CSE590B Lecture 6
Cubic Polynomials

Covariants and Resultants

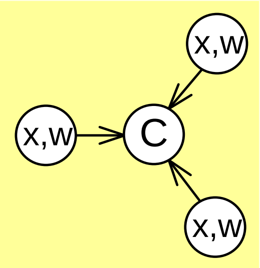
James F. Blinn

JimBlinn.Com

http://courses.cs.washington.edu/courses/cse590b/13au/

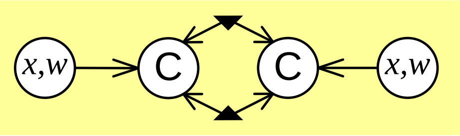
Covariants of Cubic

$\mathbf{C}(x,w) =$



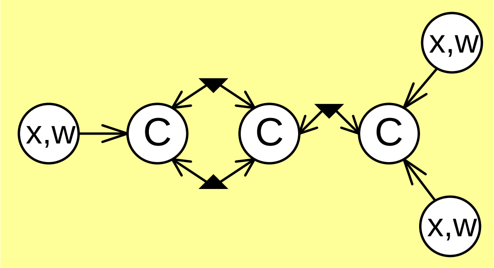
$= Ax^3 + 3Bx^2w + 3Cxw^2 + Dw^3$

$\mathbf{H}(x,w) =$



$$= \begin{bmatrix} x & w \end{bmatrix} \begin{bmatrix} -2(AC - BB) & -AD + BC \\ -AD + BC & -2(DB - CC) \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix}$$

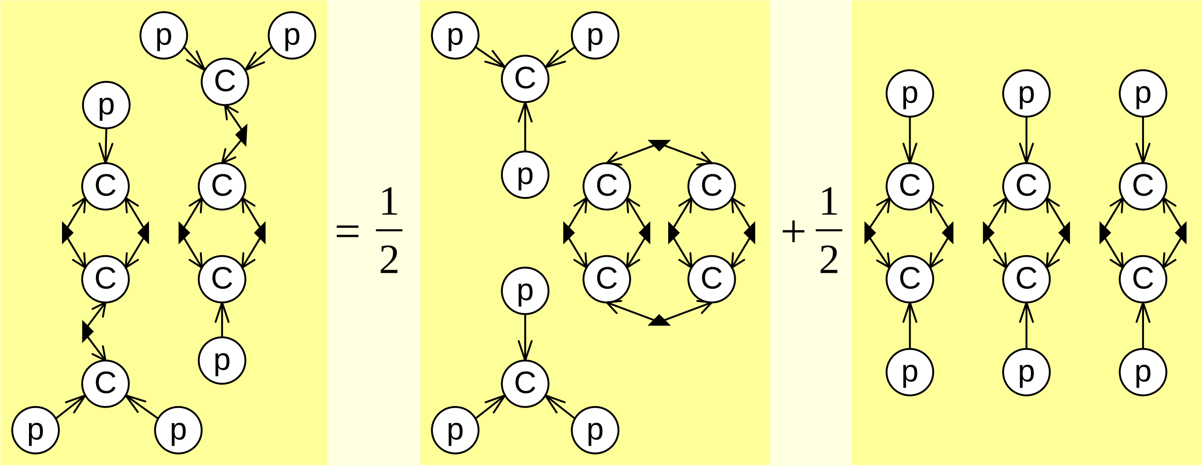
$\mathbf{J}(x,w) =$



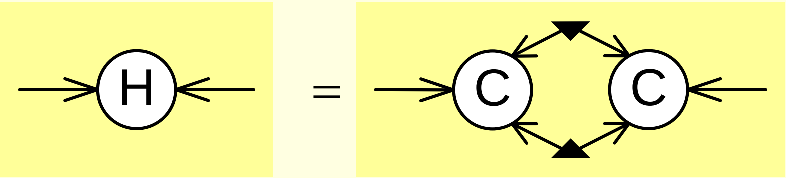
$= A_J x^3 + 3B_J x^2 w + 3C_J x w^2 + D_J w^3$

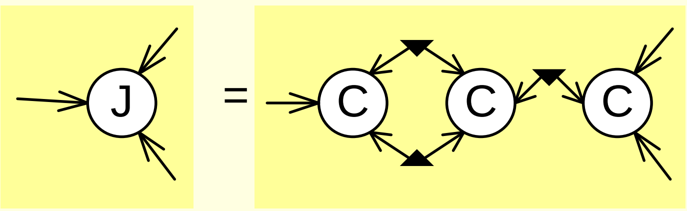
$$A_J = +A^2D - 3ABC + 2B^3$$
$$B_J = -2AC^2 + ABD + B^2C$$
$$C_J = -ACD + 2B^2D - BC^2$$
$$D_J = -AD^2 + 3BCD - 2C^3$$

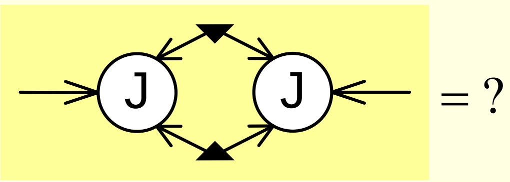
Syzygy of J



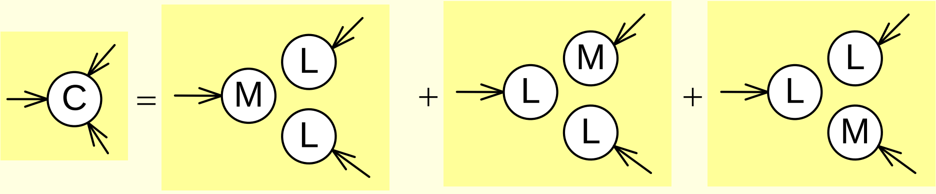
Hessian of J



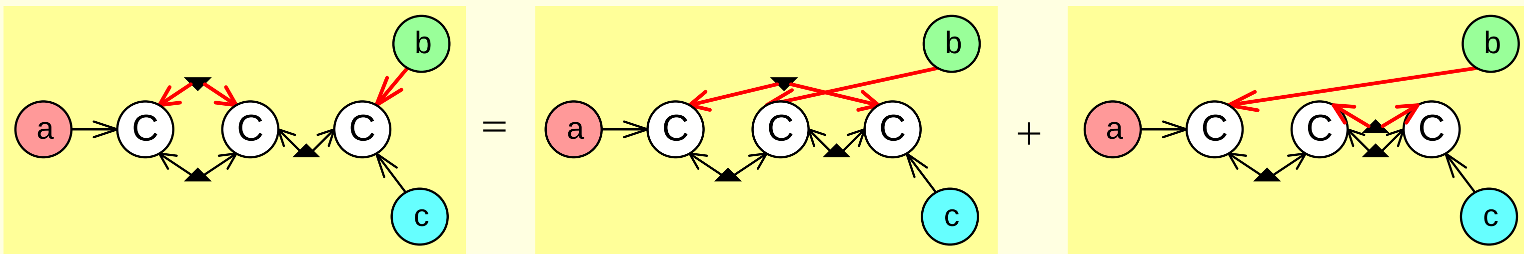


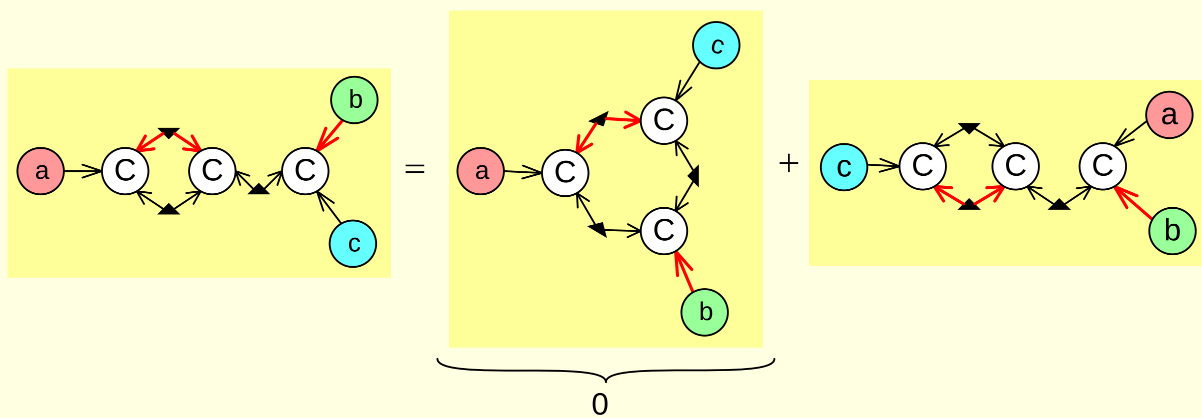


Symmetrize? Like we did with:

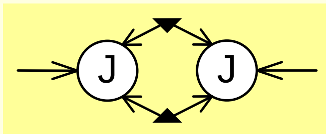


J is already symmetrical!

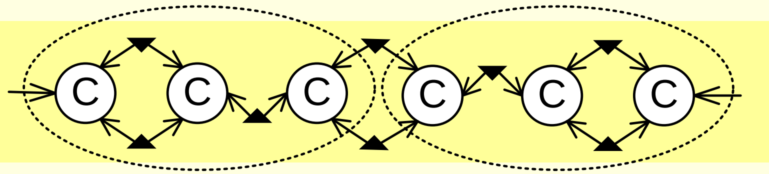




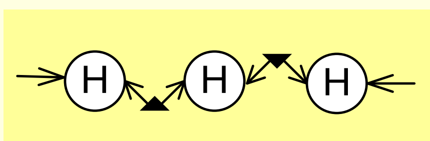
Hessian of J

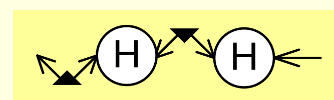


=

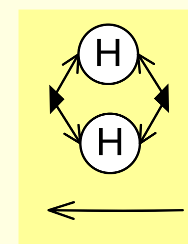


=

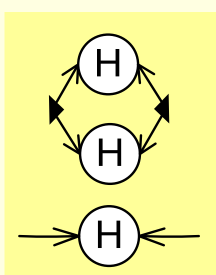




$= -2$

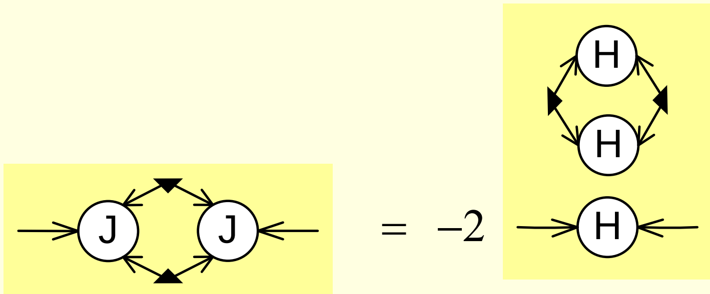


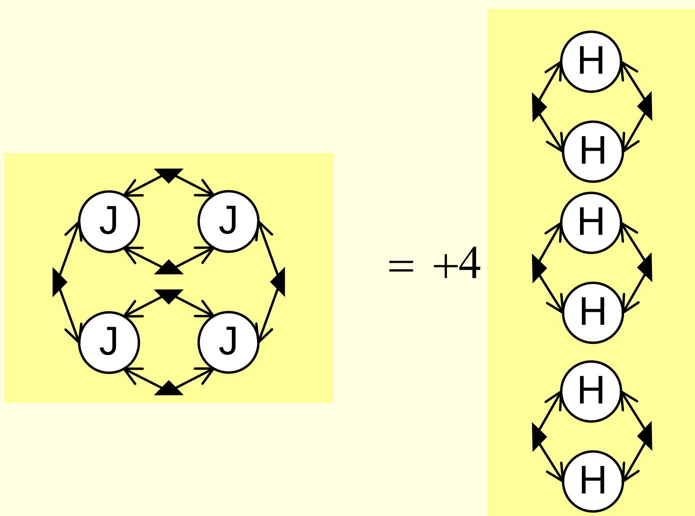
$= -2$



Hessian of J is same as
Hessian of C (up to a scale)

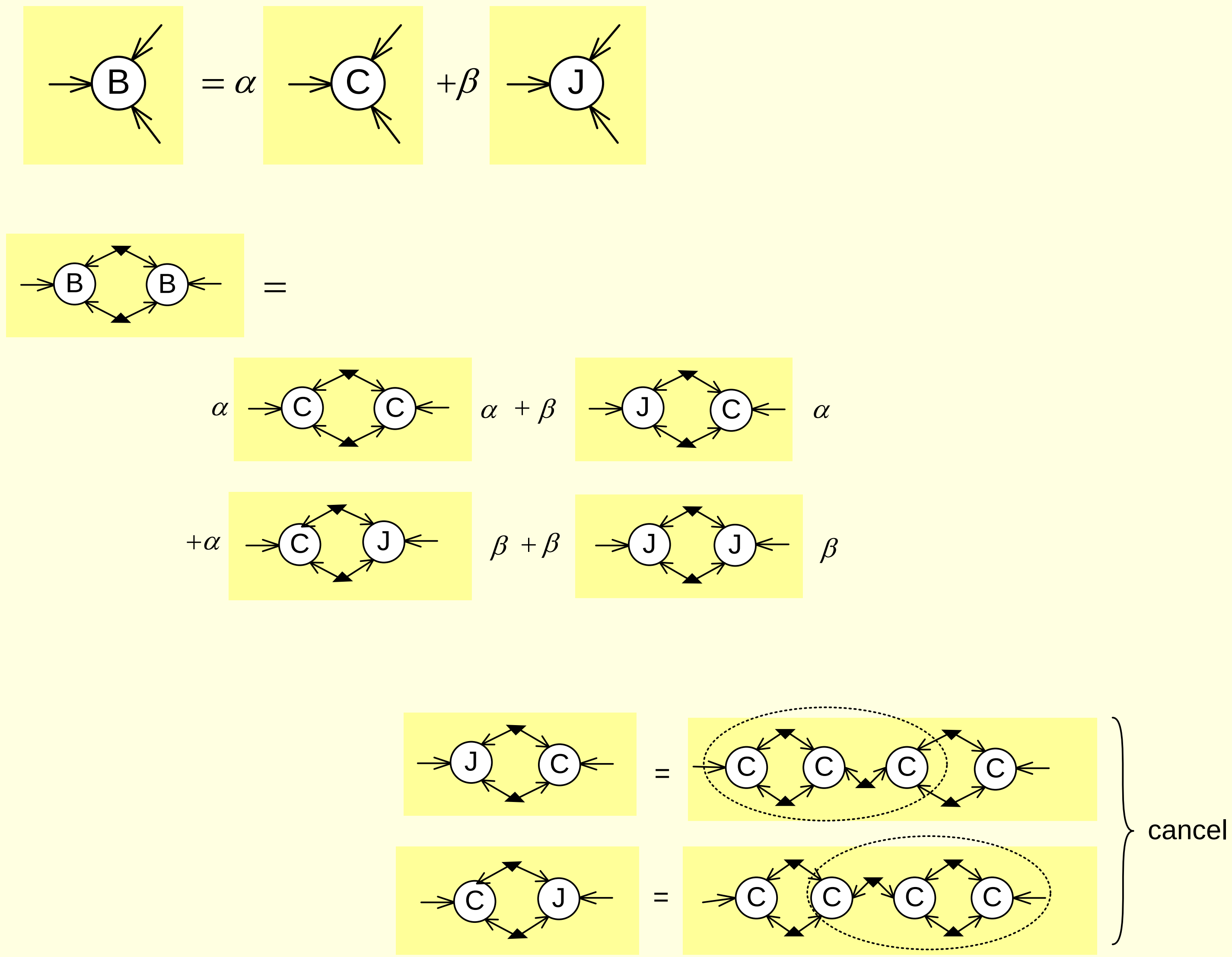
Discriminant of J



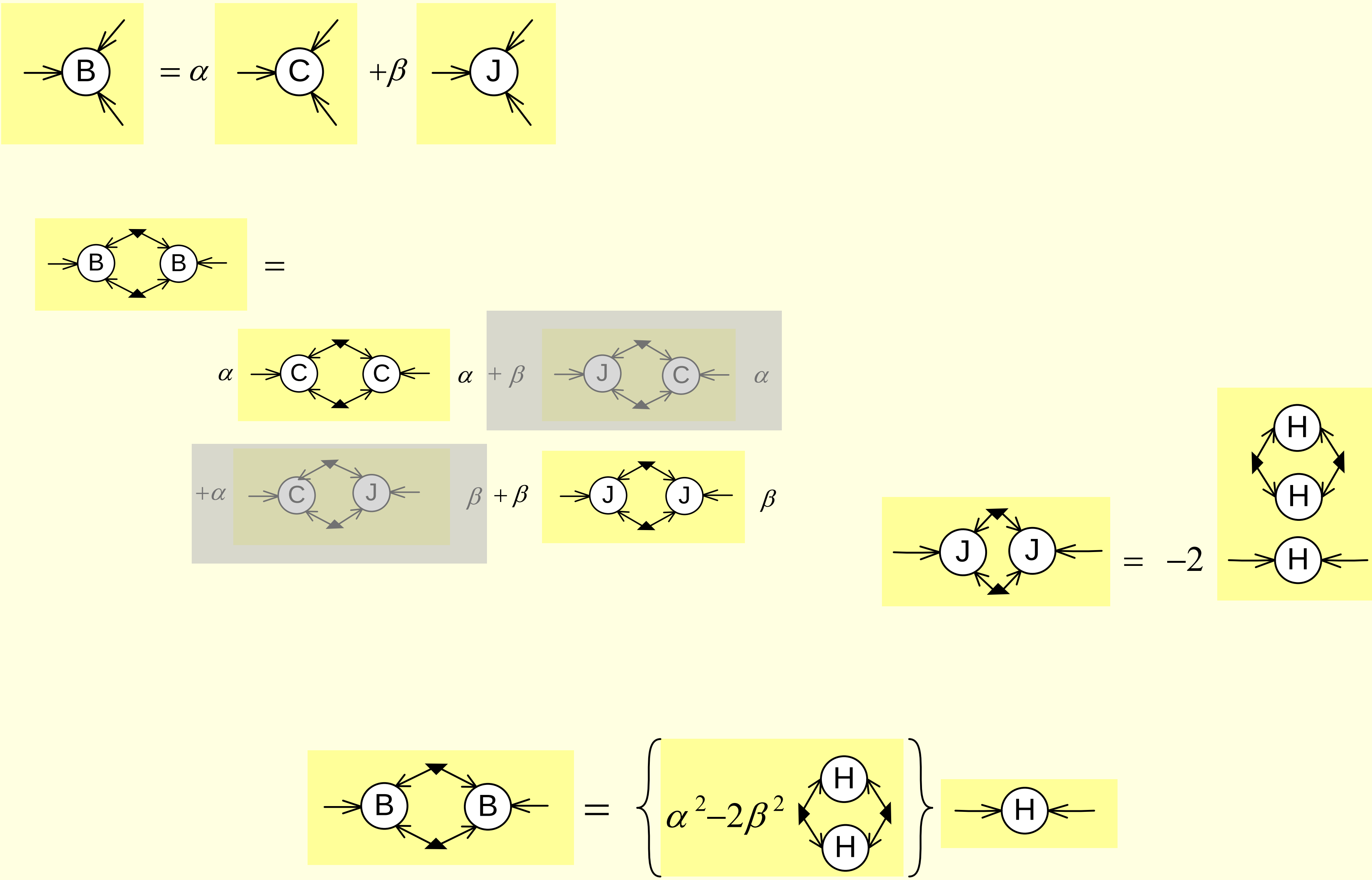


Sign of Discrim(J)
is same as
sign of Discrim(C)

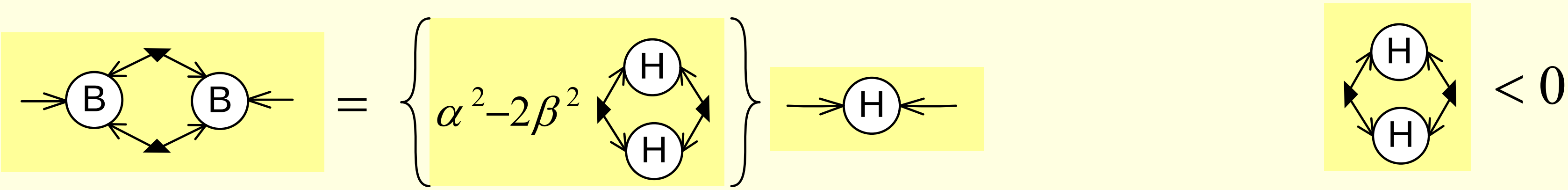
Hessian of Linear Combo



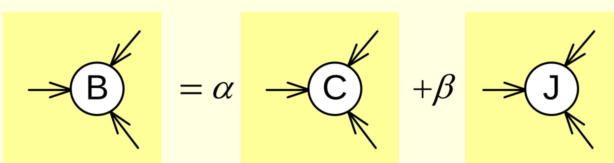
Hessian of Linear Combo

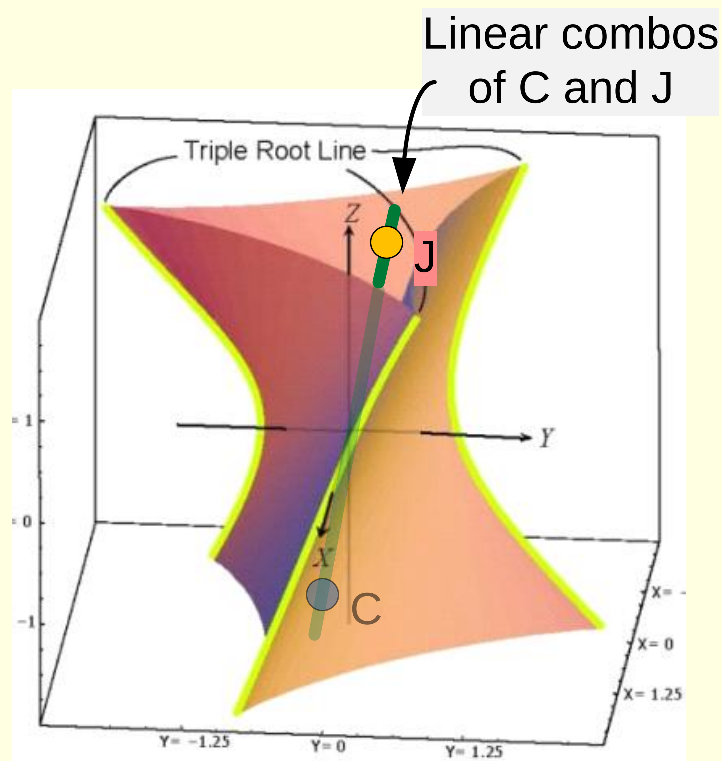


Locus of constant H for type 111

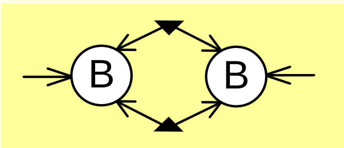


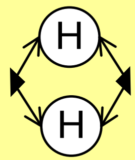
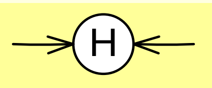
Always positive

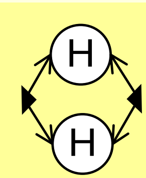




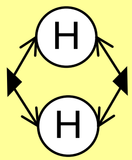
Locus of constant H for type 1 1/1

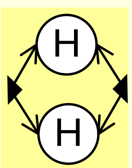


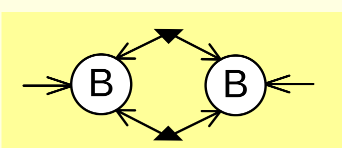
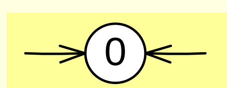
$=$ $\left\{ \alpha^2 - 2\beta^2 \right.$  $\left. \right\}$ 

 > 0

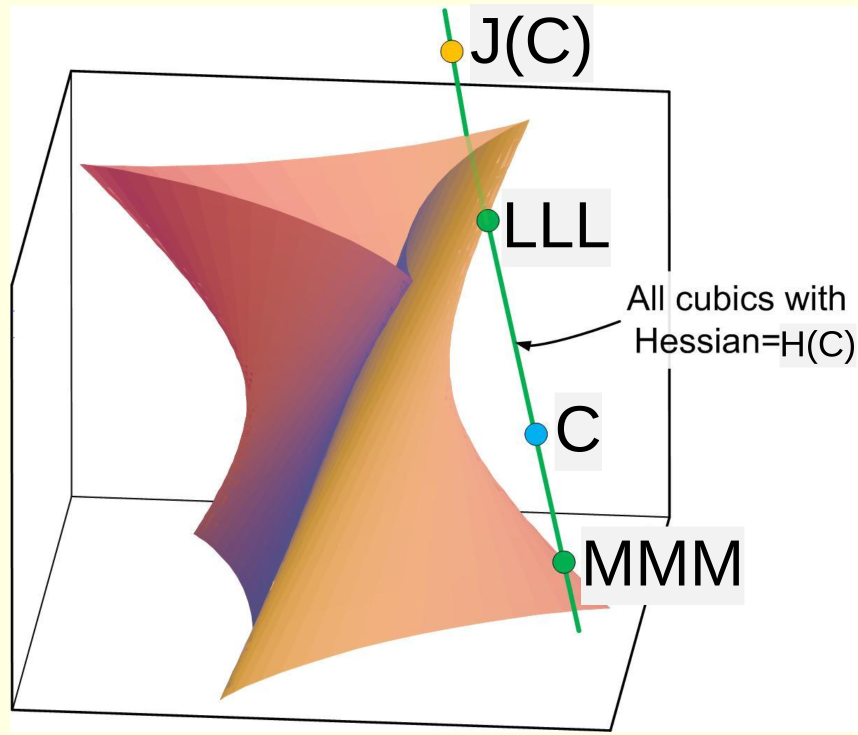
If

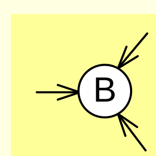
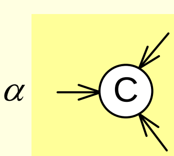
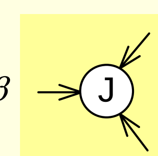
$\alpha^2 - 2\beta^2$  $= 0$

$\alpha = \pm\beta\sqrt{2}$ 

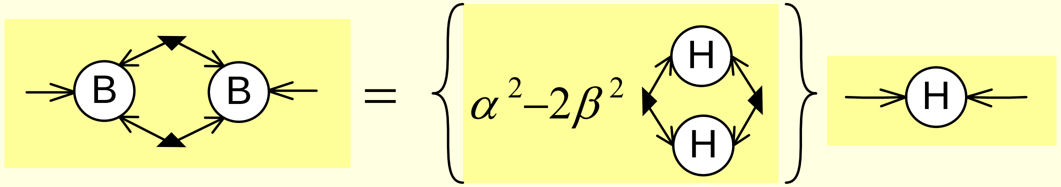
 $=$ 

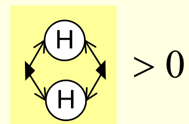
$\Rightarrow$Triple Root

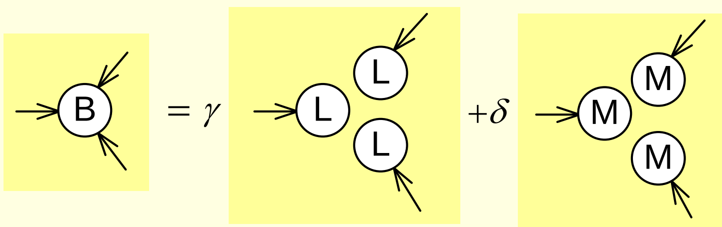


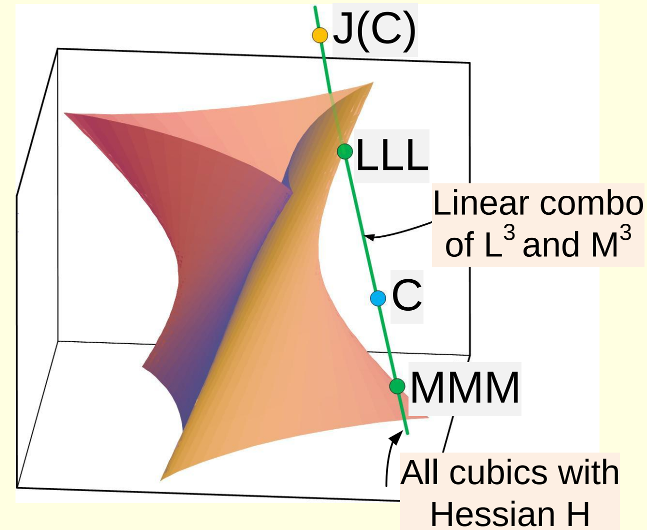
 $= \alpha$  $+\beta$ 

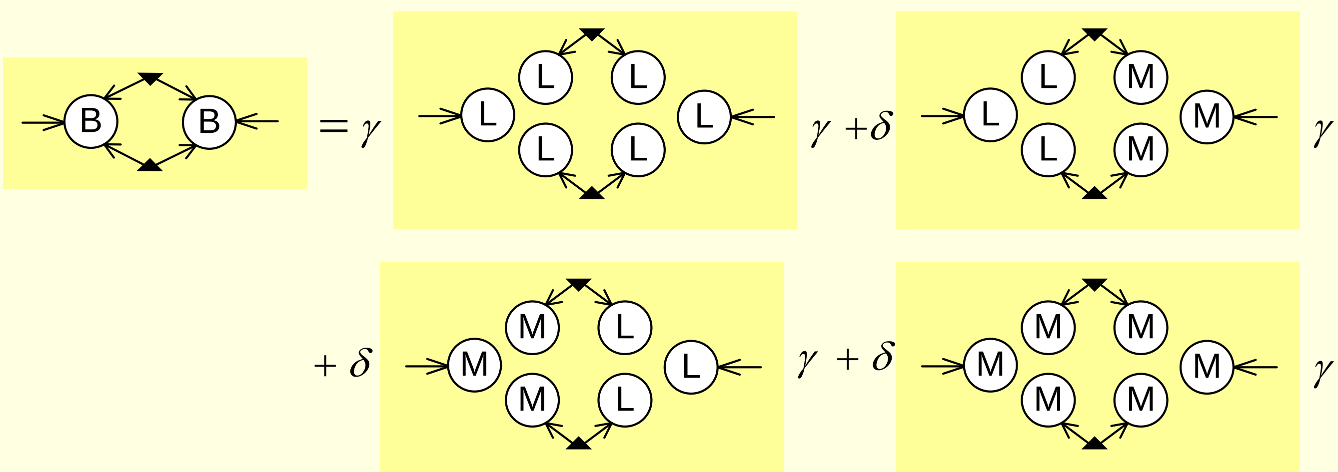
Locus of constant H for type 1 1/1



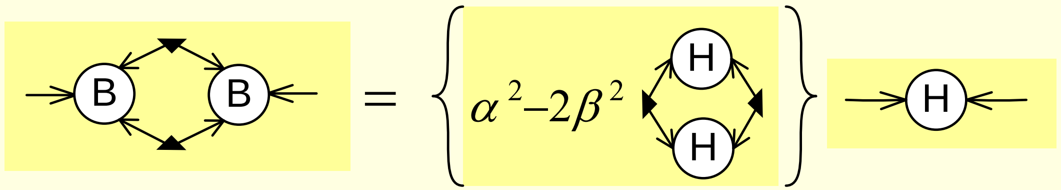


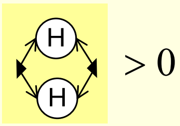


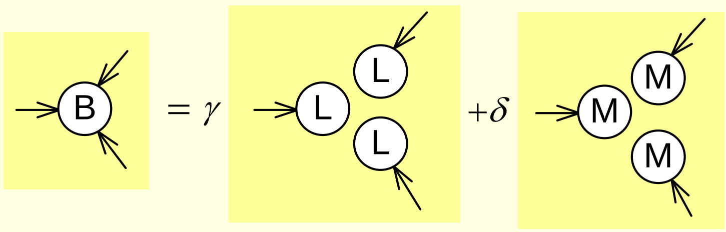


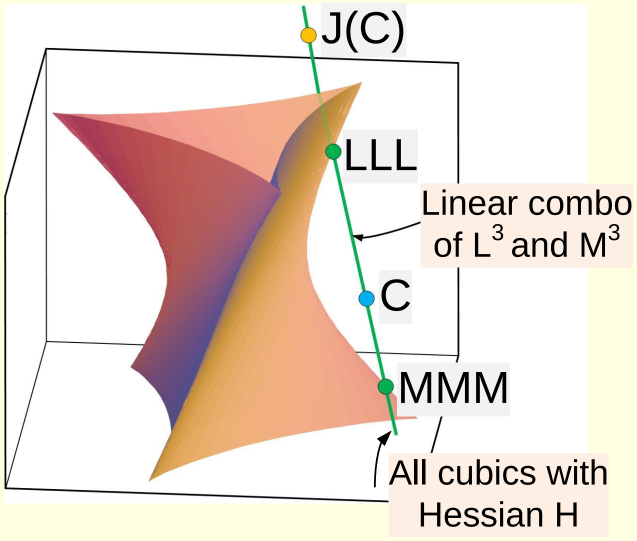


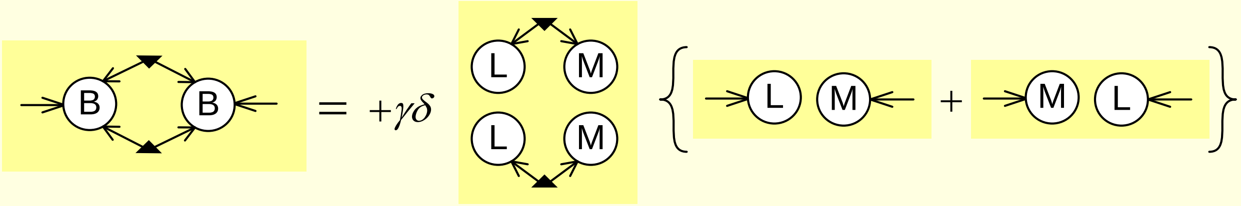
Locus of constant H for type 1 1/1





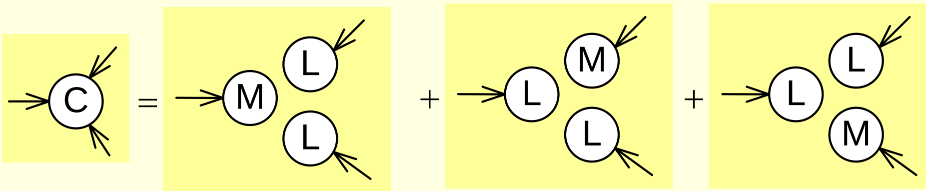


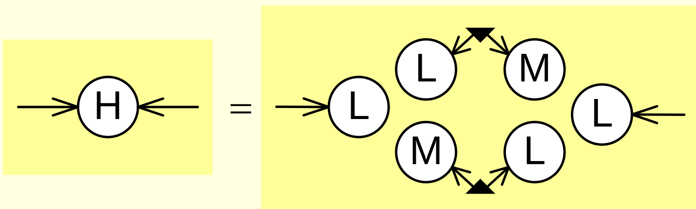


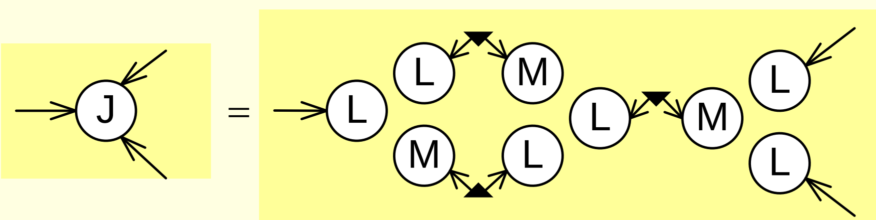


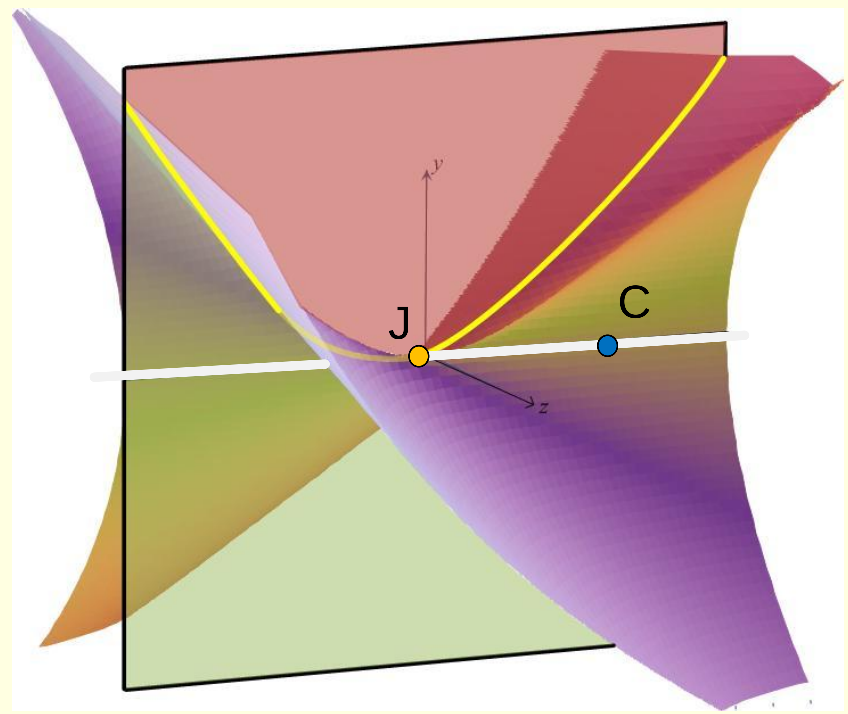
The two places the line hits the triple root curve
are the cubes of the two factors of H

Locus of constant H for type 21



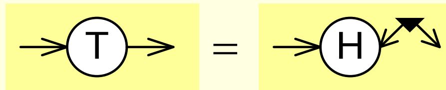




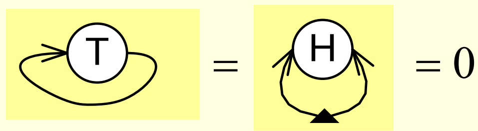


If C is type 21 (double factor L),
J is type 3 (triple factor L)

Transform based on H



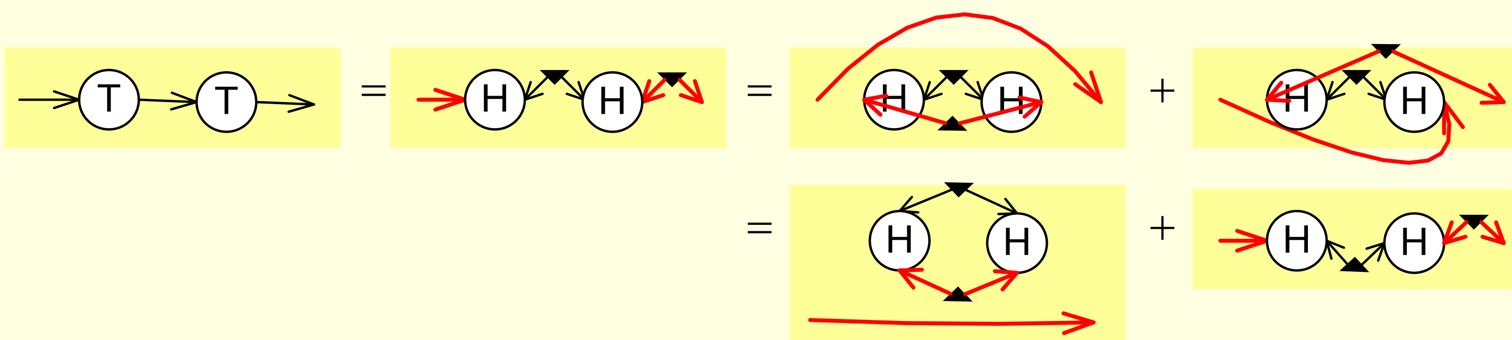
Trace:



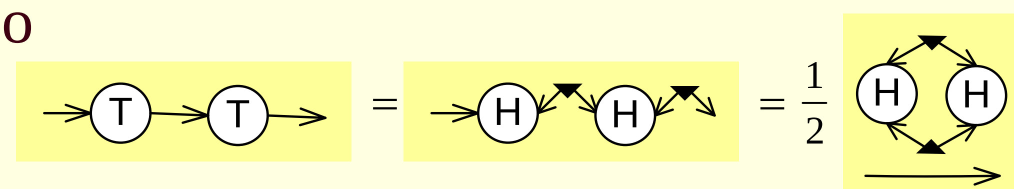
So it's an involution

Trace=0 $\Rightarrow$ involution
(true only for 2x2 matrices)

Show via arc-swap:

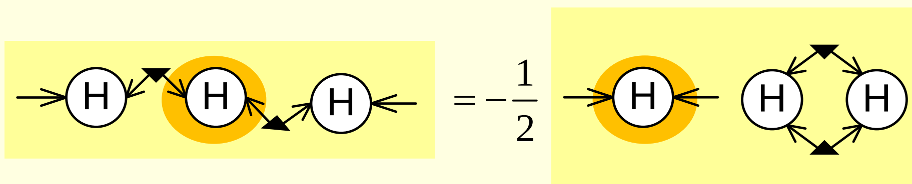


So



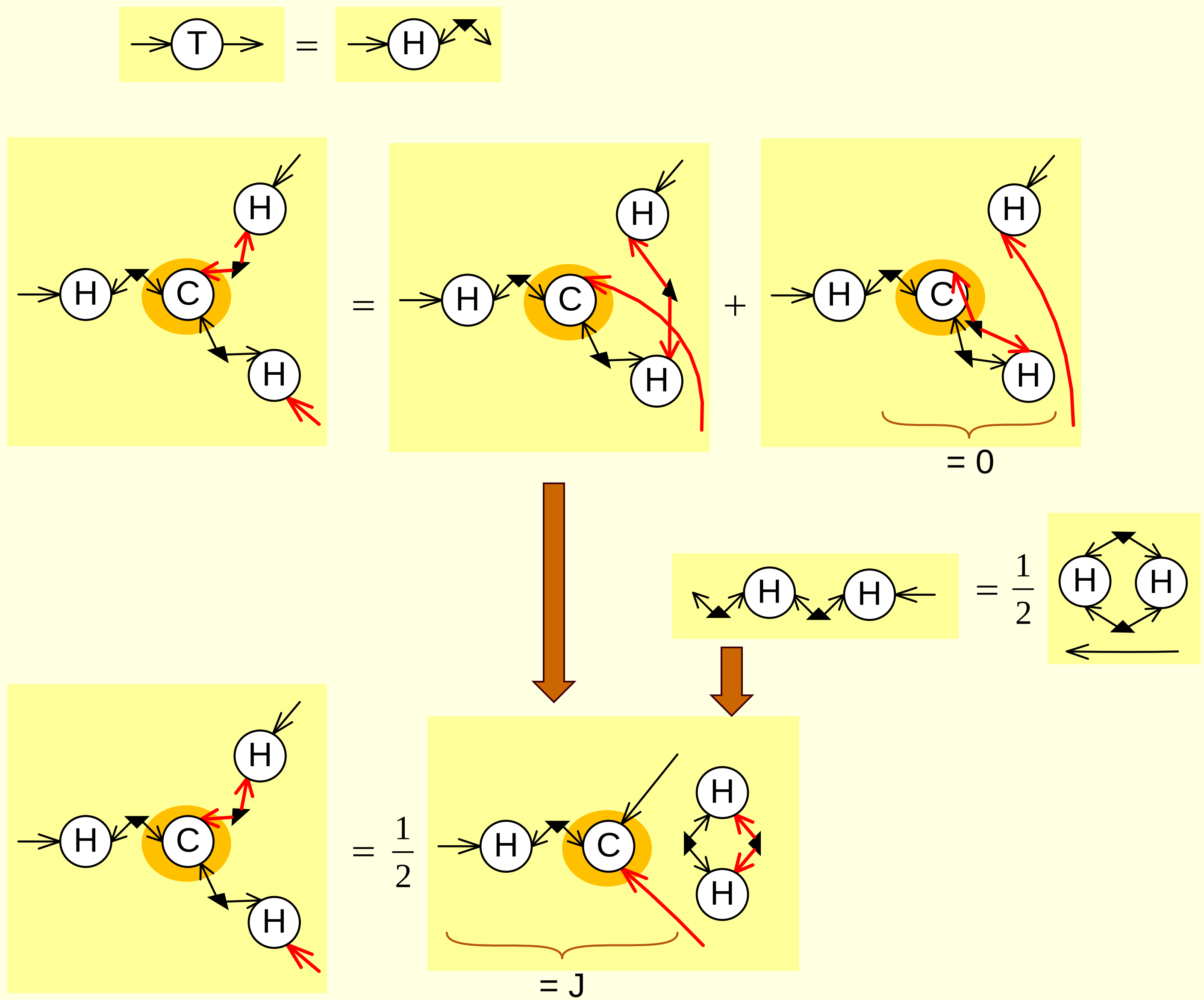
Transform based on H

Apply to H

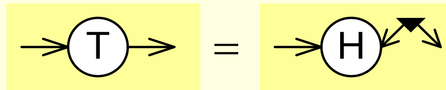


Transform based on H

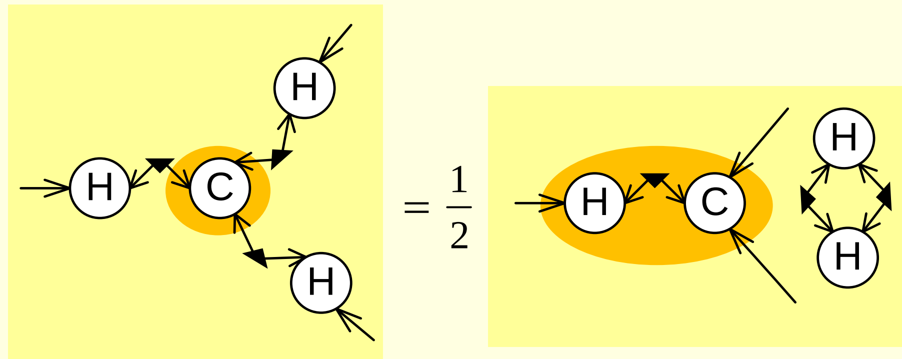
Apply to C



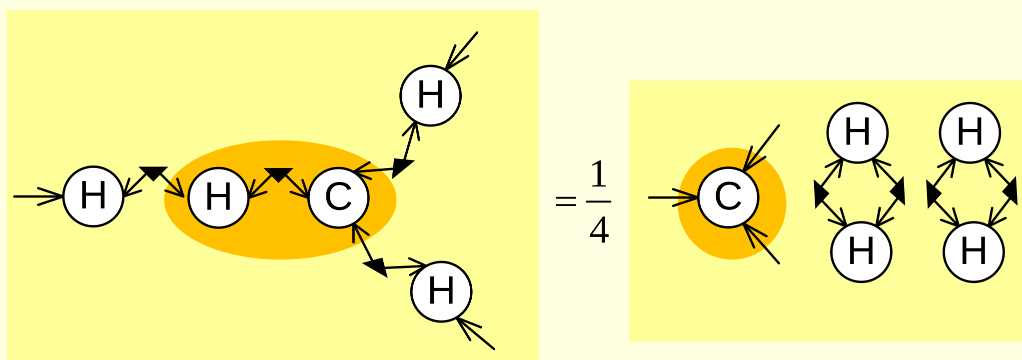
Transform based on H



Transform **C** by **T**

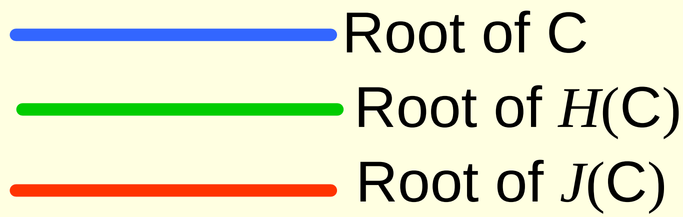


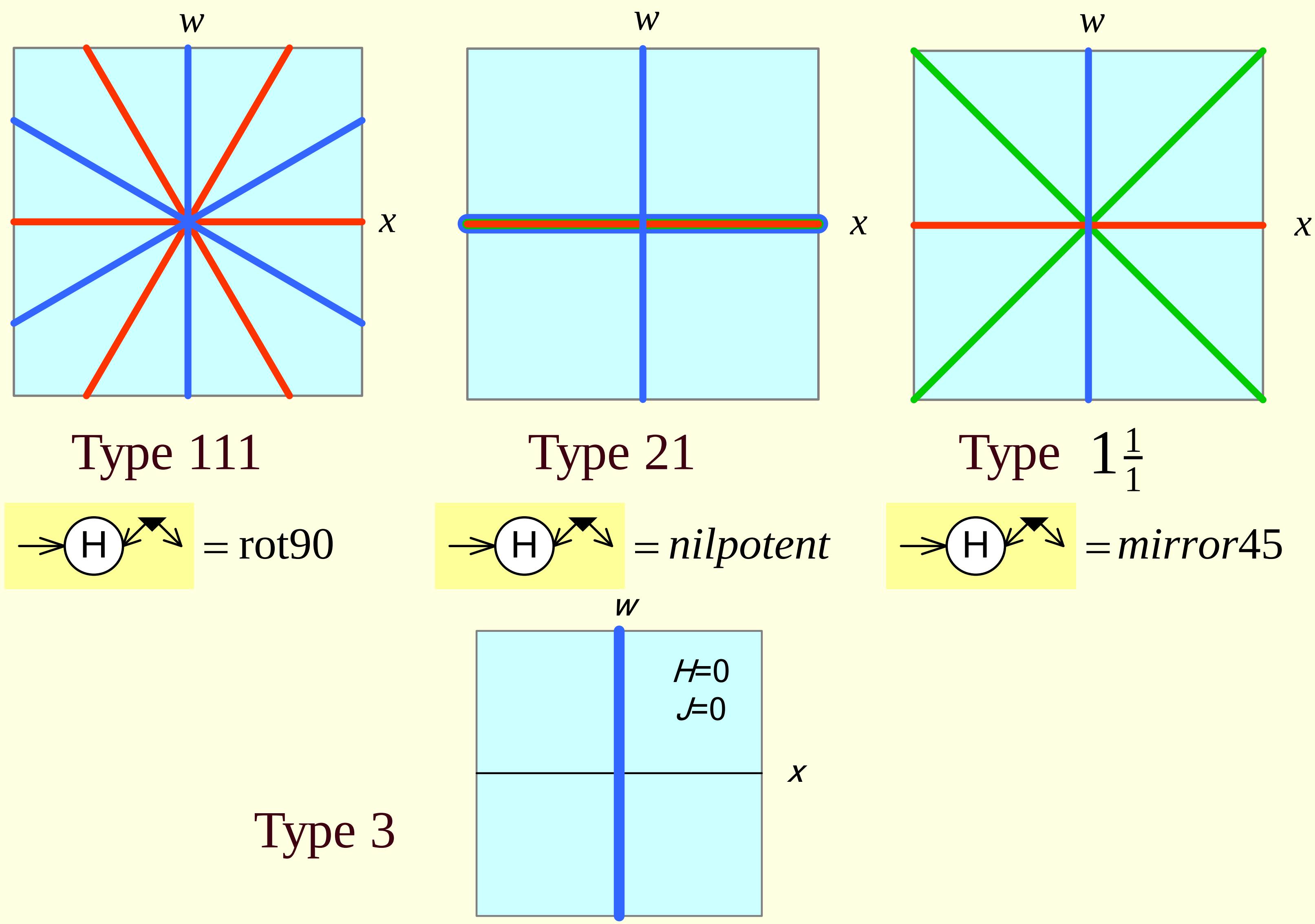
Transform **J** by **T**



$$\mathbf{C}\left(x, w\right) \overset{\mathbf{T}}{\leftrightarrow} \mathbf{J}\left(x, w\right)$$

Roots of *C*, *H*, *J*


Root of C
Root of H(C)
Root of J(C)


w
x
Type 111
H
= rot90
Type 21
= nilpotent
Type 1 1/1
= mirror45
H=0
J=0
Type 3

Rank

Physics

$$\begin{bmatrix} A \\ B \end{bmatrix}$$

1

$$\begin{bmatrix} A & B \\ B & C \end{bmatrix}$$

2

$$\begin{bmatrix} \begin{bmatrix} A & B \\ B & C \end{bmatrix} & \begin{bmatrix} B & C \\ C & D \end{bmatrix} \end{bmatrix}$$

3

Mathematics

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

1

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3

Mathematical Rank of 2-Tensor

Rank 1

$$\begin{bmatrix} x & w \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} = x^2$$

$$\begin{bmatrix} x & w \end{bmatrix} \begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} = (ax + bw)^2$$

Rank 2

$$\begin{bmatrix} x & w \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} = x^2 + w^2$$

$$\begin{bmatrix} x & w \end{bmatrix} \begin{bmatrix} A & B \\ B & C \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} = (ax + bw)^2 \pm (cx + dw)^2$$

Mathematical Rank of 3-Tensor

Rank 1

$$\left[\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\ \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\right] = x^3 \qquad \left[\begin{bmatrix}a^3 & a^2b\\ a^2b & ab^2\end{bmatrix}\ \begin{bmatrix}a^2b & ab^2\\ ab^2 & b^3\end{bmatrix}\right] = (ax+bw)^3$$

Rank 2

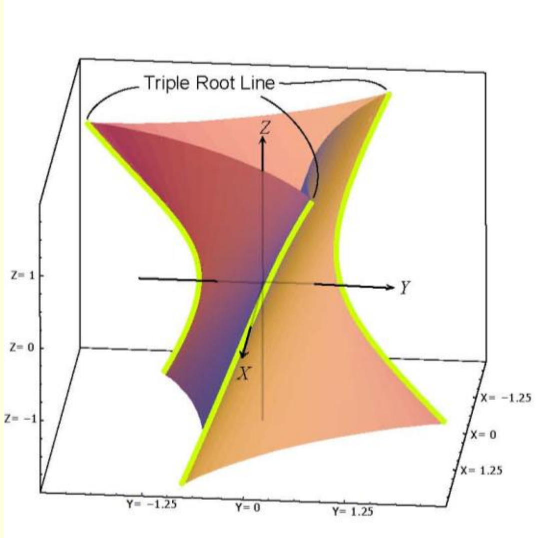
$$\left[\begin{bmatrix}\bullet & \bullet\\ \bullet & \bullet\end{bmatrix}\ \begin{bmatrix}\bullet & \bullet\\ \bullet & \bullet\end{bmatrix}\right] = (ax+bw)^3 + (cx+dw)^3$$

Rank 3

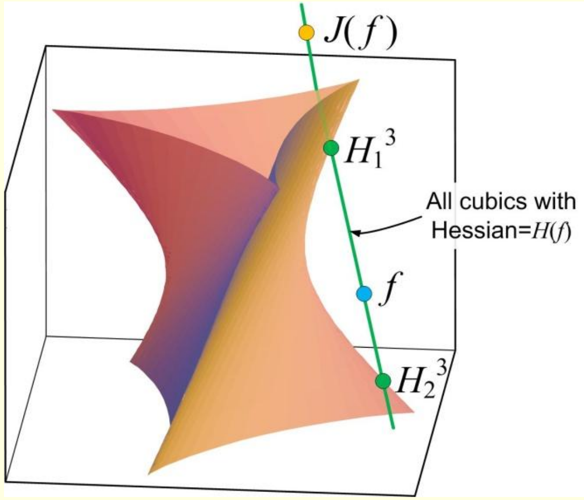
$$\left[\begin{bmatrix}\bullet & \bullet\\ \bullet & \bullet\end{bmatrix}\ \begin{bmatrix}\bullet & \bullet\\ \bullet & \bullet\end{bmatrix}\right] = (ax+bw)^3 + (cx+dw)^3 + (ex+fw)^3$$

Rank of Cubic

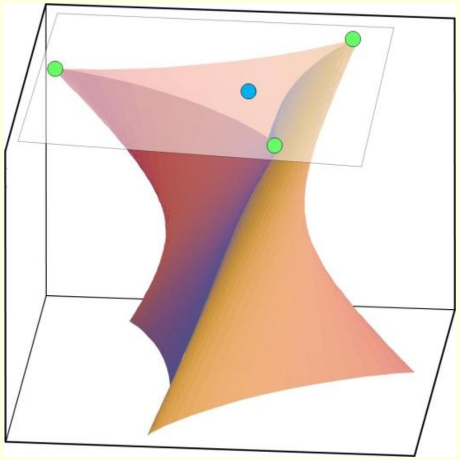
Rank 1
Type 3



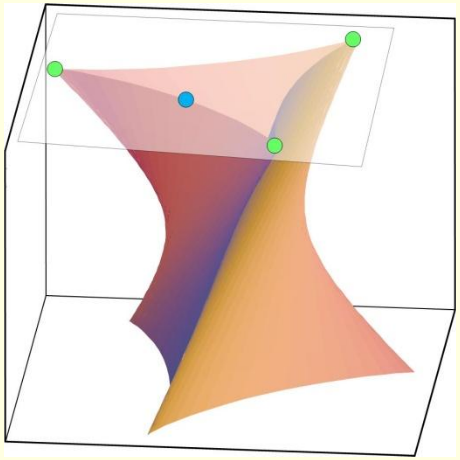
Rank 2
Type $1\frac{1}{1}$



Rank 3
Type 111
(Not unique)



Rank 3
Type 21
(Not unique)



IEEE CG&A Articles

How to Solve a Cubic Equation

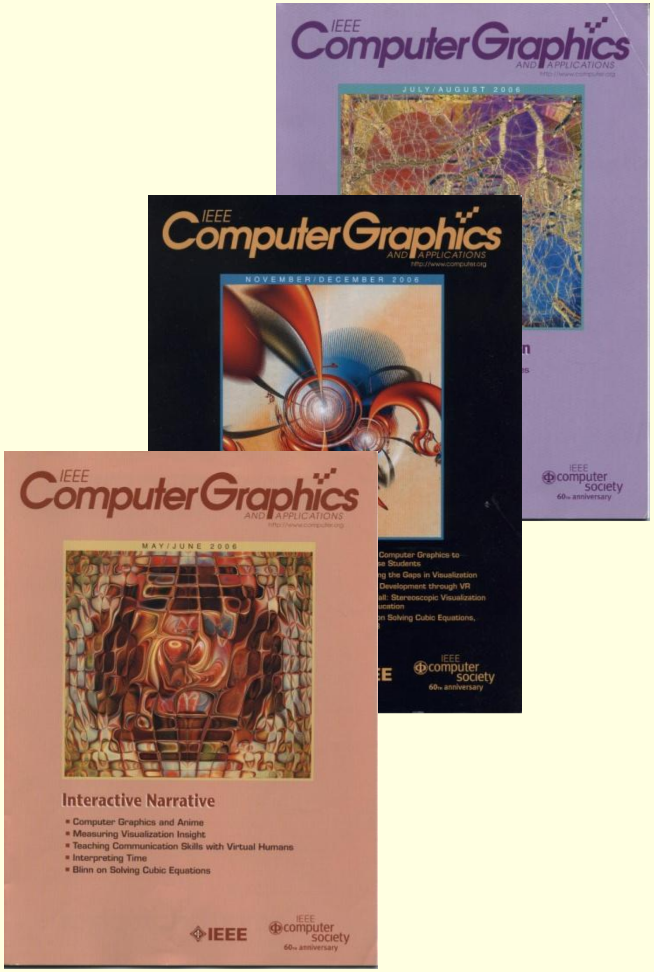
Part 1 – The Shape of the Discriminant
May/Jun 2006, pages 84–93

Part 2 – The 11bar Case
Jul/Aug 2006, pages 90–100

Part 3 – General Depression and a New Covariant
Nov/Dec 2006, pages 92–102

Part 4 – The 111 case
Jan/Feb 2007,

Part 5 – Back to Numerics
May/Jun 2007



Solving Numerically

$$\mathbf{C}(x,w) = Ax^3 + 3Bx^2w + 3Cxw^2 + Dw^3$$

Translate to get B=0

$$\begin{bmatrix} x & w \end{bmatrix} = \begin{bmatrix} \tilde{x} & \tilde{w} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -B & A \end{bmatrix}$$

"Depress" **C**

General transform to get B=0

$$\begin{bmatrix} x & w \end{bmatrix} = \begin{bmatrix} \tilde{x} & \tilde{w} \end{bmatrix} \begin{bmatrix} t & u \\ s & v \end{bmatrix}$$

IEEE CG&A article
"General Depression and a New Covariant"

High Class Spam

Dear Dr. Blinn,

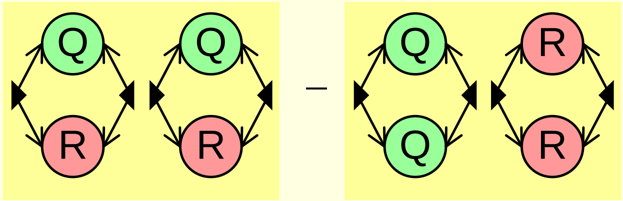
From your article titled “How to solve a cubic equation, part 3: general depression and a new covariant.” (IEEE Comput Graph Appl. 2006 Nov-Dec;26(6):92-102.), we learned of your research in the area of Depression and thought you might be interest in knowing that GenWay offers an ELISA kit for the in-vitro diagnostic quantitative determination of Serotonin in various biological fluids.

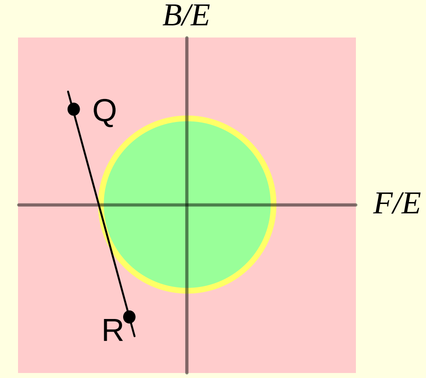
You can click the following link to our website in order to view the datasheet: **Serotonin ELISA Kit**

Resultants

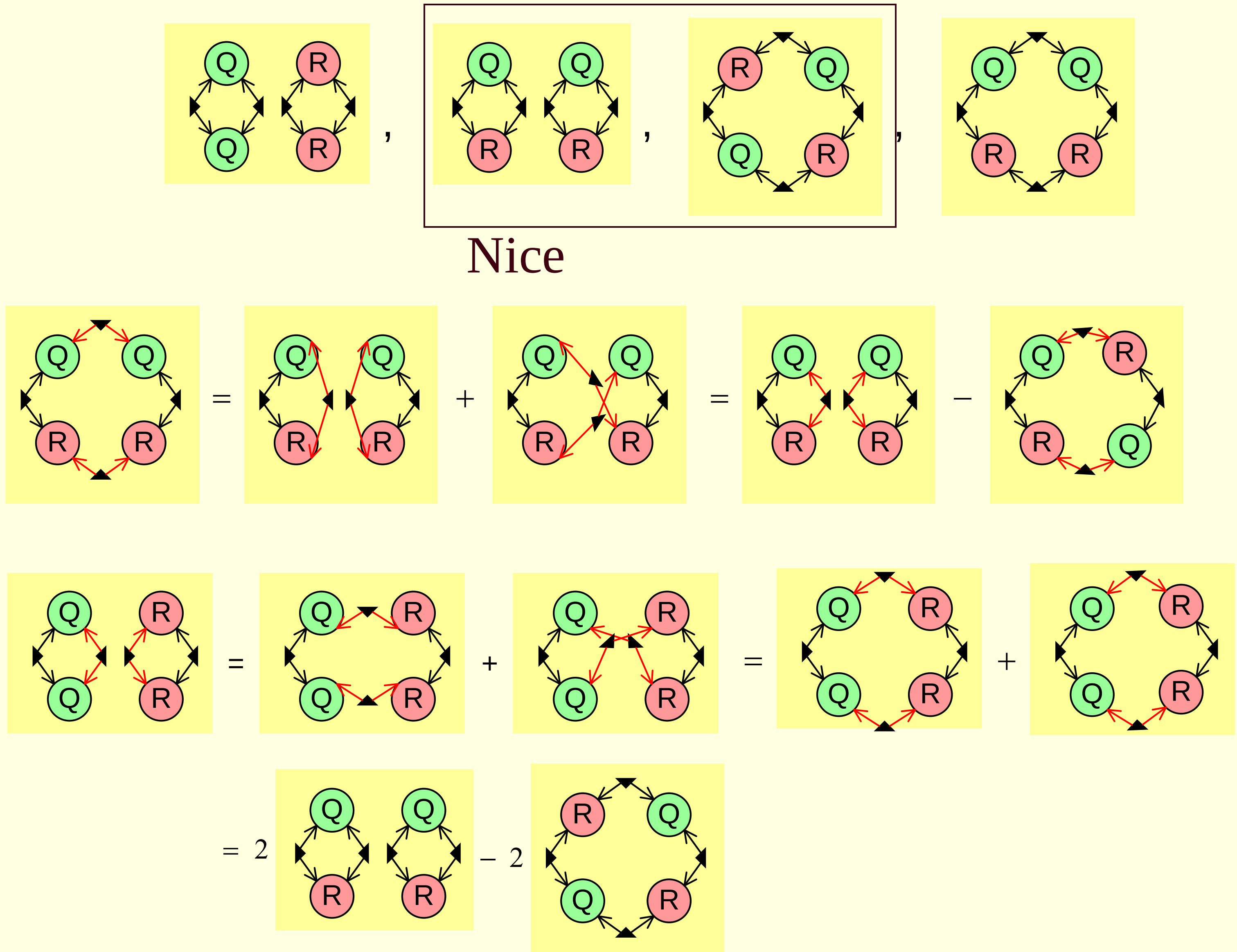
Resultant of Two Quadratics

$\rho(Q,R) =$

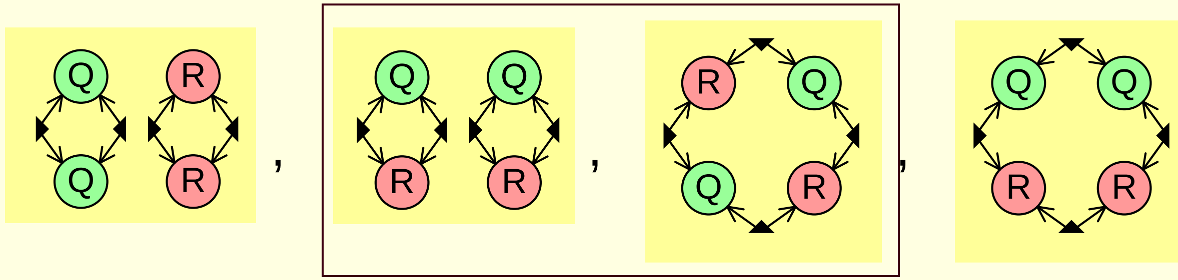




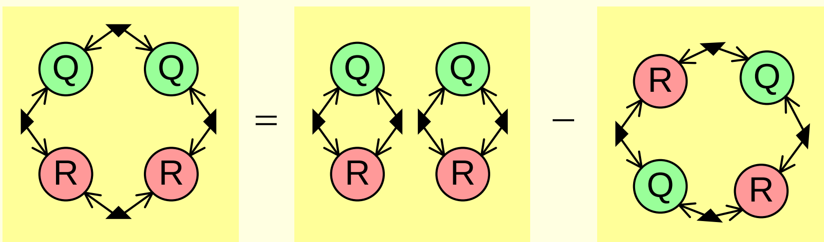
Invariants of Two Quadratics

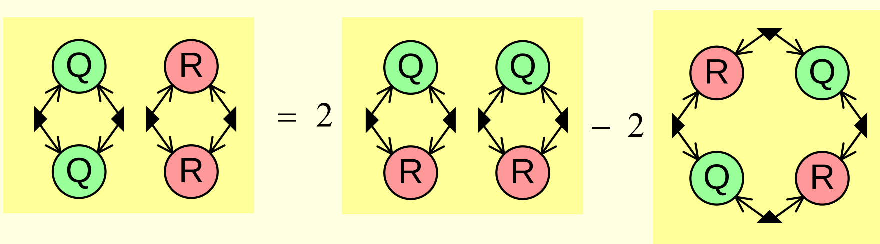


Invariants of Two Quadratics



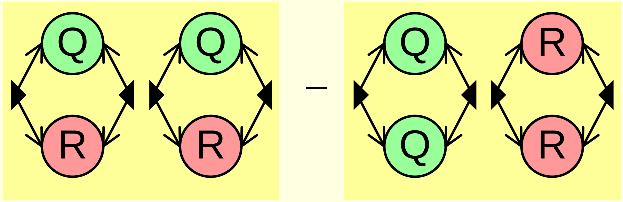
Nice

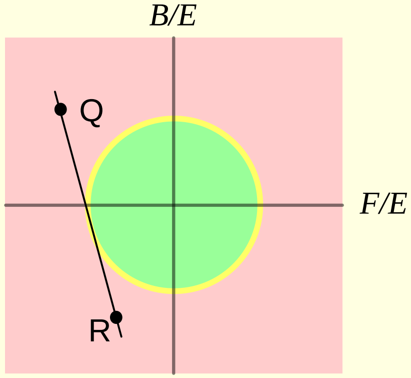


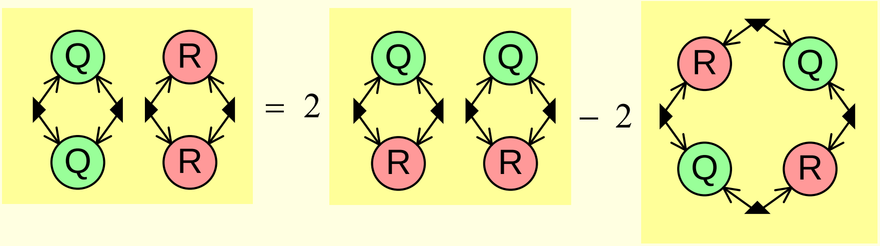


Resultant of Two Quadratics (v1)

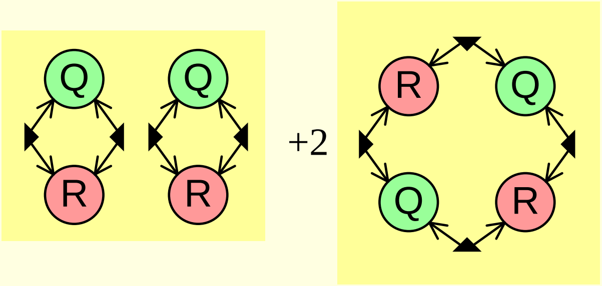
$\rho(Q,R) =$



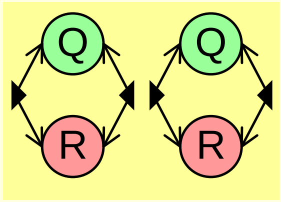


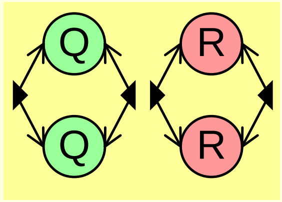


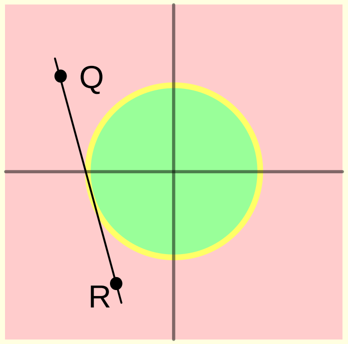
$\rho(Q,R) = -$

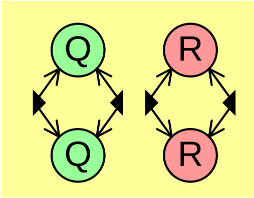


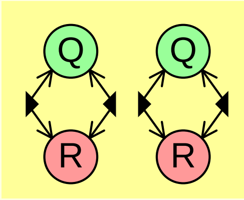
Resultant of Two Quadratics (v1)

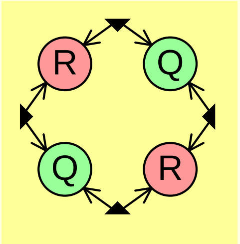
$\rho(Q,R) =$ 

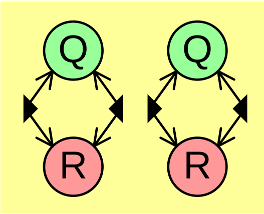
$-$ 

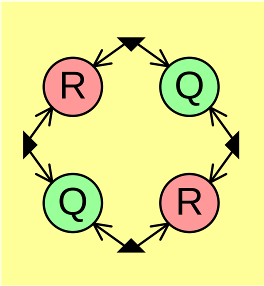




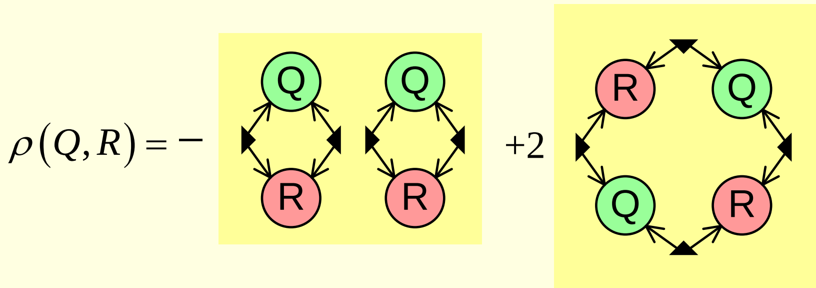
$= 2$ 

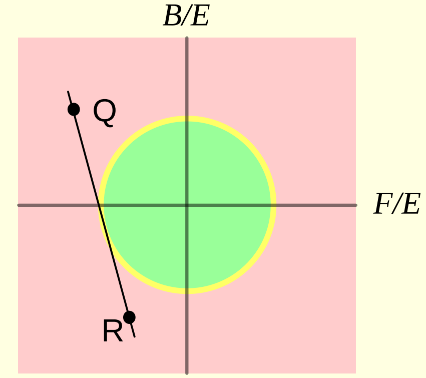
$- 2$ 

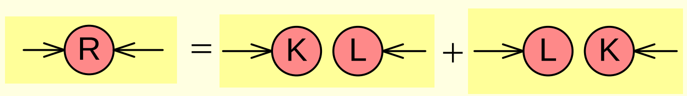
$\rho(Q,R) = -$ 

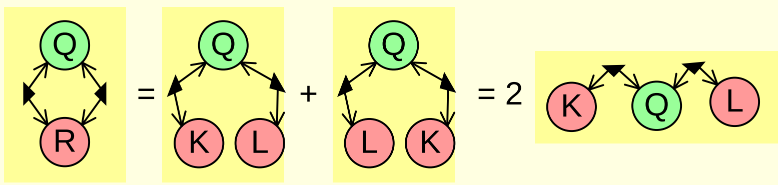
$+2$ 

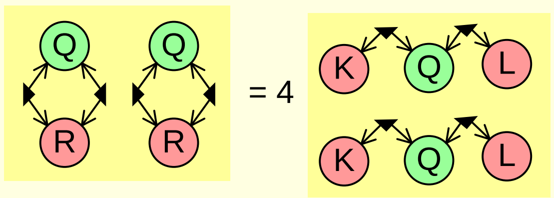
Resultant of Two Quadratics (v1)

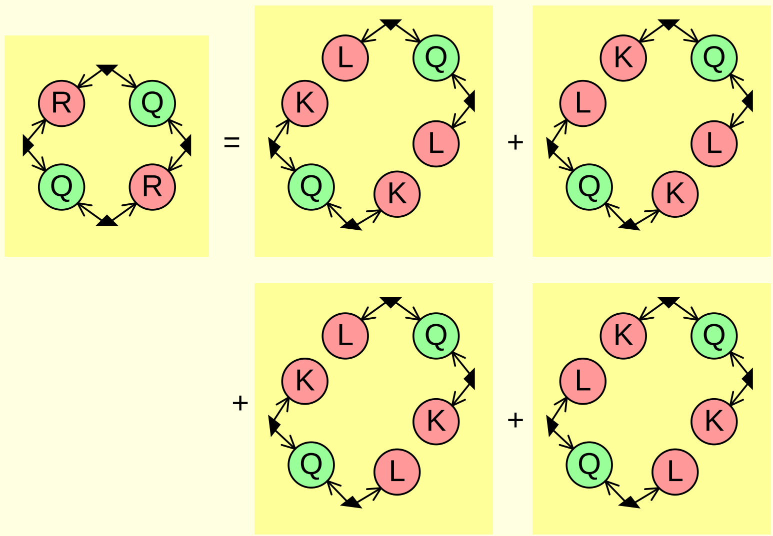




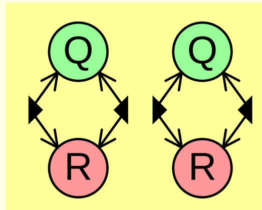
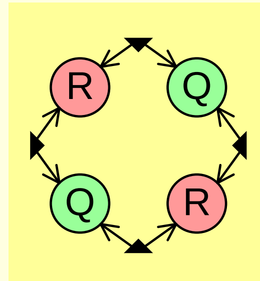


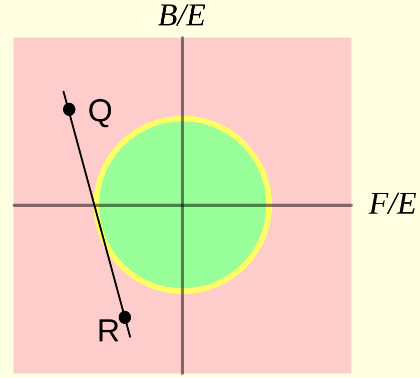


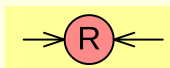
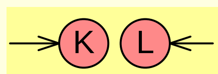
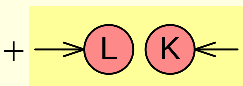


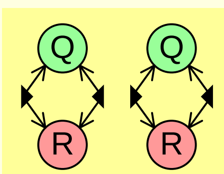
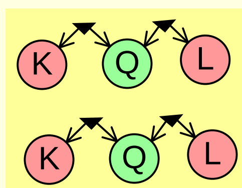


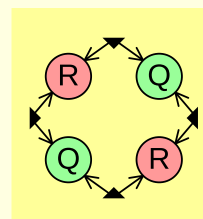
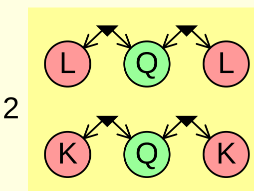
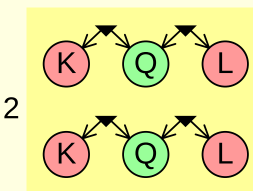
Resultant of Two Quadratics (v1)

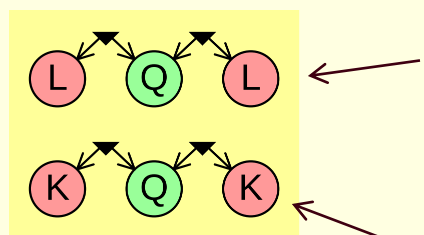
$\rho(Q,R) = -$  $+2$ 



 =  + 

 = 4 

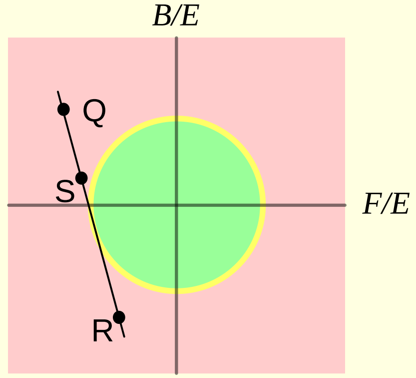
 = 2  + 2 

$\rho(Q,R) =$ 4 

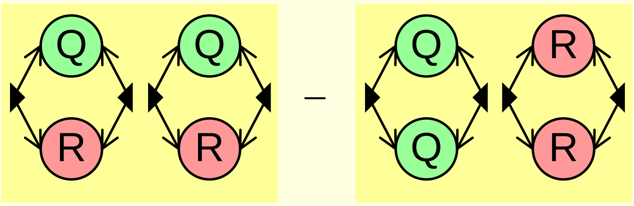
L is a factor of Q

K is a factor of Q

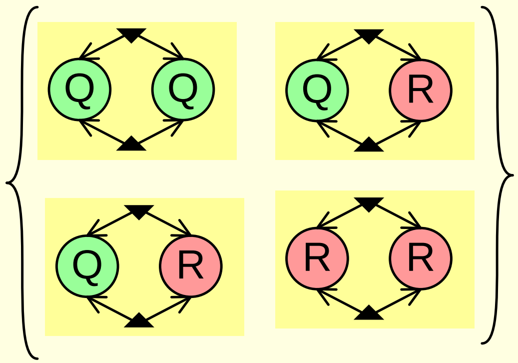
Resultant of Two Quadratics (v2)



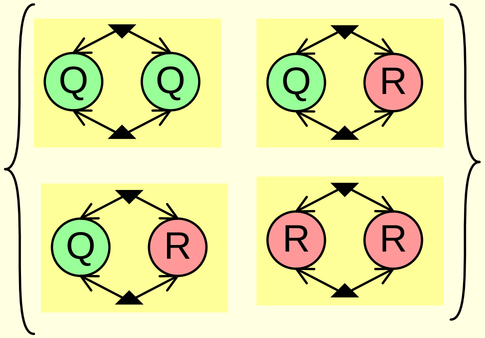
$\rho(Q,R) =$

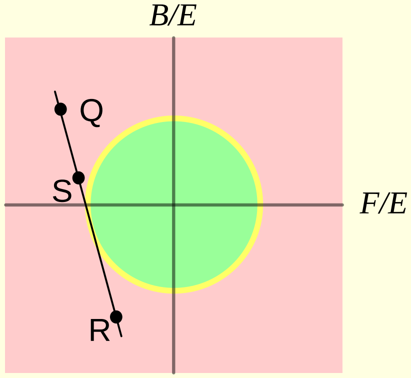


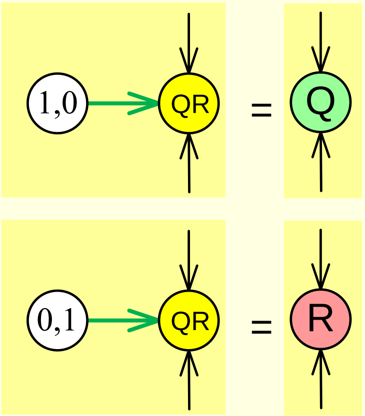
$\rho(Q,R) = -\det$

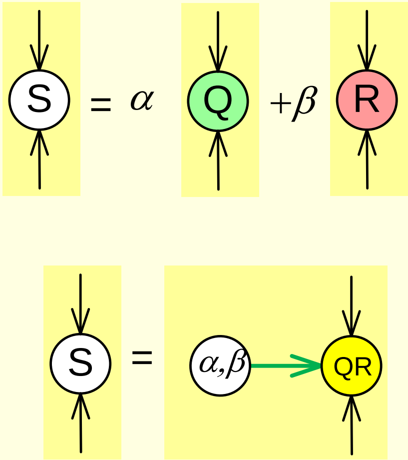


Resultant of Two Quadratics (v2)

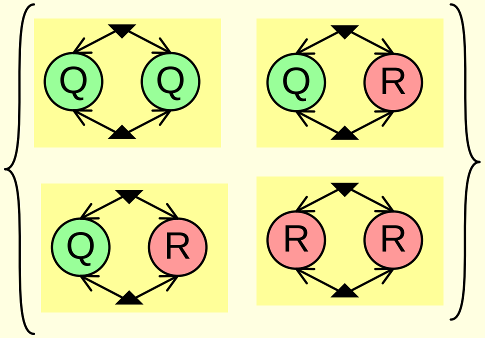
$\rho(Q,R) = -\det$ 

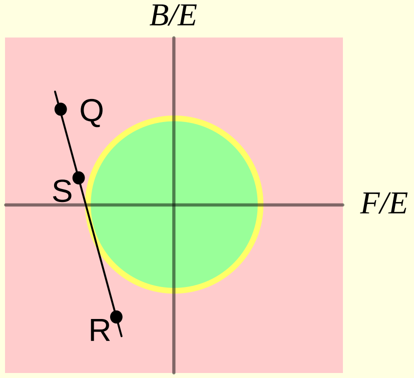


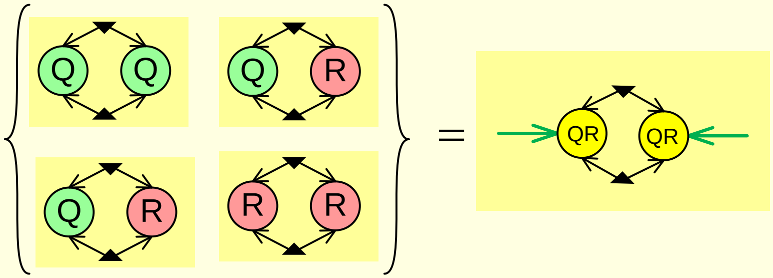


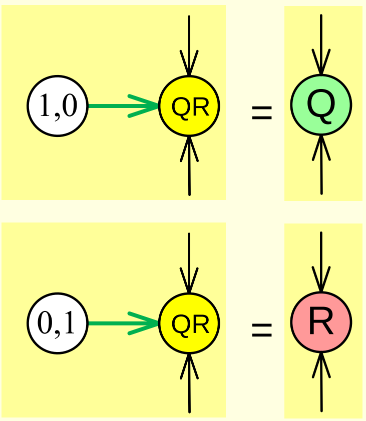


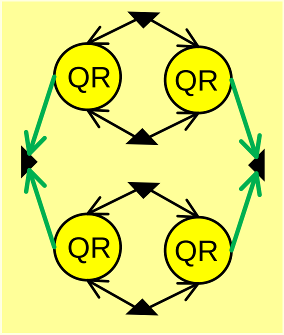
Resultant of Two Quadratics (v2)

$\rho(Q,R) = -\det$ 



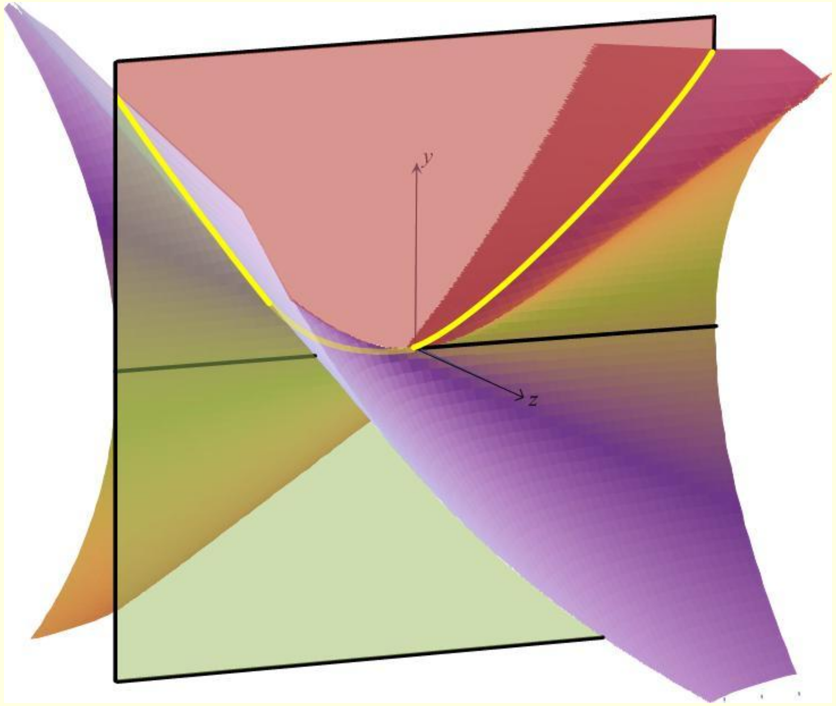


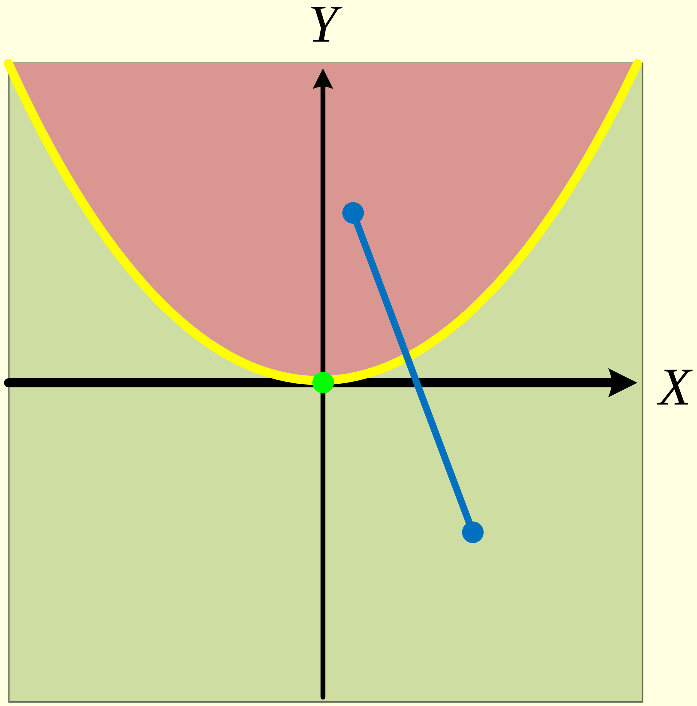


$\rho(Q,R) =$ 

Resultant of Two Cubics

Two Cubics, both with a root at 0





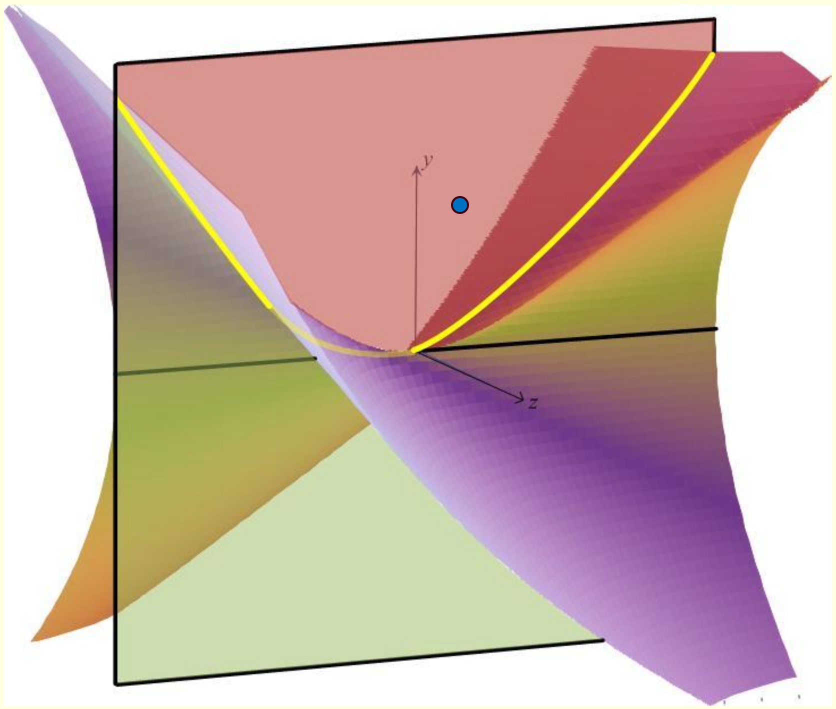
Slice thru X,Y,Z space at $Z=D/A=0$

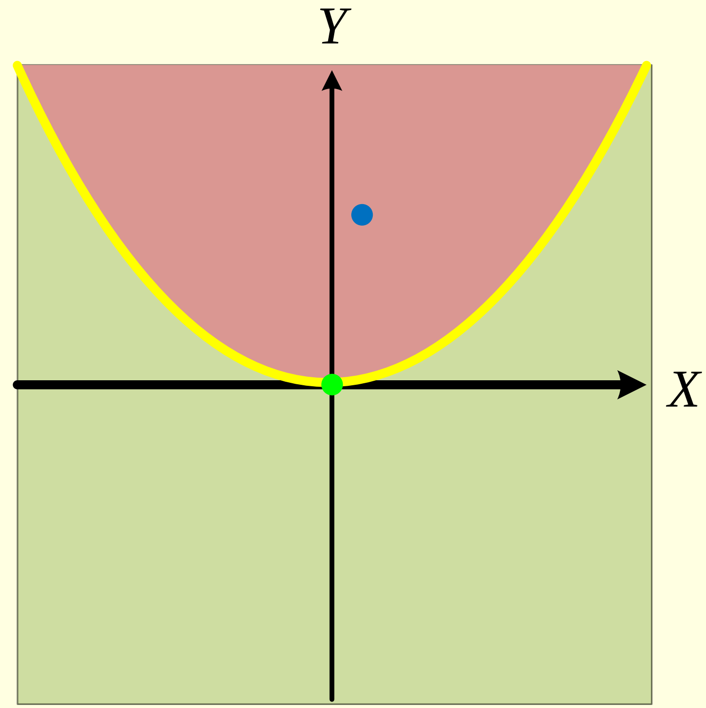
Two Cubics have a common (real) root

iff

The line connecting them is tangent to the discrim surface

Tangent to surface from each point



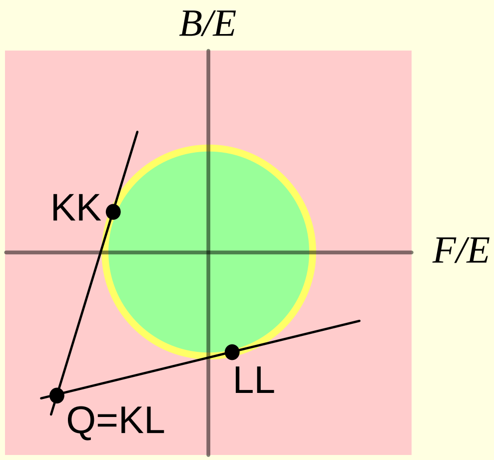


Slice thru X,Y,Z space at $Z=D/A=0$

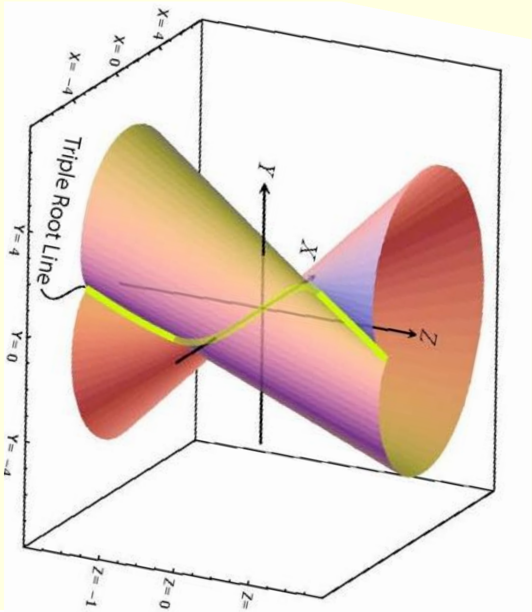
From each point **C**
There will be one plane tangent to the surface
for each real root of **C**

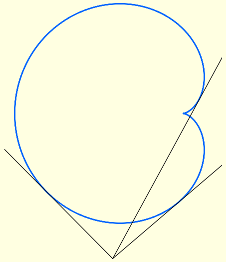
Geometry of Resultants

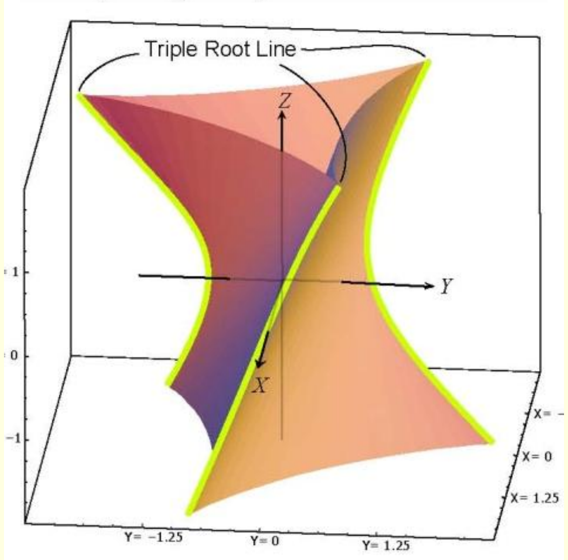
Quadratic

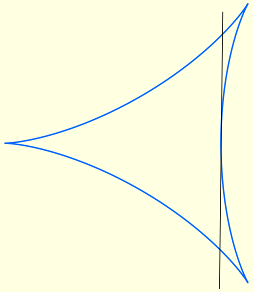


Cubic

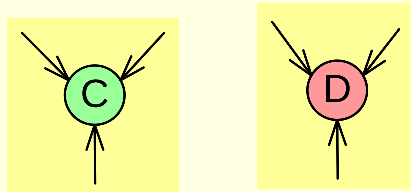


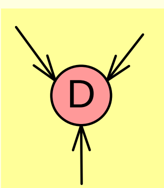




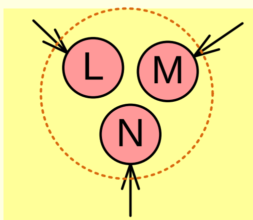


Resultant of Two Cubics

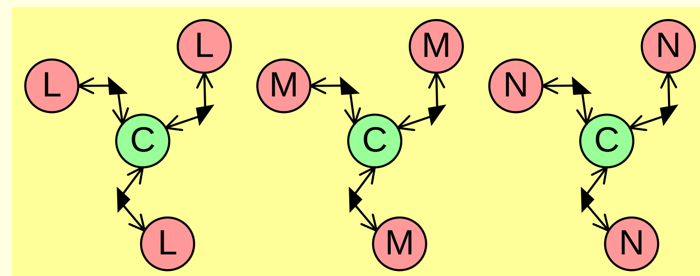




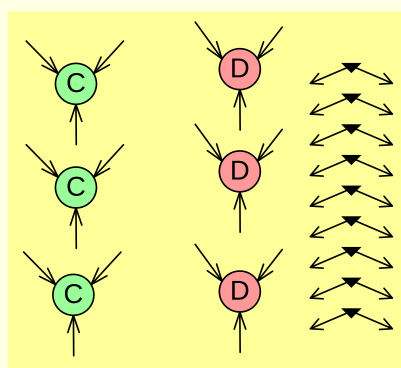
=



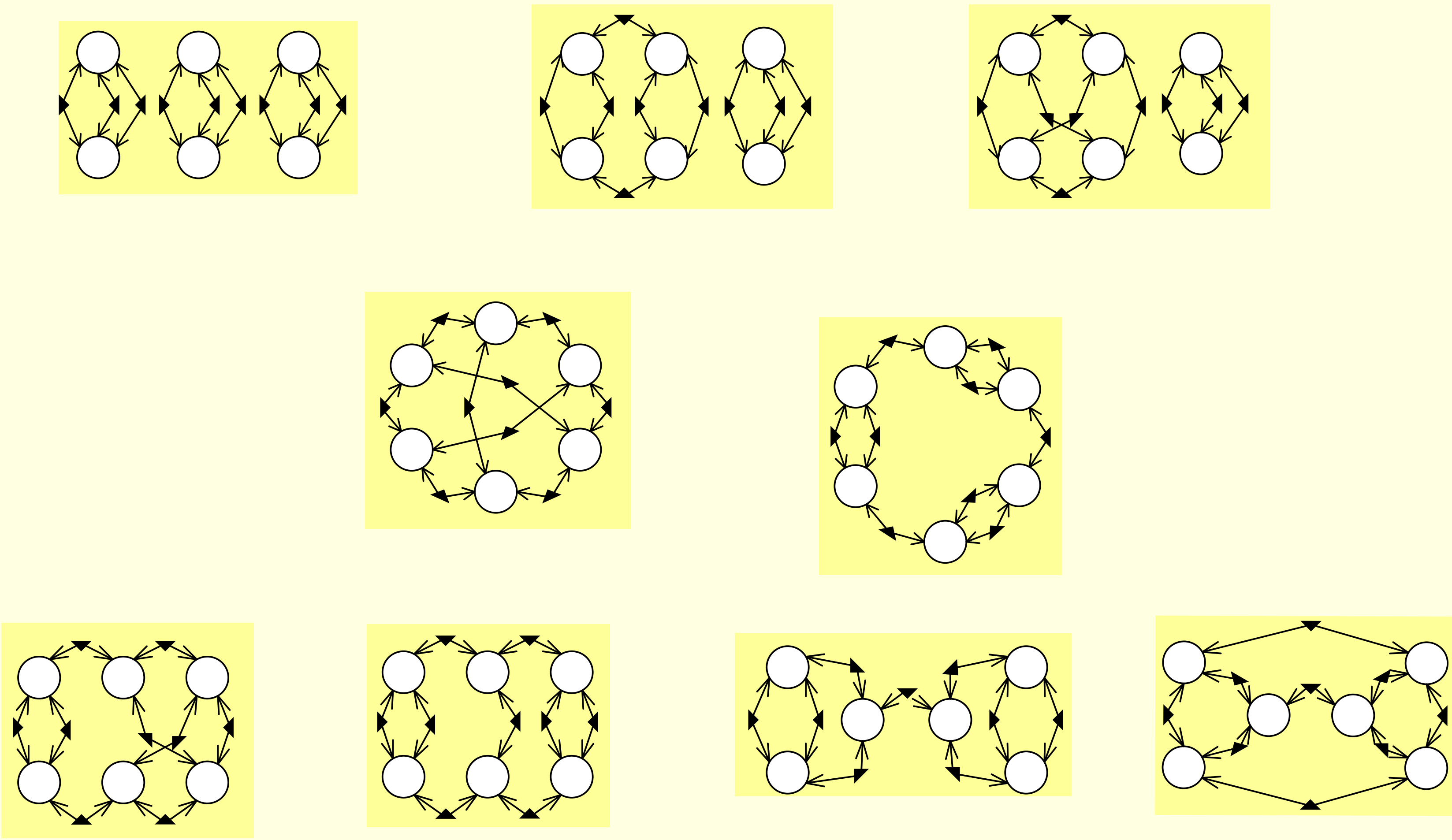
ρ(C,D) =



ρ(C,D) =

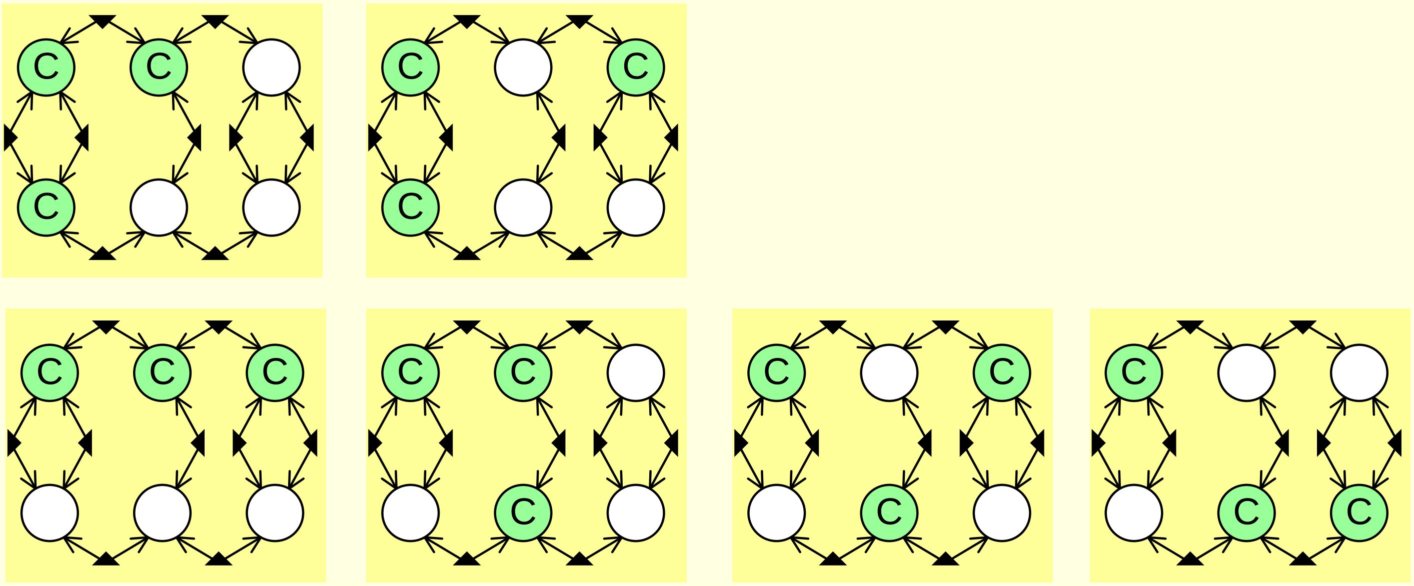


Possible Topologies

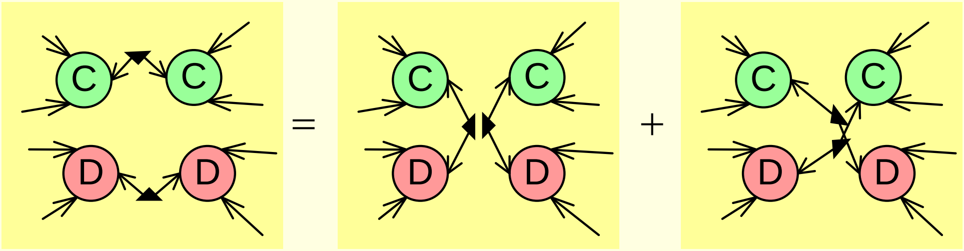


Distribute 3 each C and D

Examples:

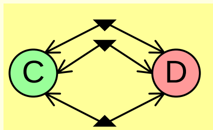


But only need CD connected diagrams since:

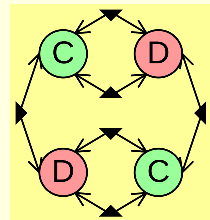


Five Possible CD diagrams

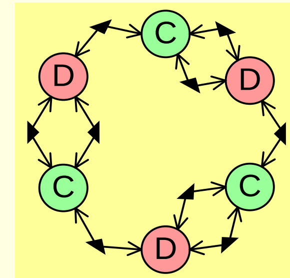
$I_2 =$



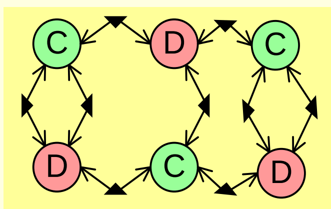
$D_3 =$



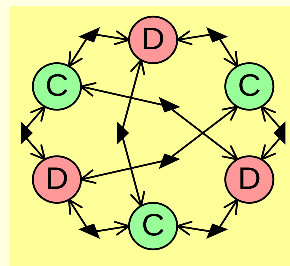
$T_3 =$



$T_{2a} =$



$T_{0b} =$



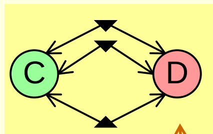
It can be shown that:

$$3I_2D_3 = I_2^{\,3} - 2T_3$$

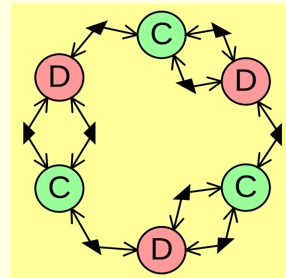
$$3T_{2a} = T_{0b} - 2T_3$$

Resultant of Two Cubics

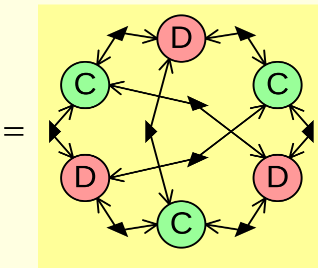
$\mathbf{I}_2 =$

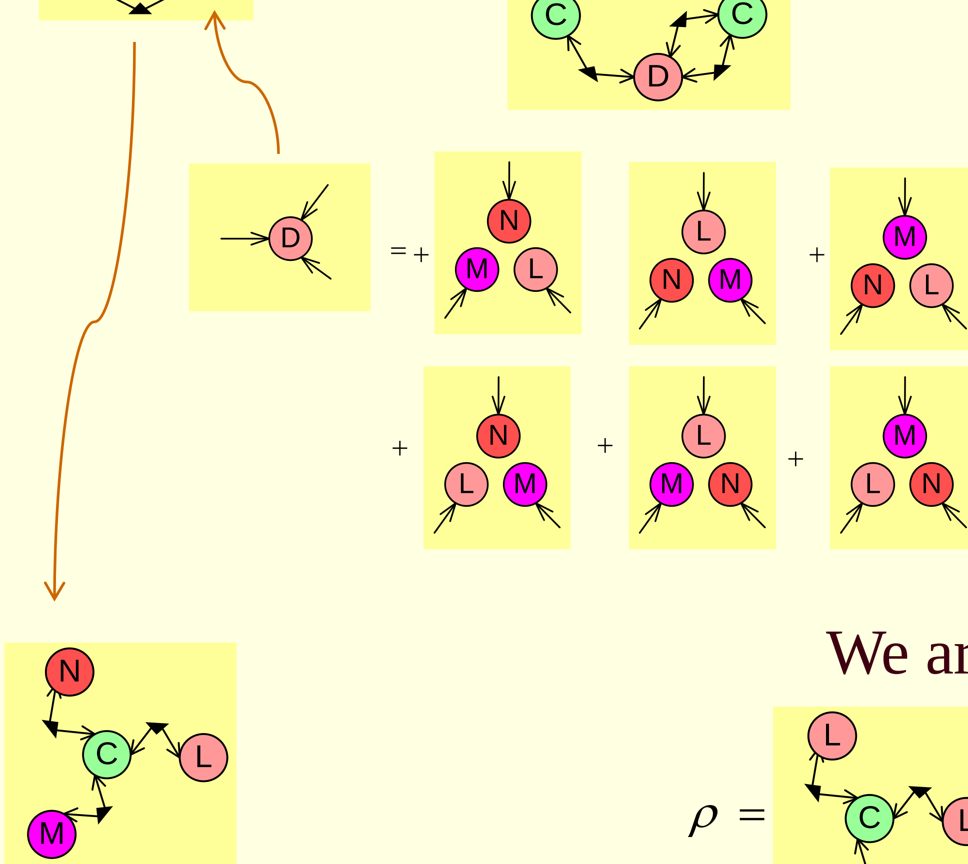


$T_3 =$



$T_{0b} =$

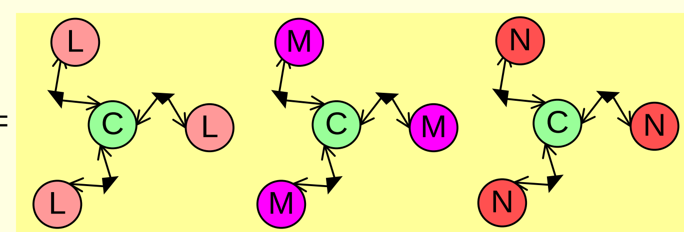




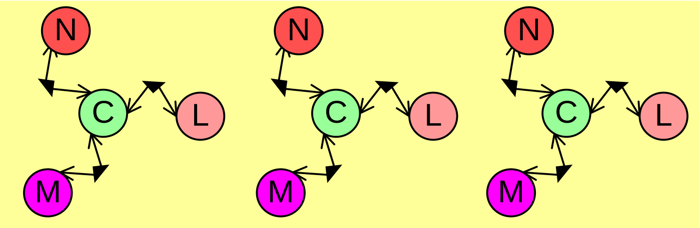
$\mathbf{I}_2 = 6$

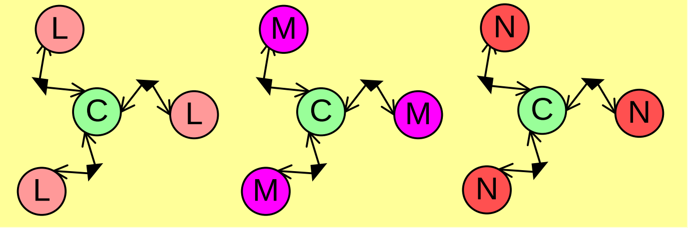
We are looking for:

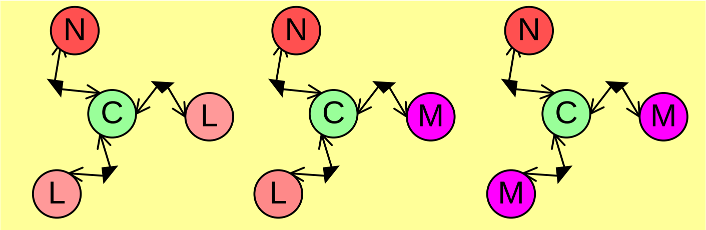
$\rho =$

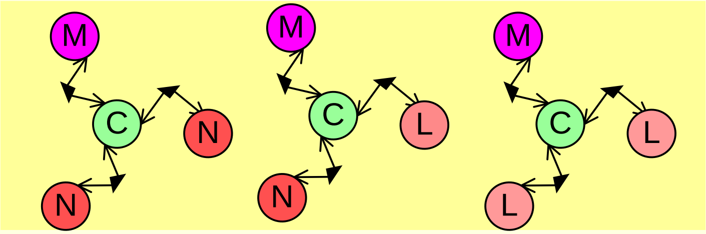


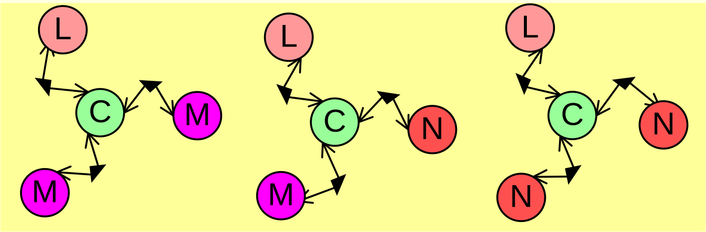
After some symbolic programming

$\alpha =$ 

$\rho =$ 

$\gamma =$ 

$+$ 

$+$ 

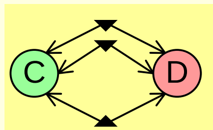
$$I_2^{\,3} = +216\alpha$$

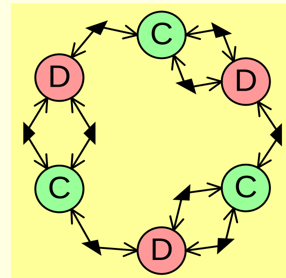
$$T_3 = \qquad -72\gamma$$

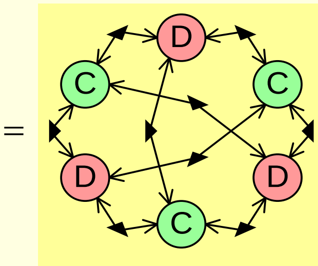
$$T_{0b} = -24\alpha + 72\gamma + 24\rho$$

$$6^3\,\rho = \left(I_2\right)^3 + 9\left(T_{0b} + T_3\right)$$

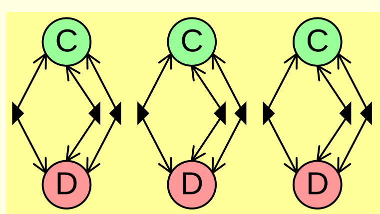
Resultant of Two Cubics

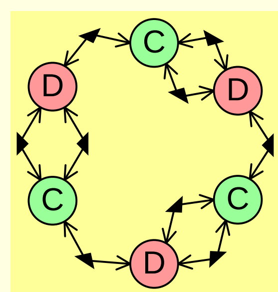
$\mathbf{I}_2 =$ 

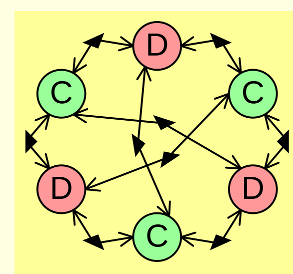
$T_3 =$ 

$T_{0b} =$ 

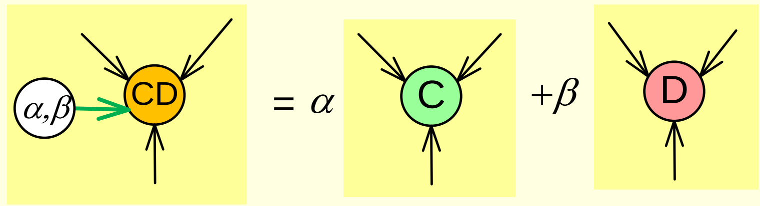
$$6^3\,\rho = \left(I_2\right)^3 + 9\left(T_{0b} + T_3\right)$$

$216\,\rho =$ 

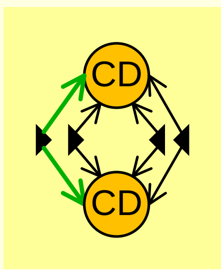
$+ 9$ 

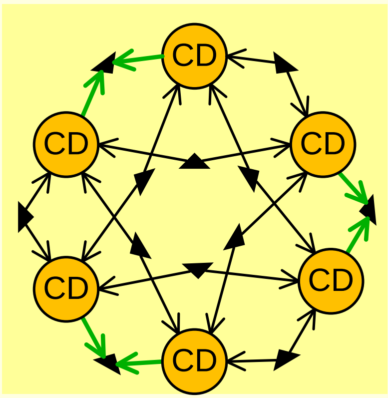
$+ 9$ 

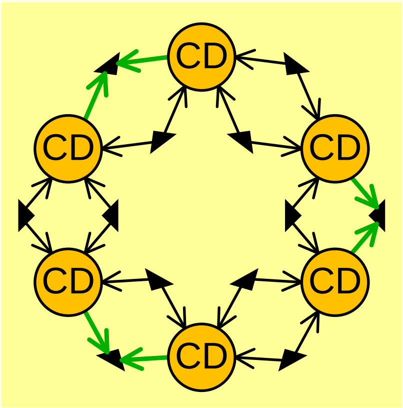
Resultant of Two Cubics



Among the many possible diagrams:

$D_2 =$ 

$D_0 =$ 

$D_6 =$ 

It can be shown that:

$$D_2^{\,3} - 8D_0 - 4D_6 = 0$$

$$\rho\left(\mathbf{C},\mathbf{D}\right) = -11D_2^{\,3} + 36\,D_6$$

$$\rho\left(\mathbf{C},\mathbf{D}\right) = -\;\;2D_2^{\,3} - 72D_0$$