{1}$

Note: can always flip sign with 180° rotation

2D Cubic Polynomial Covariants





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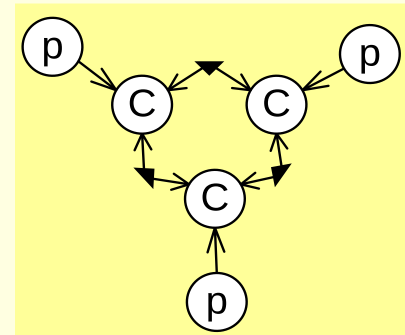

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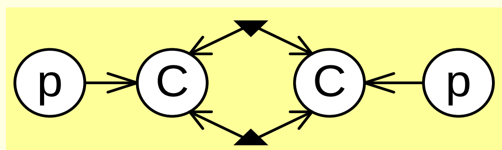
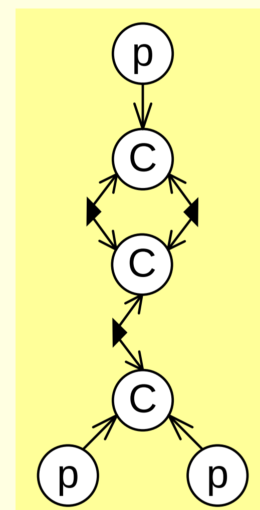
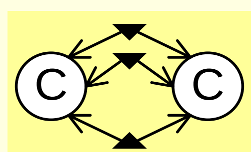


= H



= J

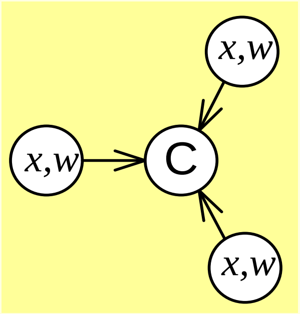


= 0

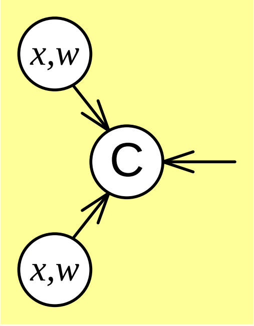
Derivatives

$$\mathbf{C}(x,w) = Ax^3 + 3Bx^2w + 3Cxw^2 + Dw^3$$

$$= \begin{bmatrix} x & w \end{bmatrix} \left\{ \begin{bmatrix} \begin{bmatrix} A & B \\ B & C \end{bmatrix} & \begin{bmatrix} B & C \\ C & D \end{bmatrix} \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \right\} \begin{bmatrix} x \\ w \end{bmatrix} =$$



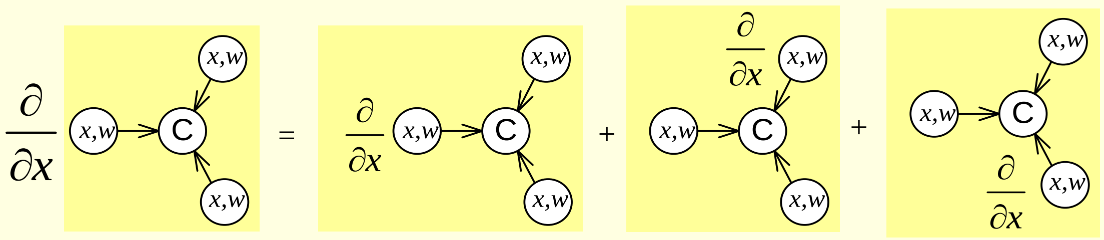
$$\begin{bmatrix} \mathbf{C}_x \\ \mathbf{C}_w \end{bmatrix} = 3 \begin{bmatrix} Ax^2 + 2Bxw + Cw^2 \\ Bx^2 + 2Cxw + Dw^2 \end{bmatrix} = 3$$

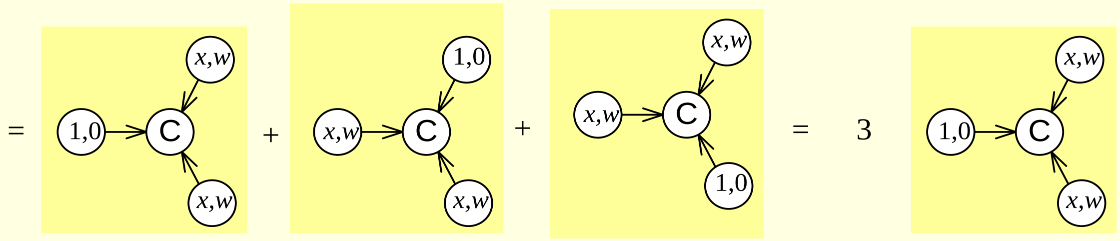


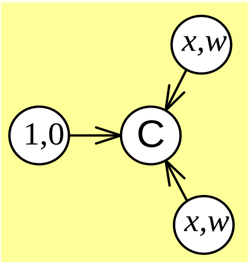
Derivatives

$$\frac{d}{dx}x^3 = \frac{d}{dx}(xxx) = \left(\frac{d}{dx}x\right)xx + x\left(\frac{d}{dx}x\right)x + xx\left(\frac{d}{dx}x\right)$$

$$= 1xx + x1x + xx1 = 3x^2$$







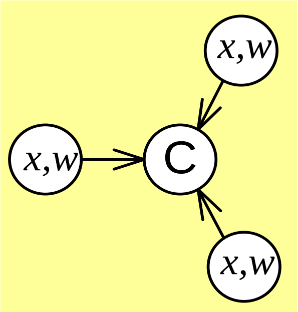
$$= \begin{bmatrix} x & w \end{bmatrix} \left\{ \begin{bmatrix} \begin{bmatrix} A & B \\ B & C \end{bmatrix} & \begin{bmatrix} B & C \\ C & D \end{bmatrix} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} \begin{bmatrix} x \\ w \end{bmatrix}$$

$$= \begin{bmatrix} x & w \end{bmatrix} \begin{bmatrix} A & B \\ B & C \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix}$$

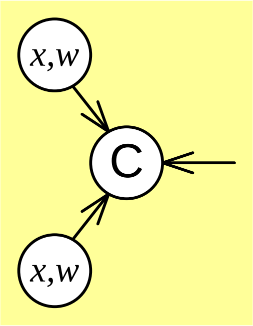
Derivatives

$$\mathbf{C}(x,w) = Ax^3 + 3Bx^2w + 3Cxw^2 + Dw^3$$

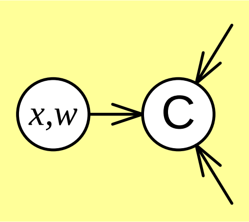
$$= \begin{bmatrix} x & w \end{bmatrix} \left\{ \begin{bmatrix} \begin{bmatrix} A & B \\ B & C \end{bmatrix} & \begin{bmatrix} B & C \\ C & D \end{bmatrix} \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \right\} \begin{bmatrix} x \\ w \end{bmatrix} =$$



$$\begin{bmatrix} \mathbf{C}_x \\ \mathbf{C}_w \end{bmatrix} = 3 \begin{bmatrix} Ax^2 + 2Bxw + Cw^2 \\ Bx^2 + 2Cxw + Dw^2 \end{bmatrix} = 3$$



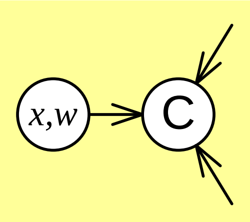
$$\begin{bmatrix} \mathbf{C}_{xx} & \mathbf{C}_{xw} \\ \mathbf{C}_{xw} & \mathbf{C}_{ww} \end{bmatrix} = 6 \begin{bmatrix} Ax + Bw & Bx + Cw \\ Bx + Cw & Cx + Dw \end{bmatrix} = 6$$



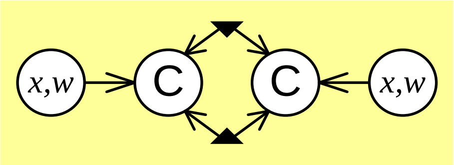
Hessian of cubic

$$\text{Hessian}(\mathbf{C}) \equiv \det\begin{bmatrix} \mathbf{C}_{xx} & \mathbf{C}_{xw} \\ \mathbf{C}_{xw} & \mathbf{C}_{ww} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{C}_{xx} & \mathbf{C}_{xw} \\ \mathbf{C}_{xw} & \mathbf{C}_{ww} \end{bmatrix} = 6\begin{bmatrix} Ax+Bw & Bx+Cw \\ Bx+Cw & Cx+Dw \end{bmatrix} = 6$$

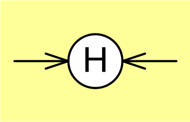


$$\text{Hessian}(\mathbf{C}) = \det\begin{bmatrix} \mathbf{C}_{xx} & \mathbf{C}_{xw} \\ \mathbf{C}_{xw} & \mathbf{C}_{ww} \end{bmatrix} = -18$$

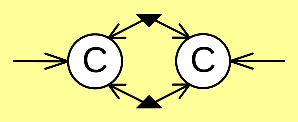


$$\det\left\{6\begin{bmatrix} Ax+Bw & Bx+Cw \\ Bx+Cw & Cx+Dw \end{bmatrix}\right\} = 36\left\{(AC-BB)x^2 + (AD-BC)xw + (DB-CC)w^2\right\}$$

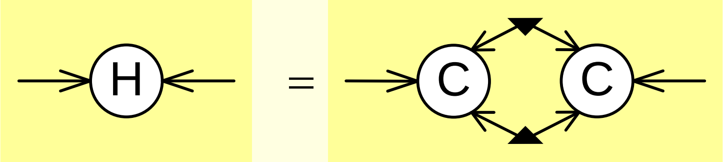
$$= -18\begin{bmatrix} x & w \end{bmatrix}\begin{bmatrix} -2(AC-BB) & -AD+BC \\ -AD+BC & -2(DB-CC) \end{bmatrix}\begin{bmatrix} x \\ w \end{bmatrix}$$



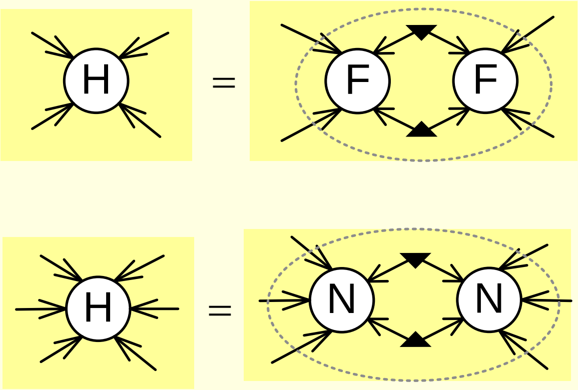
$\equiv$



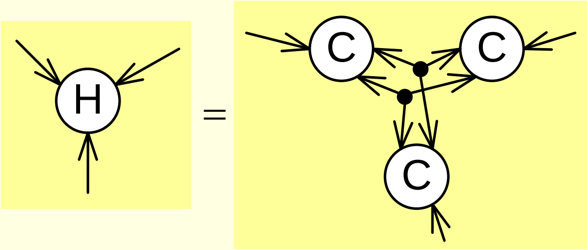
Other Hessians



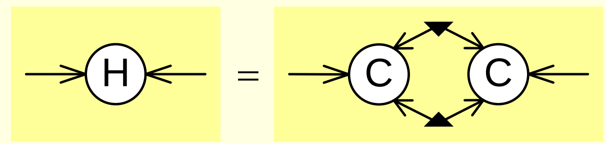
Higher orders



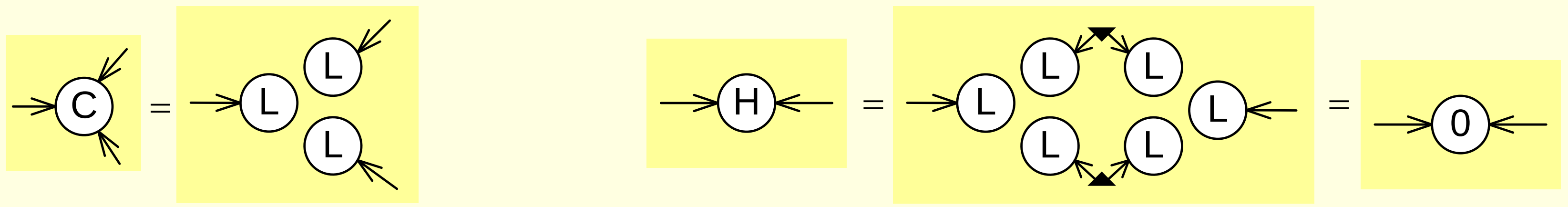
Higher Dimension



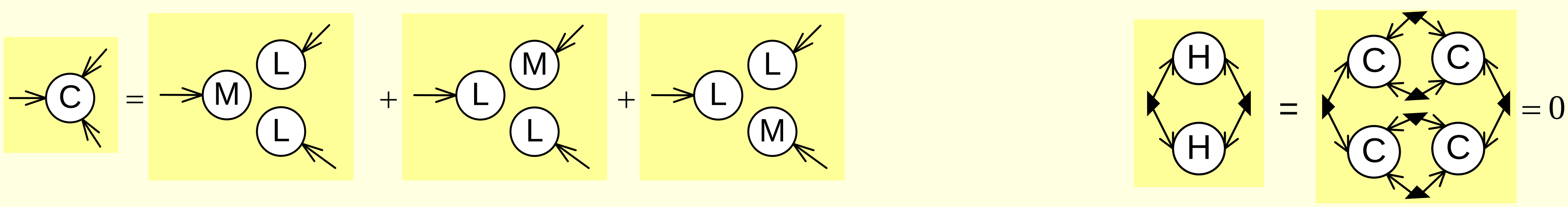
Geometric Meaning of Hessian

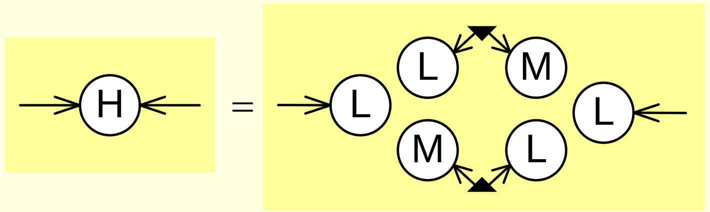


Type 3



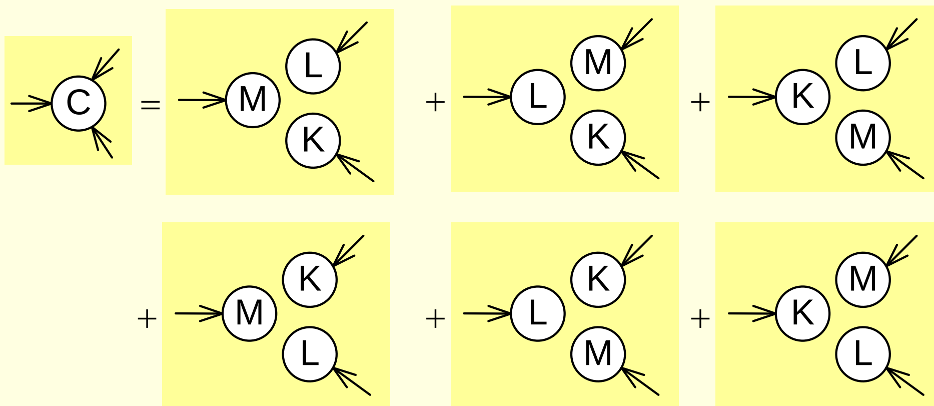
Type 21

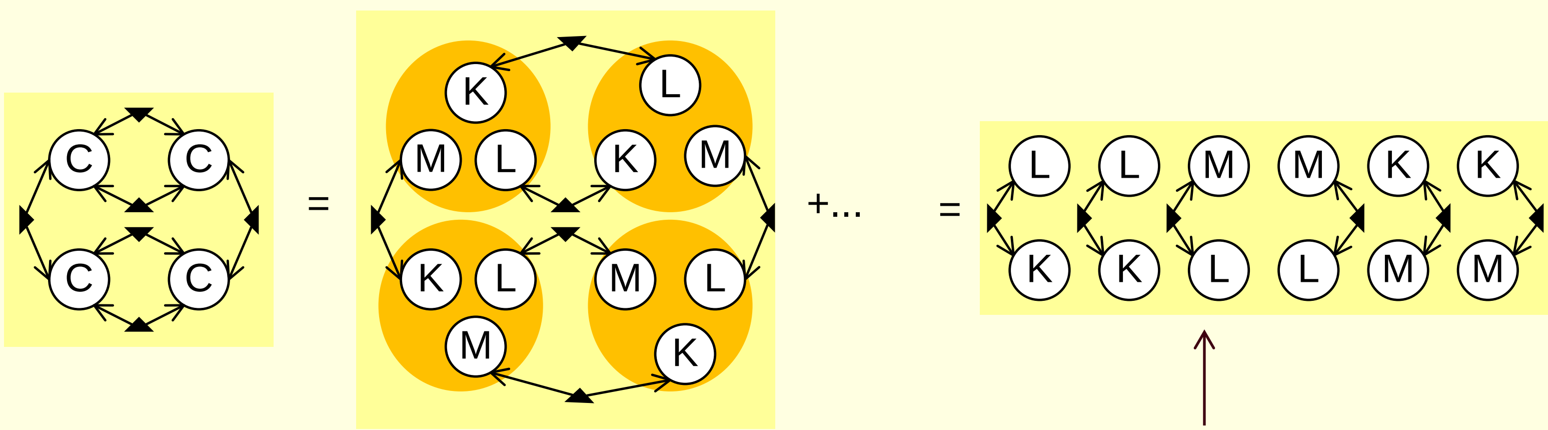




Geometric Meaning of Discriminant

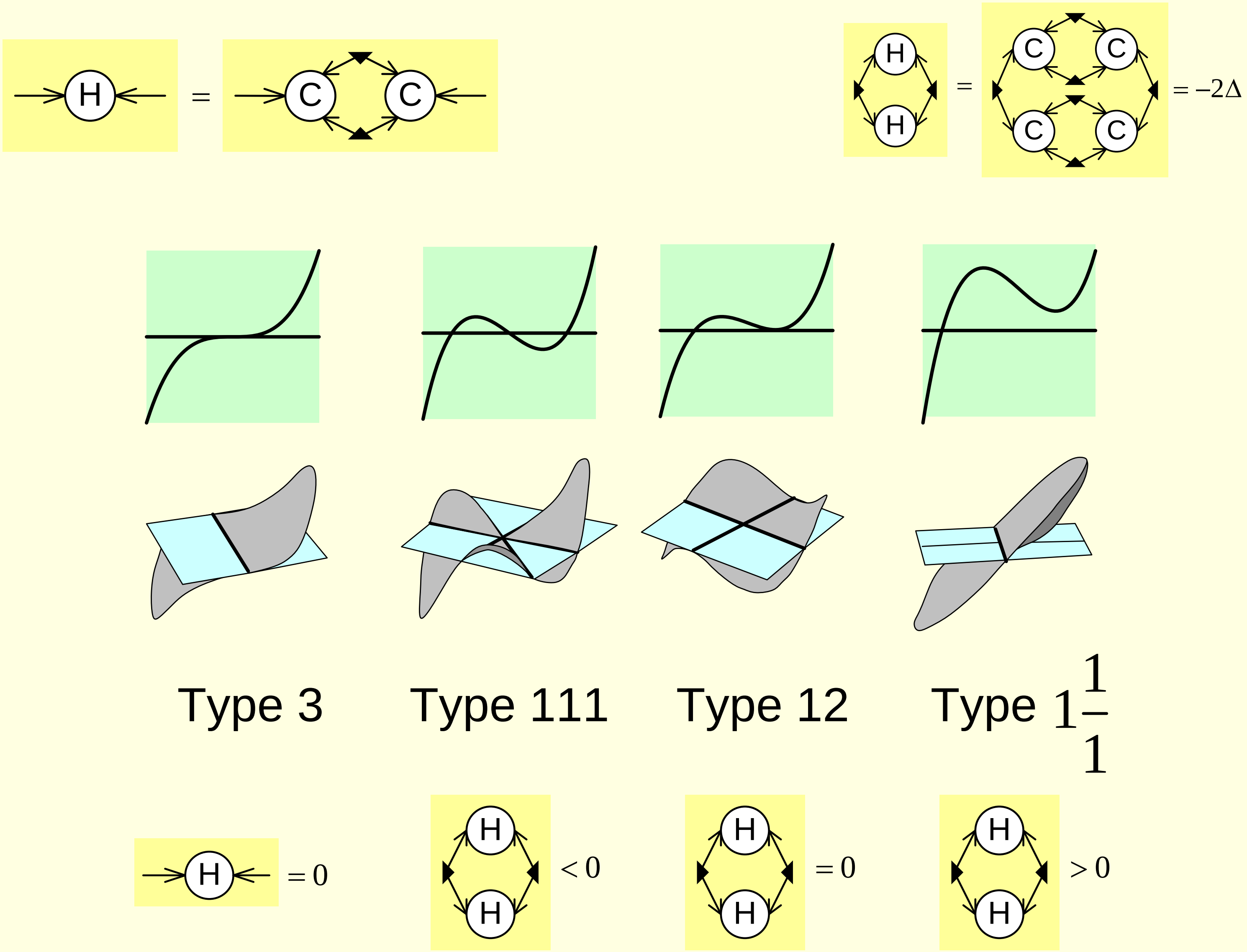
Type 111





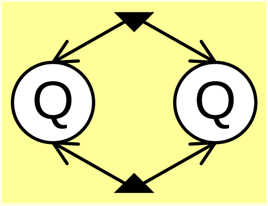
Negative

Types of Cubic

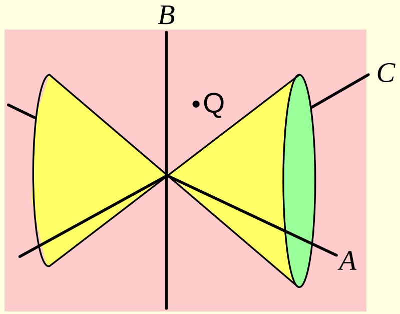


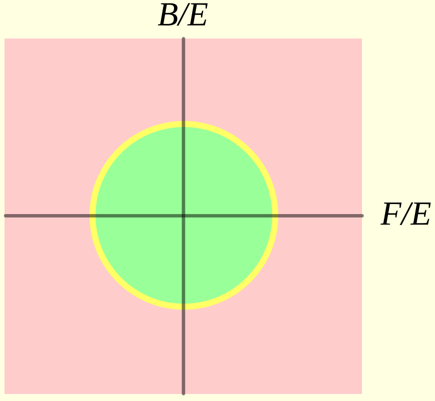
Discriminant Surfaces

Quadratic

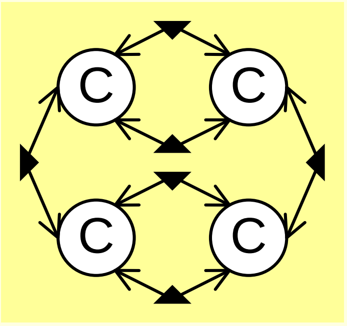


$$= -2\left(AC - B^2\right)$$





Cubic



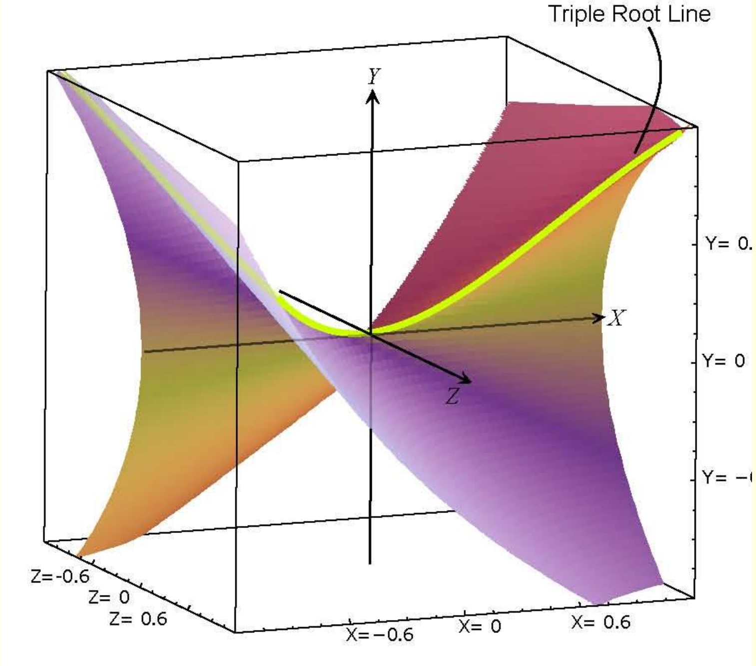
$$= -2\begin{pmatrix} -A^2D^2 + 6ABCD \\ -4AC^3 - 4B^3D + 3B^2C^2 \end{pmatrix}$$

?

Discriminant Surface

$$-A^2D^2 + 6ABCD - 4AC^3 - 4B^3D + 3B^2C^2 = 0$$

$$-\left(\frac{D}{A}\right)^2 + 6\frac{B}{A}\frac{C}{A}\frac{D}{A} - 4\left(\frac{C}{A}\right)^3 - 4\left(\frac{B}{A}\right)^3\frac{D}{A} + 3\left(\frac{B}{A}\right)^2\left(\frac{C}{A}\right)^2 = 0$$

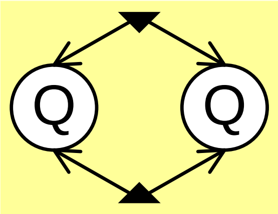


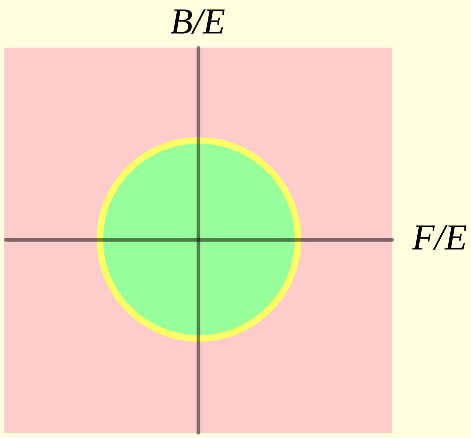
$$\frac{B}{A} \equiv X,\ \frac{C}{A} \equiv Y,\ \frac{D}{A} \equiv Z$$

$$-Z^2 + 6XYZ - 4Y^3 - 4X^3Z + 3X^2Y^2 = 0$$

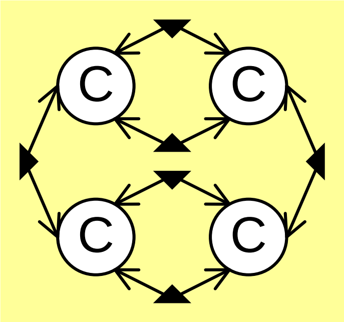
Discriminant Surfaces

Quadratic

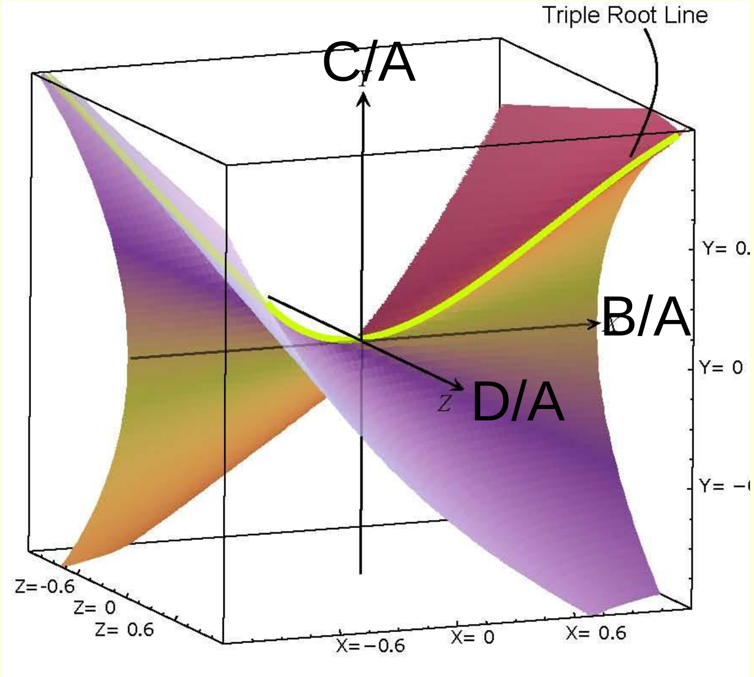




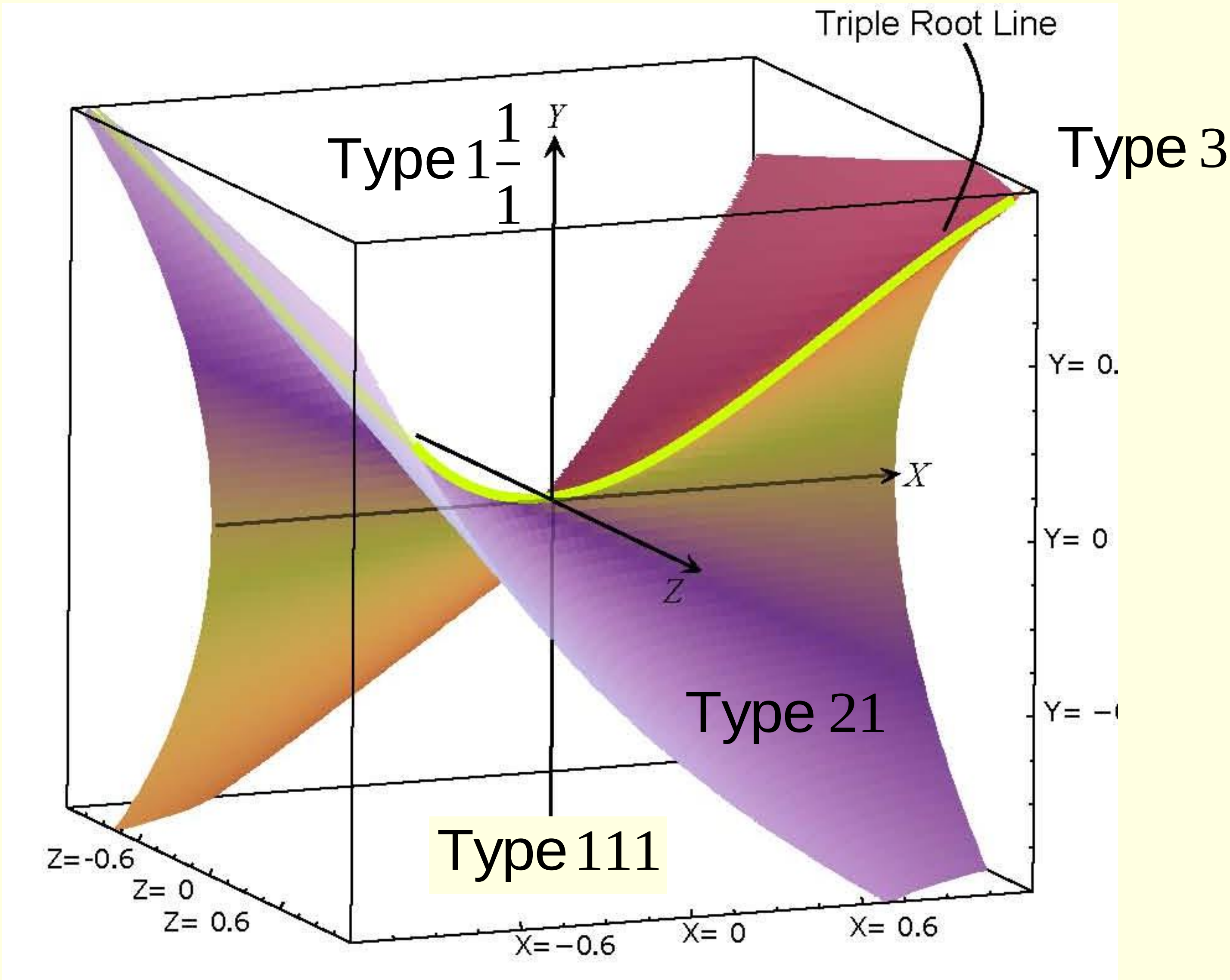
Cubic



$$= -2\begin{pmatrix} -A^2D^2 + 6ABCD \\ -4AC^3 - 4B^3D + 3B^2C^2 \end{pmatrix}$$



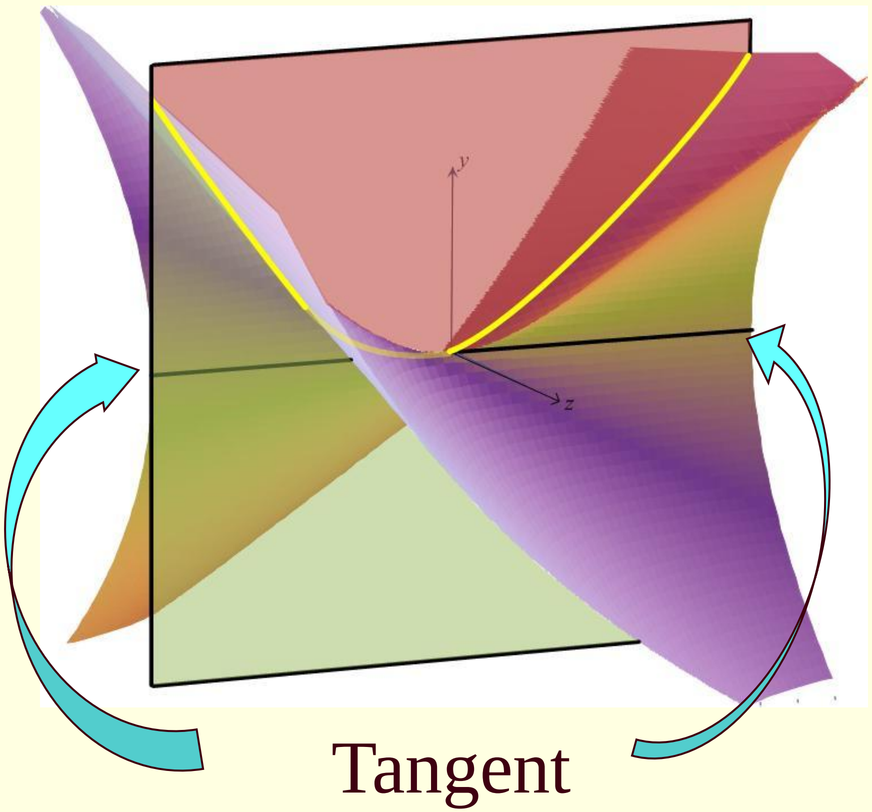
Types of Cubics

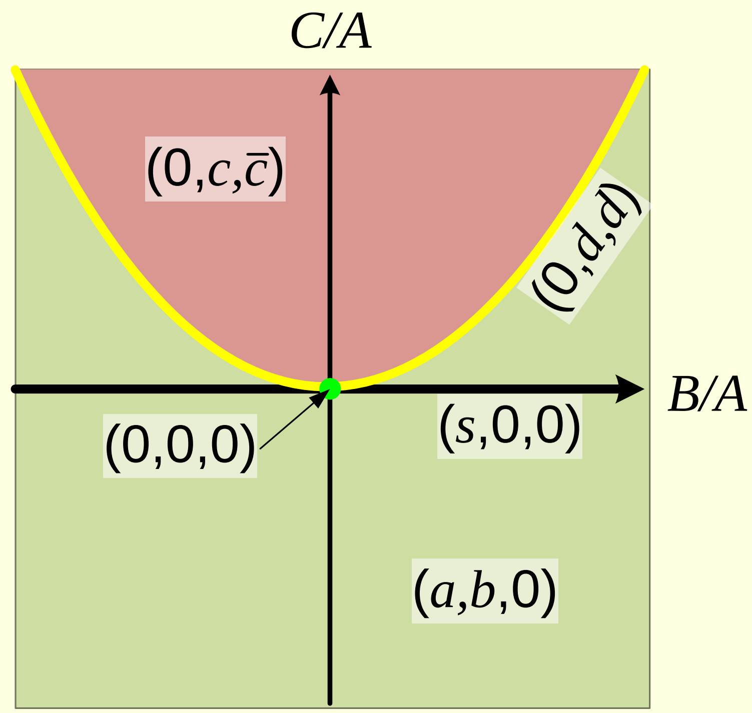


Locus of Cubics with one (or more) roots at (x,w)=(0,1)

$$f(x,w) = Ax^3 + 3Bx^2w + 3Cxw^2 + \boxed{0}w^3$$

Three roots: (r_1,r_2,r_3), $r_1=0$

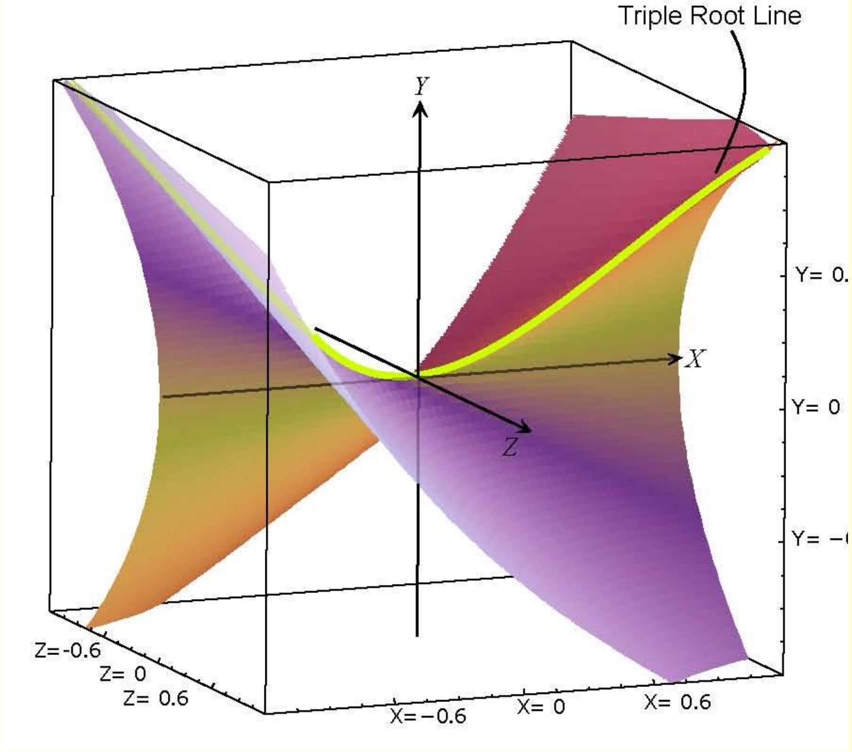




Slice thru space at $D/A=0$

Plane rolls along surface as root=0 changes

Less obvious property



Triple root line is tangent to plane at infinity

A New Coordinate System

$$\begin{bmatrix} x & w \end{bmatrix} = \begin{bmatrix} r\cos\alpha & r\sin\alpha \end{bmatrix}$$

$$Ax^3 + 3Bx^2w + 3Cxw^2 + Dw^3 = r^3 \begin{pmatrix} +\tfrac{3}{4}(A+C)\cos\alpha + \tfrac{3}{4}(B+D)\sin\alpha \\ +\tfrac{1}{4}(A-3C)\cos 3\alpha + \tfrac{1}{4}(3B-D)\sin 3\alpha \end{pmatrix}$$

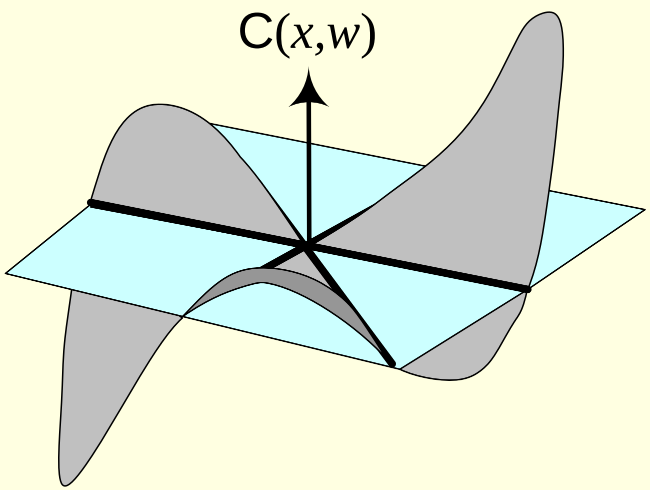
$$= r^3 \begin{pmatrix} +3E\cos\alpha + 3F\sin\alpha \\ +G\cos 3\alpha + H\sin 3\alpha \end{pmatrix}$$

$$E = (A+C)/4$$

$$F = (B+D)/4$$

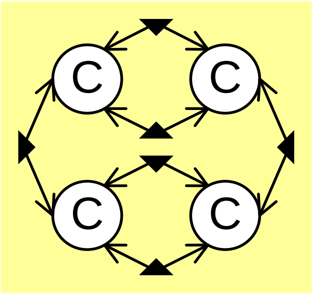
$$G = (A-3C)/4$$

$$H = (3B-D)/4$$



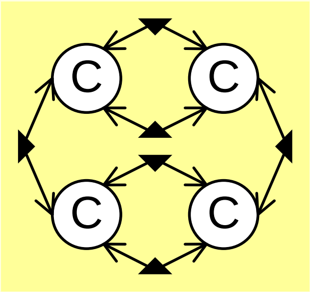
Discriminant in new system

$$3E + G = A$$
$$E - G = C$$
$$F + H = B$$
$$3F - H = D$$

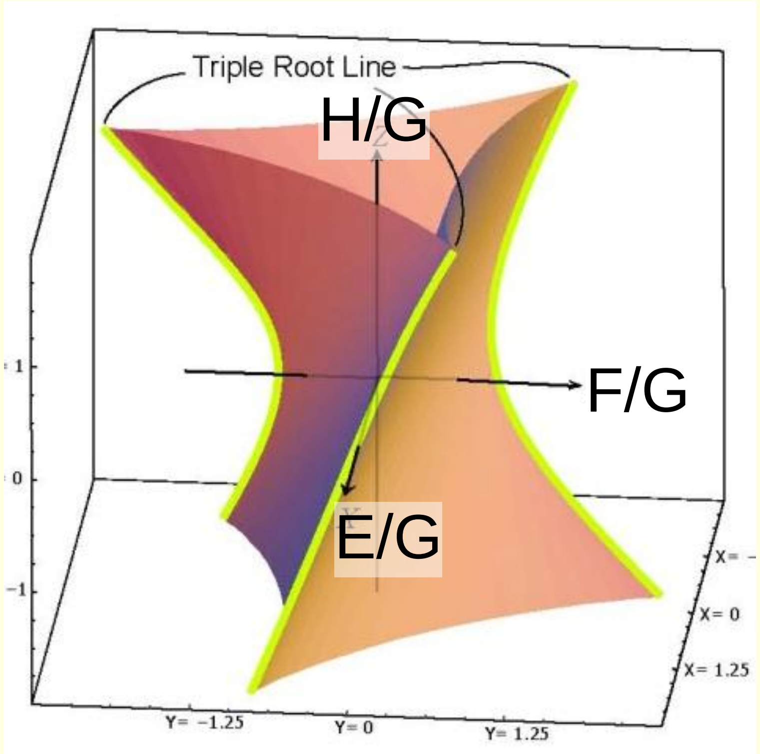


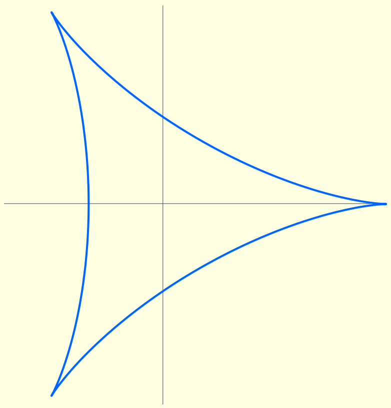
$$= -2\begin{pmatrix} -A^2D^2 + 6ABCD \\ -4AC^3 - 4B^3D + 3B^2C^2 \end{pmatrix} = 4\begin{pmatrix} -3\left(F^2 + E^2\right)^2 \\ +8HF\left(3E^2 - F^2\right) - 8GE\left(3FF - E^2\right) \\ -6\left(F^2 + E^2\right)\left(G^2 + H^2\right) \\ +\left(G^2 + H^2\right)^2 \end{pmatrix}$$

Discriminant Plot in new system



$$= 4\begin{pmatrix} -3\left(F^2+E^2\right)^2 \\ +8HF\left(3E^2-F^2\right)-8GE\left(3FF-E^2\right) \\ -6\left(F^2+E^2\right)\left(G^2+H^2\right) \\ +\left(G^2+H^2\right)^2 \end{pmatrix}$$

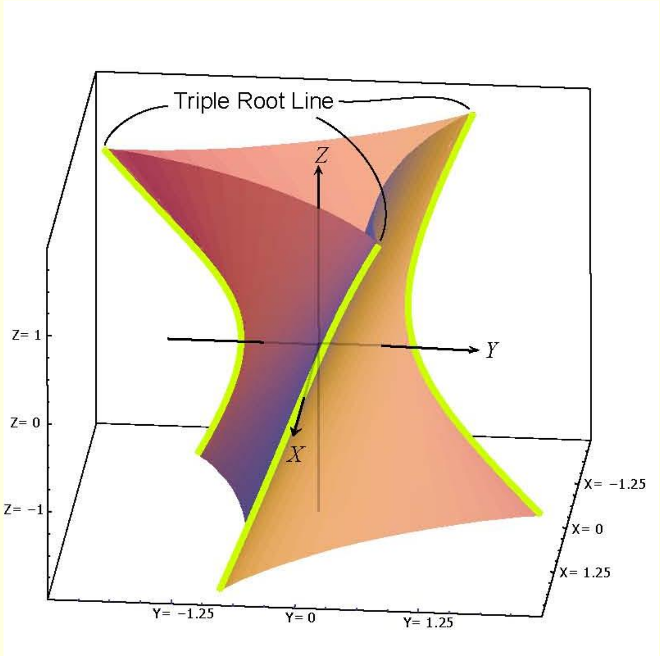


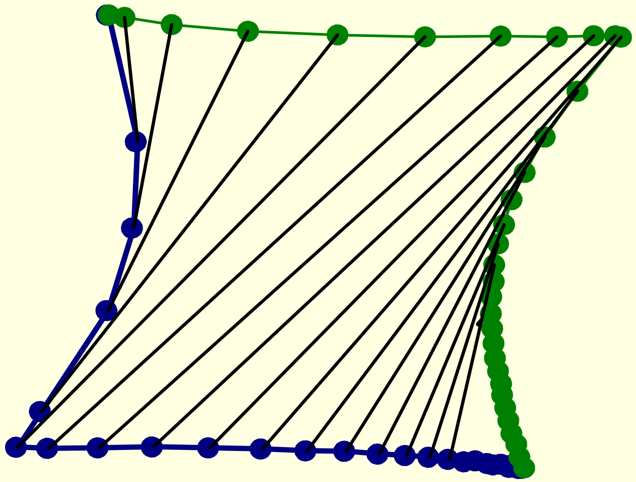


Discriminant in EFGH space

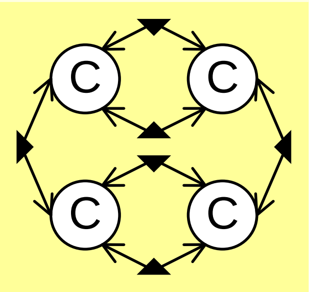
Zeeman 1976

Helamon Fergusen

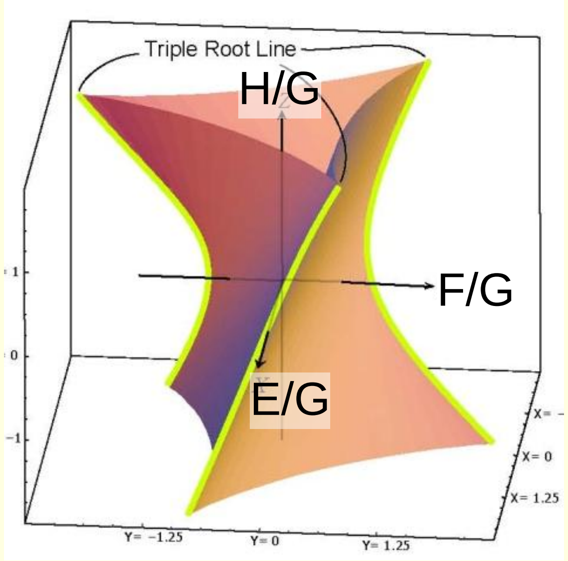


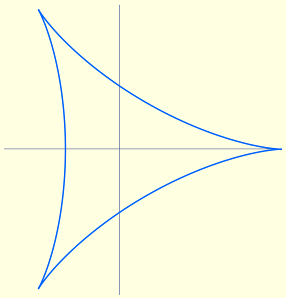


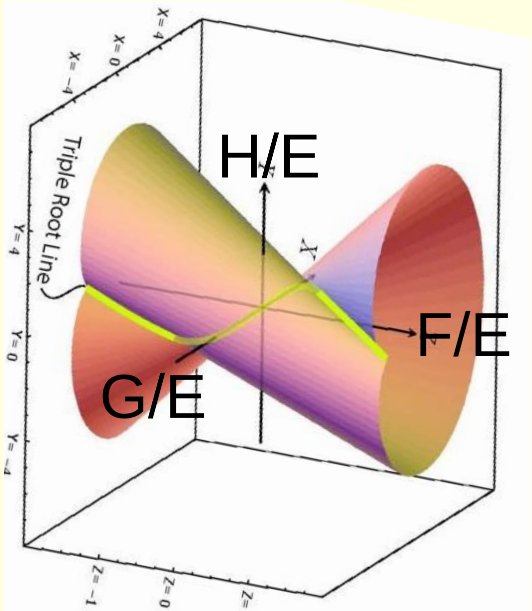
Another View of Discriminant

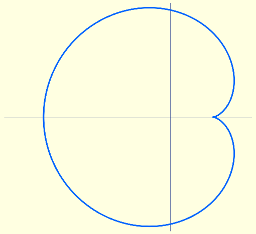


$$
= 4\begin{pmatrix}
-3\left(F^2+E^2\right)^2 \\
+8HF\left(3E^2-F^2\right)-8GE\left(3FF-E^2\right) \\
-6\left(F^2+E^2\right)\left(G^2+H^2\right) \\
+\left(G^2+H^2\right)^2
\end{pmatrix}
$$

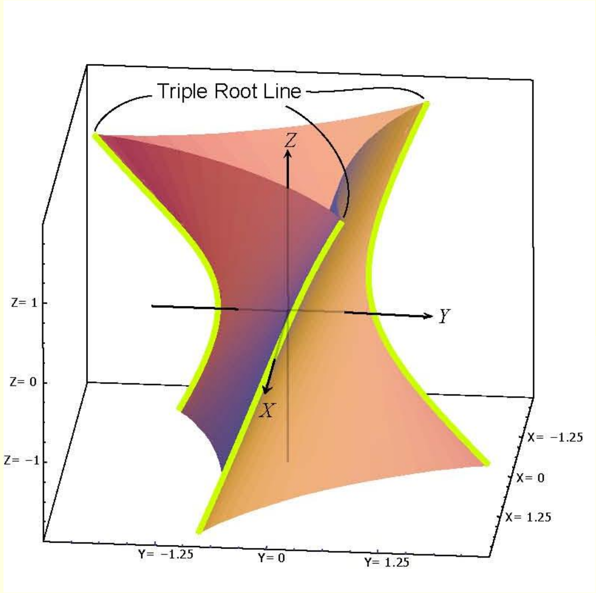


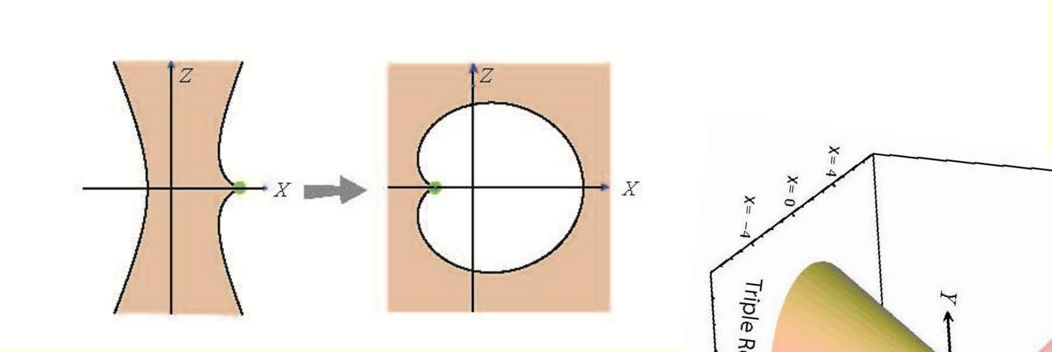


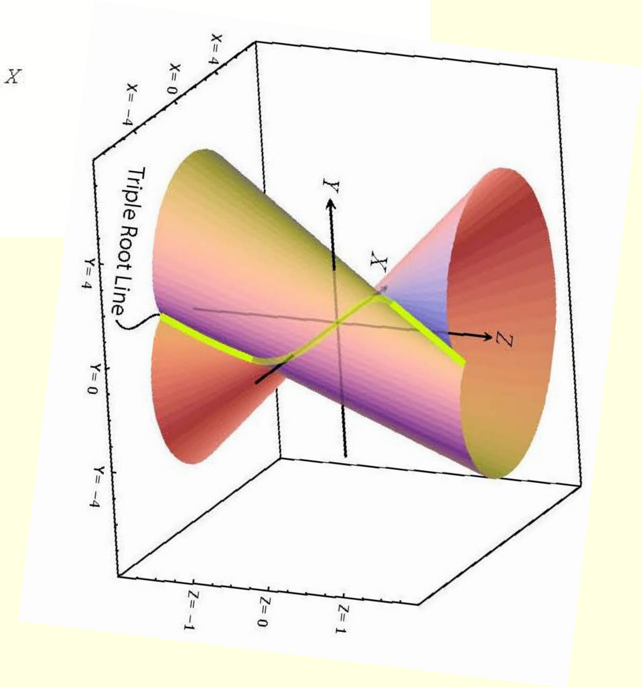




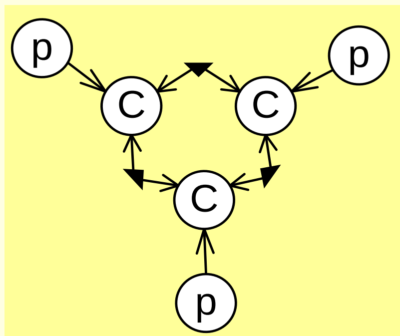
Transforming Deltoid to Cardioid



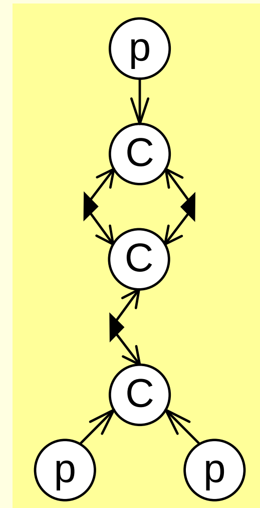




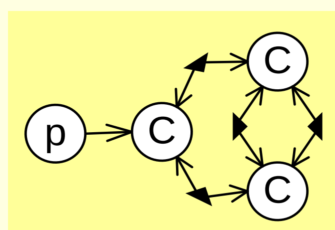
Degree 3 Diagrams



= 0

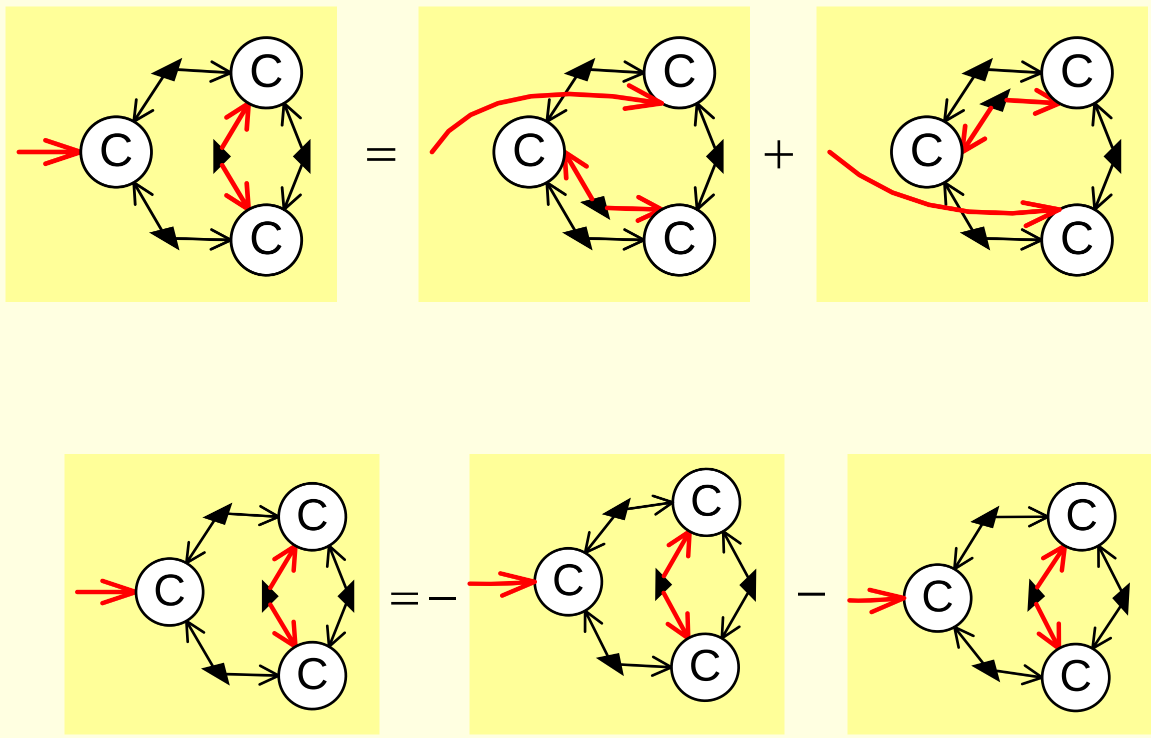


= J

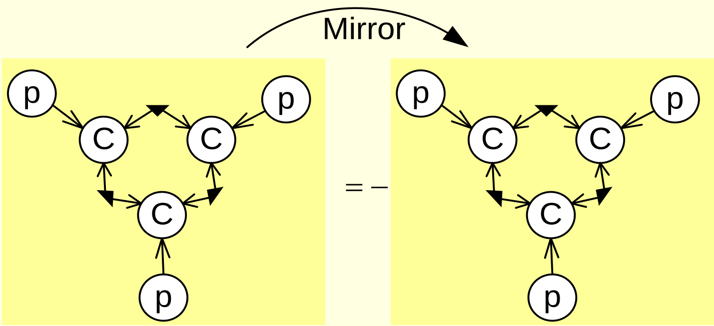


= 0

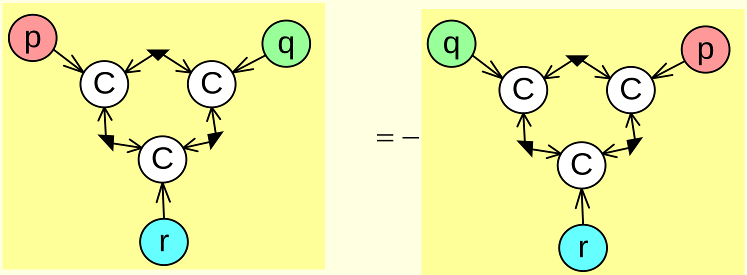
Proof of 0



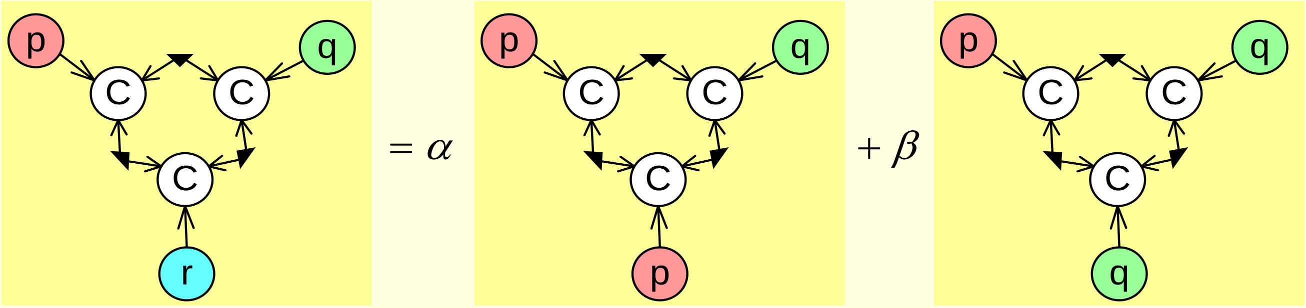
Proof of 0



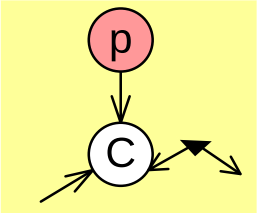
What about?



Write **r** in terms of **p**,**q**



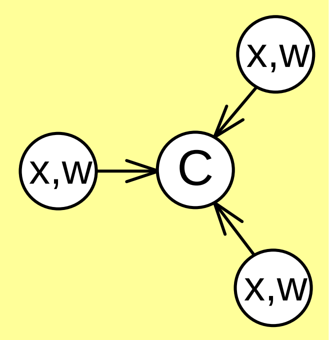
Or: brute force expansion



$$= \begin{bmatrix} -Bx_p - Cw_p & Ax_p + Bw_p \\ -Cx_p - Dw_p & Bx_p + Cw_p \end{bmatrix}$$

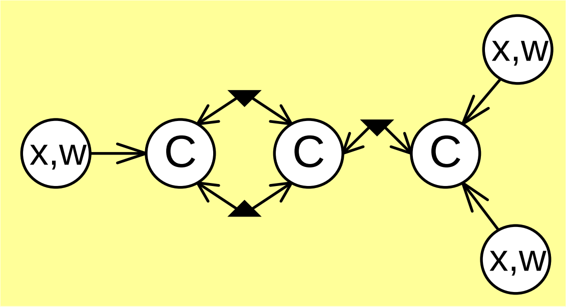
Skew covariant

$$\mathbf{C}(x,w) =$$



$$= Ax^3 + 3Bx^2w + 3Cxw^2 + Dw^3$$

$$\mathbf{J}(x,w) =$$

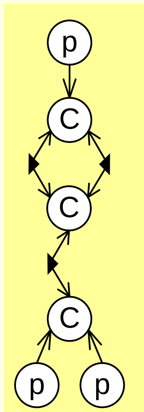


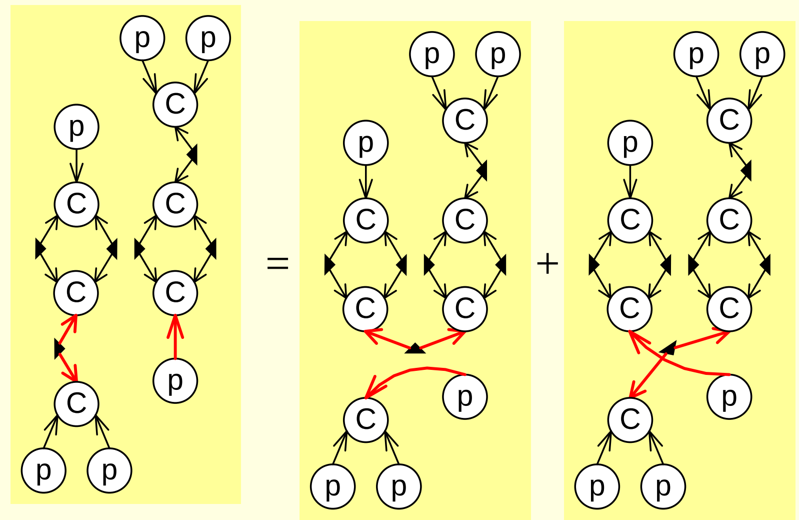
$$= A_J x^3 + 3B_J x^2 w + 3C_J xw^2 + D_J w^3$$

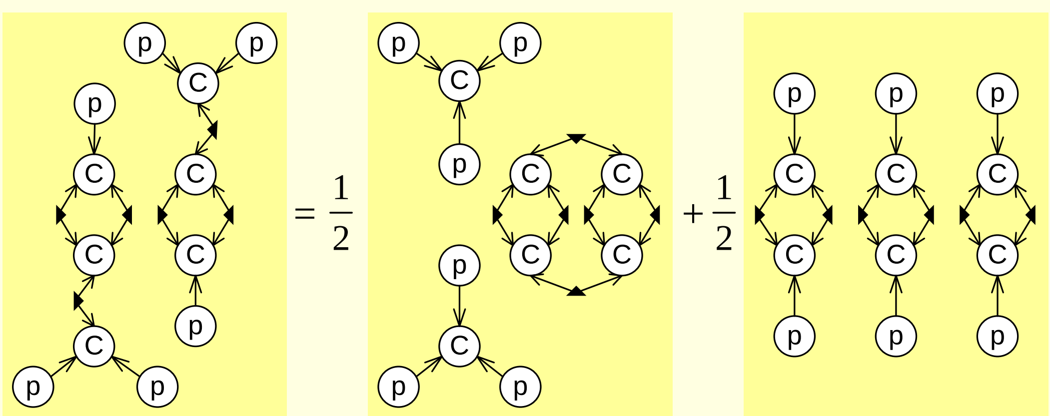
$$\begin{bmatrix} x & w \end{bmatrix} \begin{bmatrix} -2(AC-BB) & -AD+BC \\ -AD+BC & -2(DB-CC) \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} Ax^2+2Bxw+Cw^2 \\ Bx^2+2Cxw+Dw^2 \end{bmatrix}^T$$

$$\begin{aligned} A_J &= +A^2D - 3ABC + 2B^3 \\ B_J &= -2AC^2 + ABD + B^2C \\ C_J &= -ACD + 2B^2D - BC^2 \\ D_J &= -AD^2 + 3BCD - 2C^3 \end{aligned}$$

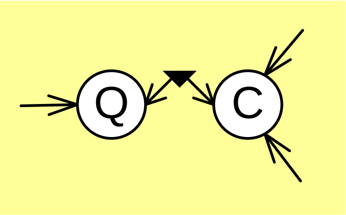
Syzygy of J





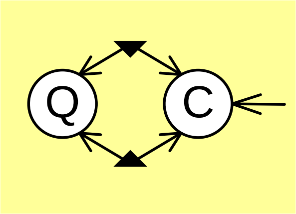


Terminology



Jacobian of Q,C

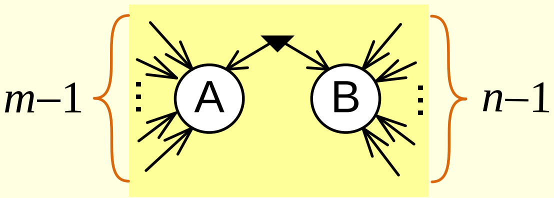
= First Transvectant of Q,C



Second Transvectant of Q,C

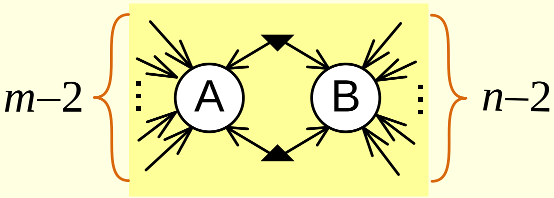
= Apolar Covariant of Q,C

Terminology

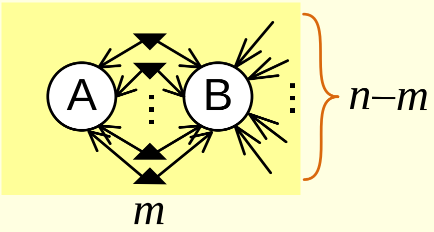


Jacobian of A,B

= First Transvectant of A,B



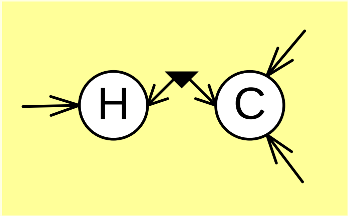
Second Transvectant of A,B



m^{th} Transvectant of A,B

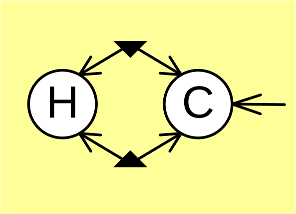
= Apolar Covariant of A,B

Terminology using H



Skew = Jacobian of H,C

= First Transvectant of H,C



Second Transvectant of H,C

= Apolar Covariant of H,C = 0