CSE590B Lecture 8
Quartics and Groups

Quartic, Quintic Polys

Quartic Curves

Cubic Group

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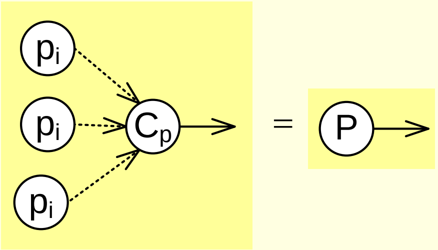
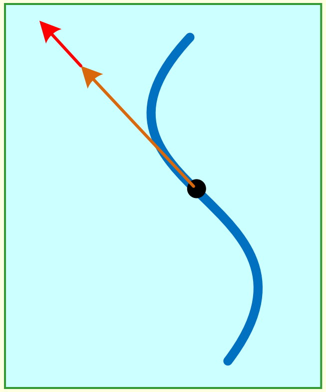
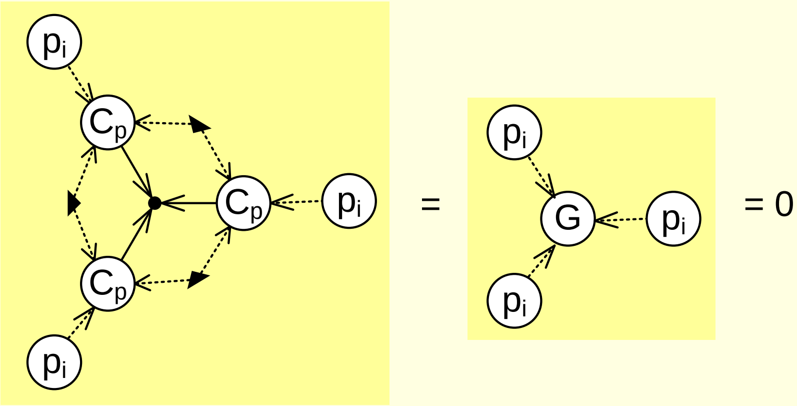
JimBlinn.Com

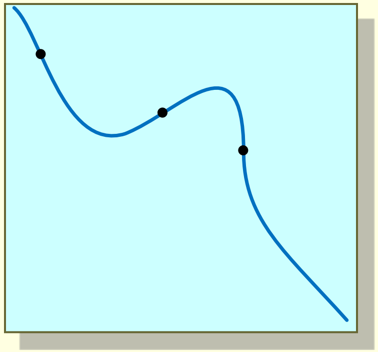
http://courses.cs.washington.edu/courses/cse590b/13au/

Inflection point

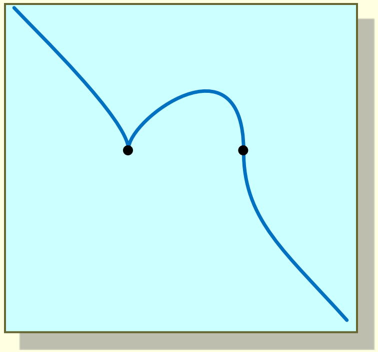




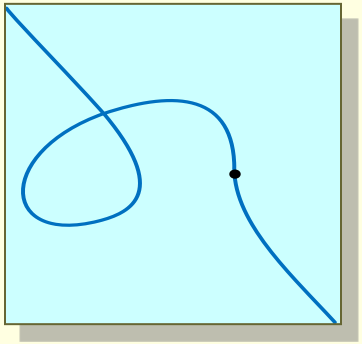




G is Type 111



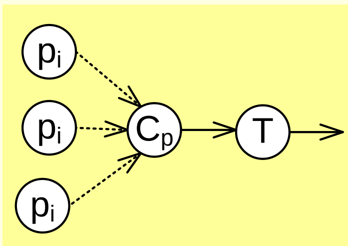
Type 12

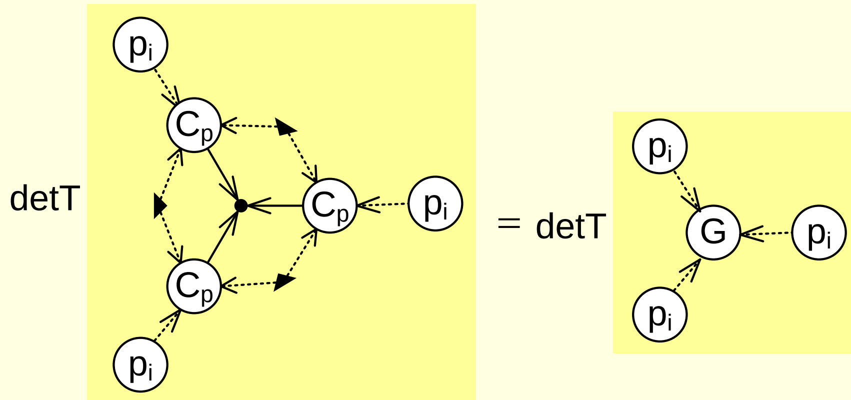


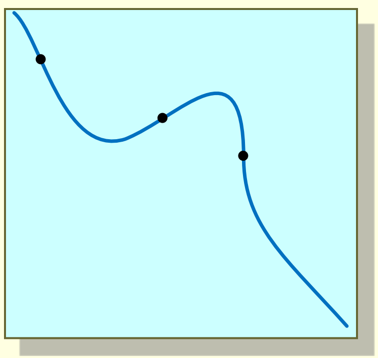
Type $1\frac{1}{1}$

Type 3?

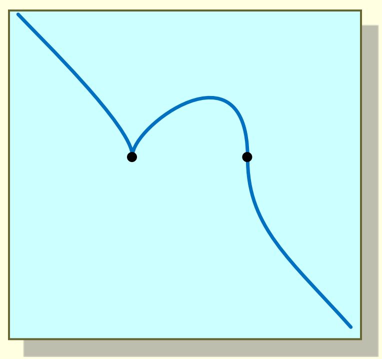
Transformational Invariance of G



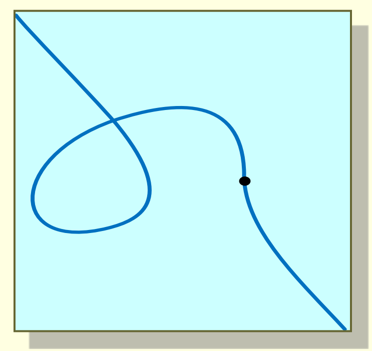




G is Type 111



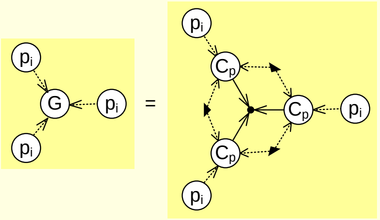
Type 12

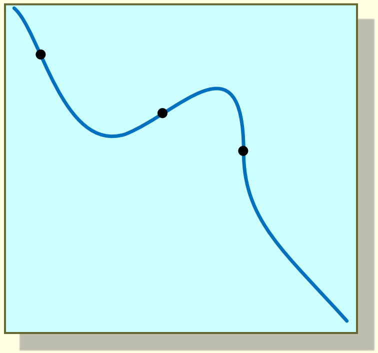


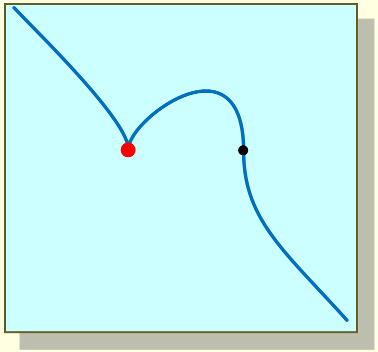
Type $1\frac{1}{1}$

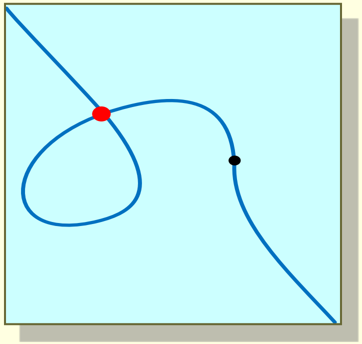
Type 3?

Double Points





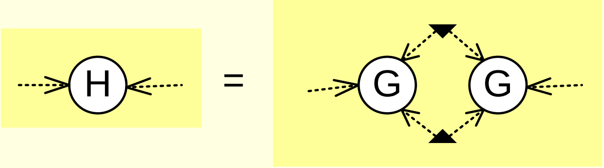




G is Type 111 | Type 12 | Type $1\frac{1}{1}$

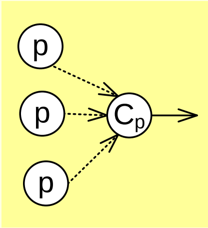
H(G) is Type $\frac{1}{1}$ | Type 2 | Type 11

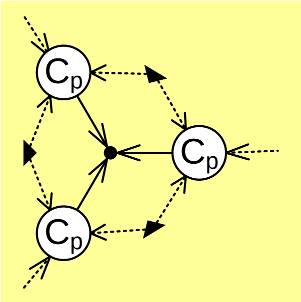
Parameters at double point are roots of Hessian(G)



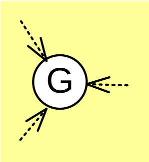
The Other Form of G

$$[t \quad s]\left[\left[\begin{bmatrix} A_x & B_x \\ B_x & C_x \end{bmatrix} \begin{bmatrix} B_x & C_x \\ C_x & D_x \end{bmatrix}\right] \left[\begin{bmatrix} A_y & B_y \\ B_y & C_y \end{bmatrix} \begin{bmatrix} B_y & C_y \\ C_y & D_y \end{bmatrix}\right] \left[\begin{bmatrix} A_w & B_w \\ B_w & C_w \end{bmatrix} \begin{bmatrix} B_w & C_w \\ C_w & D_w \end{bmatrix}\right]\right]\begin{bmatrix} t \\ s \end{bmatrix}\begin{bmatrix} t \\ s \end{bmatrix} =$$



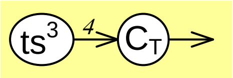


$=$

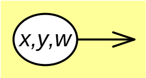


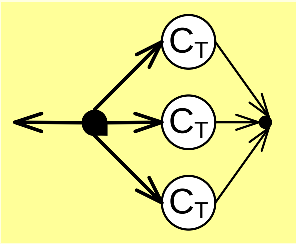
$$= \left[\begin{bmatrix} A & B \\ B & C \end{bmatrix} \begin{bmatrix} B & C \\ C & D \end{bmatrix}\right]$$

$$\begin{bmatrix} t^3 & 3t^2 s & 3ts^2 & s^3 \end{bmatrix}\begin{bmatrix} A_x & A_y & A_w \\ B_x & B_y & B_w \\ C_x & C_y & C_w \\ D_x & D_y & D_w \end{bmatrix} =$$

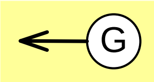


$=$





$=$



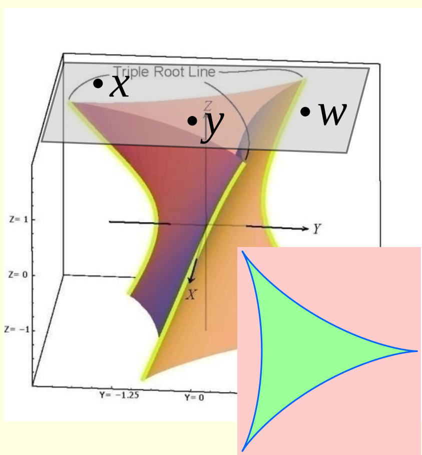
$$= \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}$$

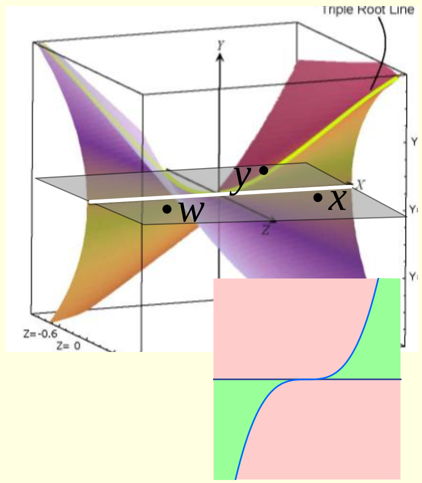
$$A = \det\begin{bmatrix} B_x & B_y & B_w \\ C_x & C_y & C_w \\ D_x & D_y & D_w \end{bmatrix},\ B = -\det\begin{bmatrix} A_x & A_y & A_w \\ C_x & C_y & C_w \\ D_x & D_y & D_w \end{bmatrix},\ C = \det\begin{bmatrix} A_x & A_y & A_w \\ B_x & B_y & B_w \\ D_x & D_y & D_w \end{bmatrix},\ D = -\det\begin{bmatrix} A_x & A_y & A_w \\ B_x & B_y & B_w \\ C_x & C_y & C_w \end{bmatrix}$$

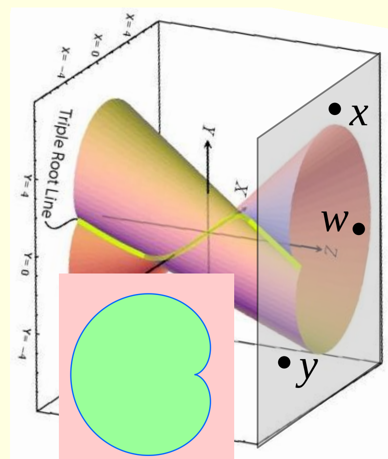
Forms and Singularities of Curve

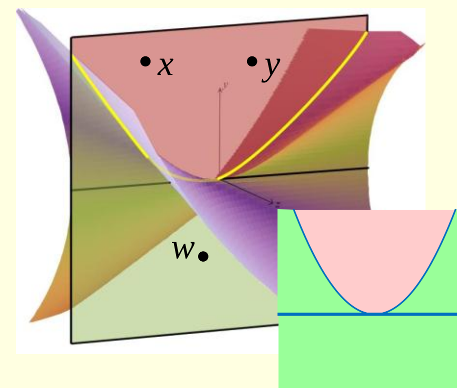
$$\begin{bmatrix} t^3 & 3t^2s & 3ts^2 & s^3 \end{bmatrix} \begin{bmatrix} A_x & A_y & A_w \\ B_x & B_y & B_w \\ C_x & C_y & C_w \\ D_x & D_y & D_w \end{bmatrix} = \begin{bmatrix} x & y & w \end{bmatrix}$$

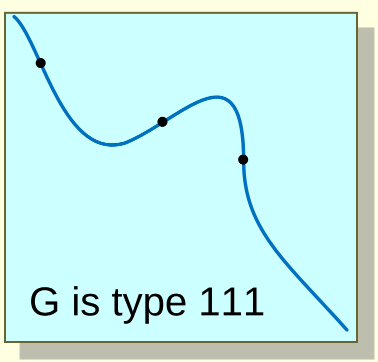
Look at plane defined by x,y,w cubics

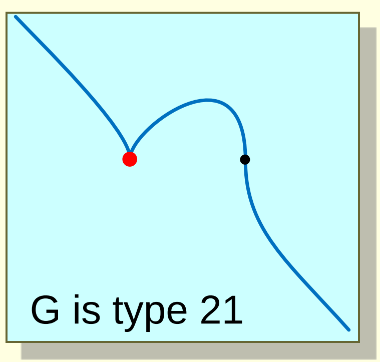


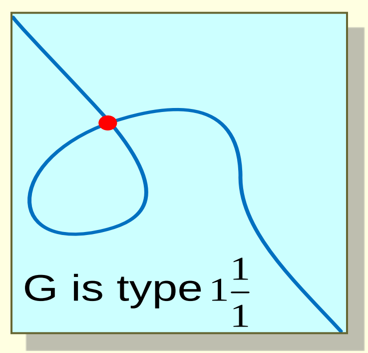


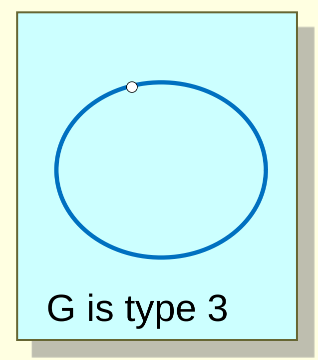




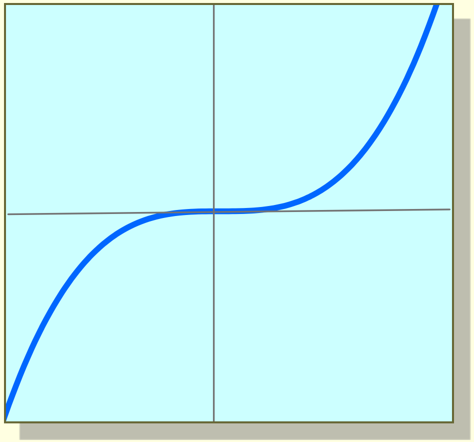

G is type 111

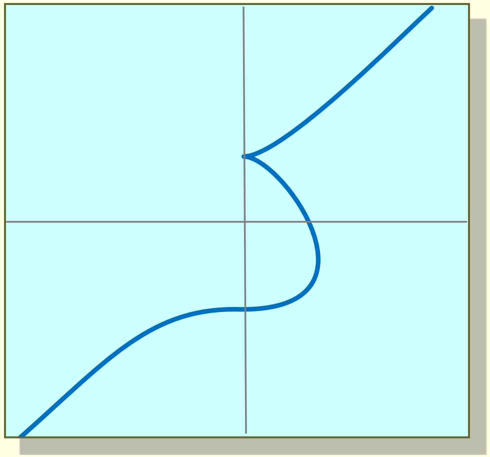

G is type 21

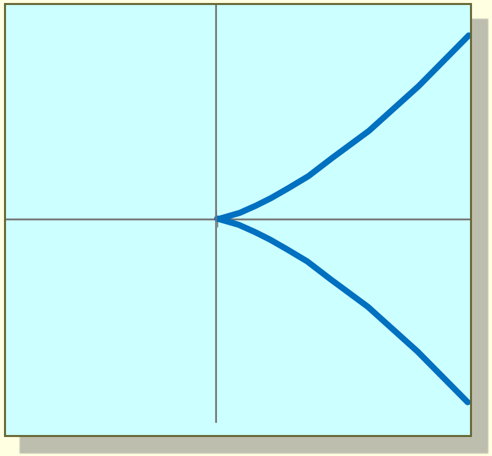

G is type $1\frac{1}{1}$


G is type 3

Cusp Cubic (review)







$$Y = X^3$$

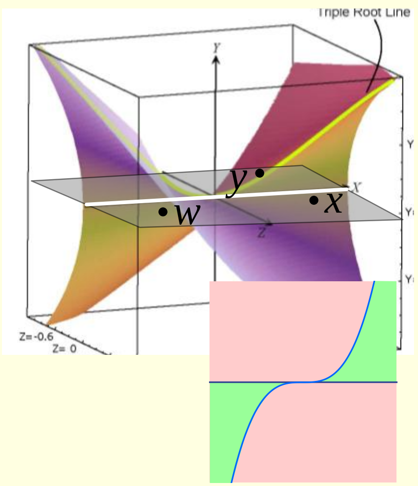
$$yw^2 = x^3$$

$$(y+w)^2(y-w) = x^3$$

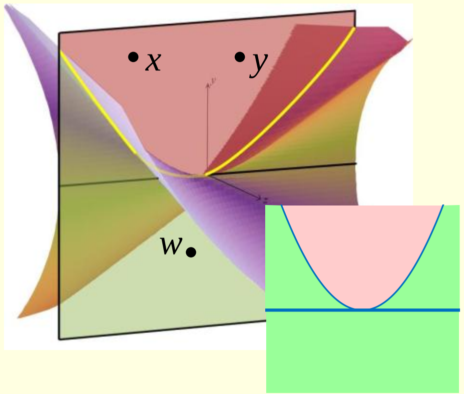
$$Y^2 = X^3$$

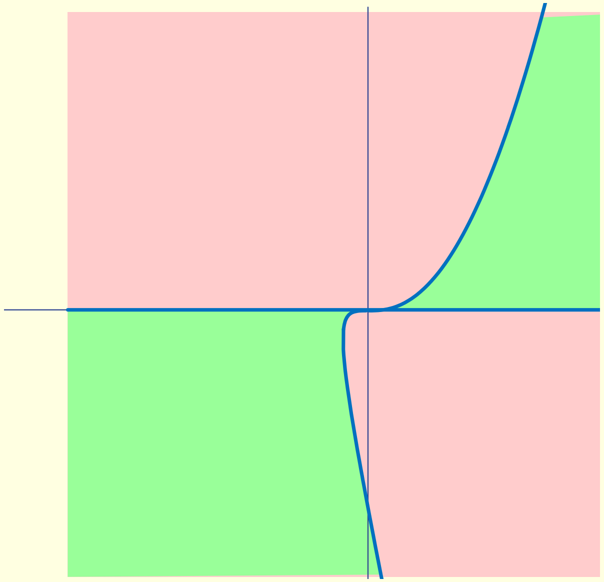
$$y^2w = x^3$$

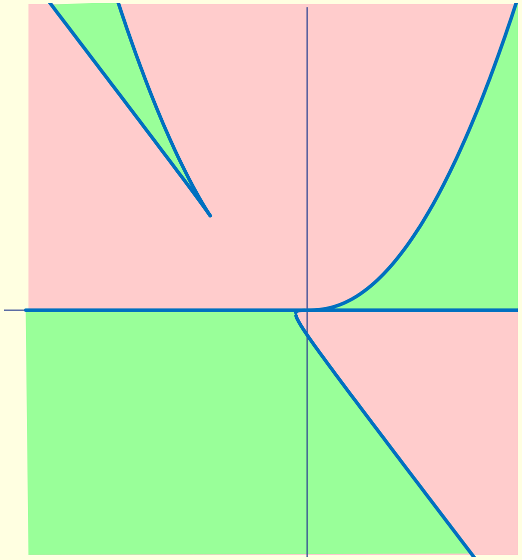
Evolution of Singularity

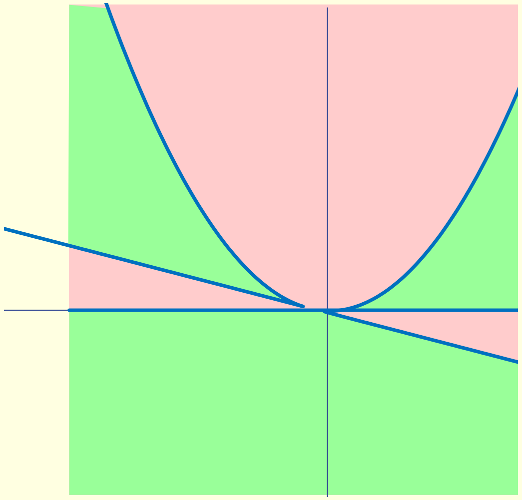


All fourth order curves



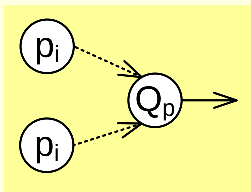


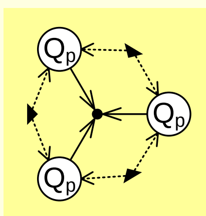


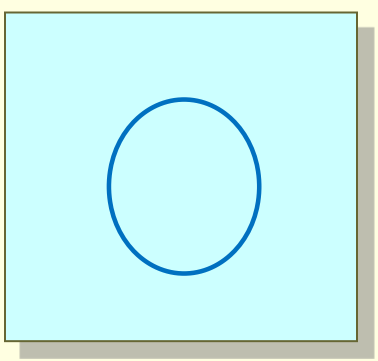


Generalization to Quartic?

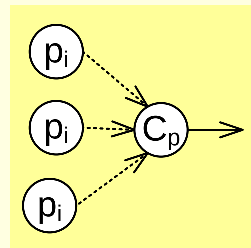
2

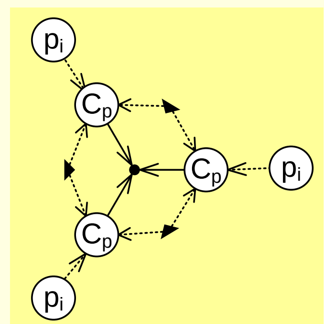


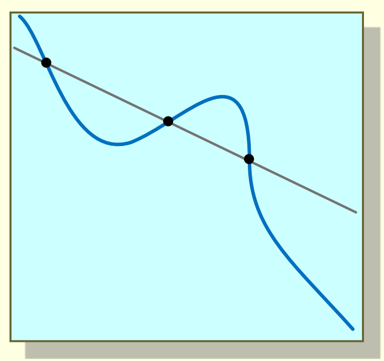




3







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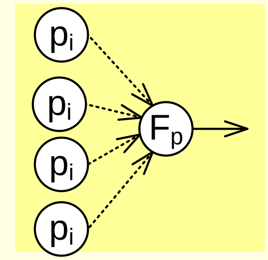
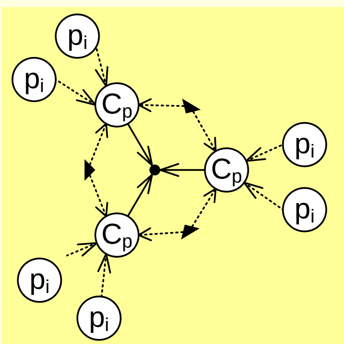
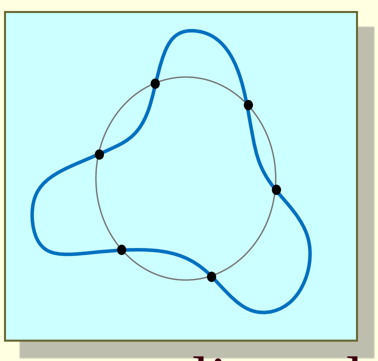




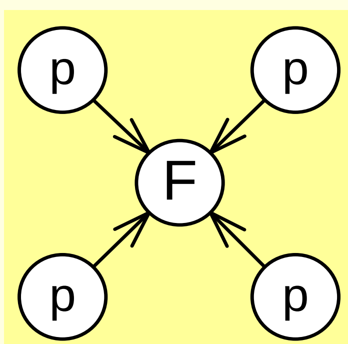
No.
More complicated



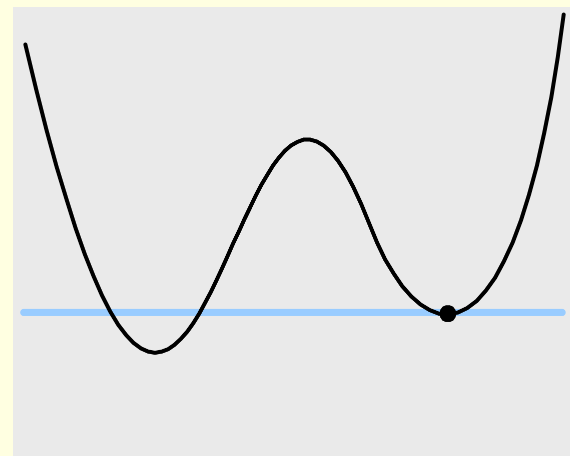
2D Quartic Polynomial

$$Ax^4 + 4Bx^3w + 6Cx^2w^2 + 4Dxw^3 + Ew^4 = 0$$

$$\begin{bmatrix} x & w \end{bmatrix} \left\{ \begin{bmatrix} x & w \end{bmatrix} \begin{bmatrix} \begin{bmatrix} A & B \\ B & C \end{bmatrix} & \begin{bmatrix} B & C \\ C & D \end{bmatrix} \\ \begin{bmatrix} B & C \\ C & D \end{bmatrix} & \begin{bmatrix} C & D \\ D & E \end{bmatrix} \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \right\} \begin{bmatrix} x \\ w \end{bmatrix} = 0$$



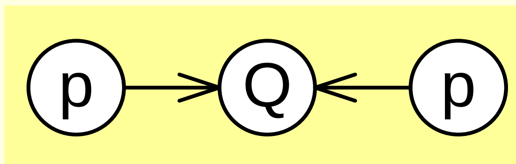
= 0

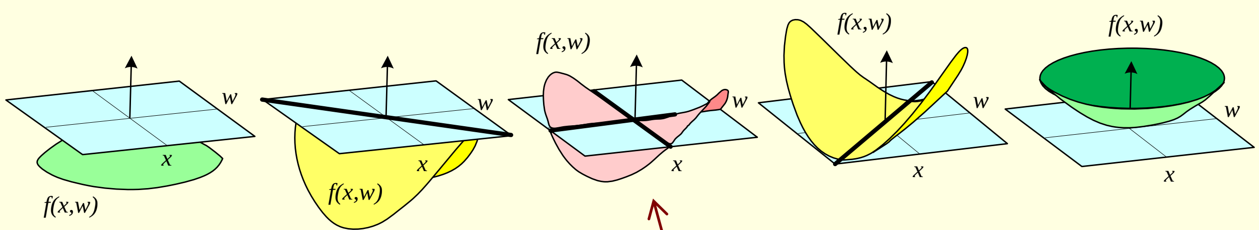


P^1 Quadratic Polynomial

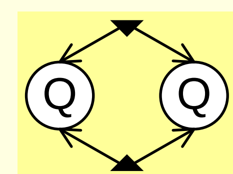
	2D=P^1 Point sets on	3D=P^2 Curves in plane	4D=P3 Surfaces in space
Linear			
Quadratic			
Cubic			
Quartic			

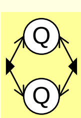
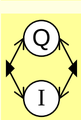
$$\mathbf{Q}(x,w) = Ax^2 + 2Bxw + Cw^2$$

$=$ 



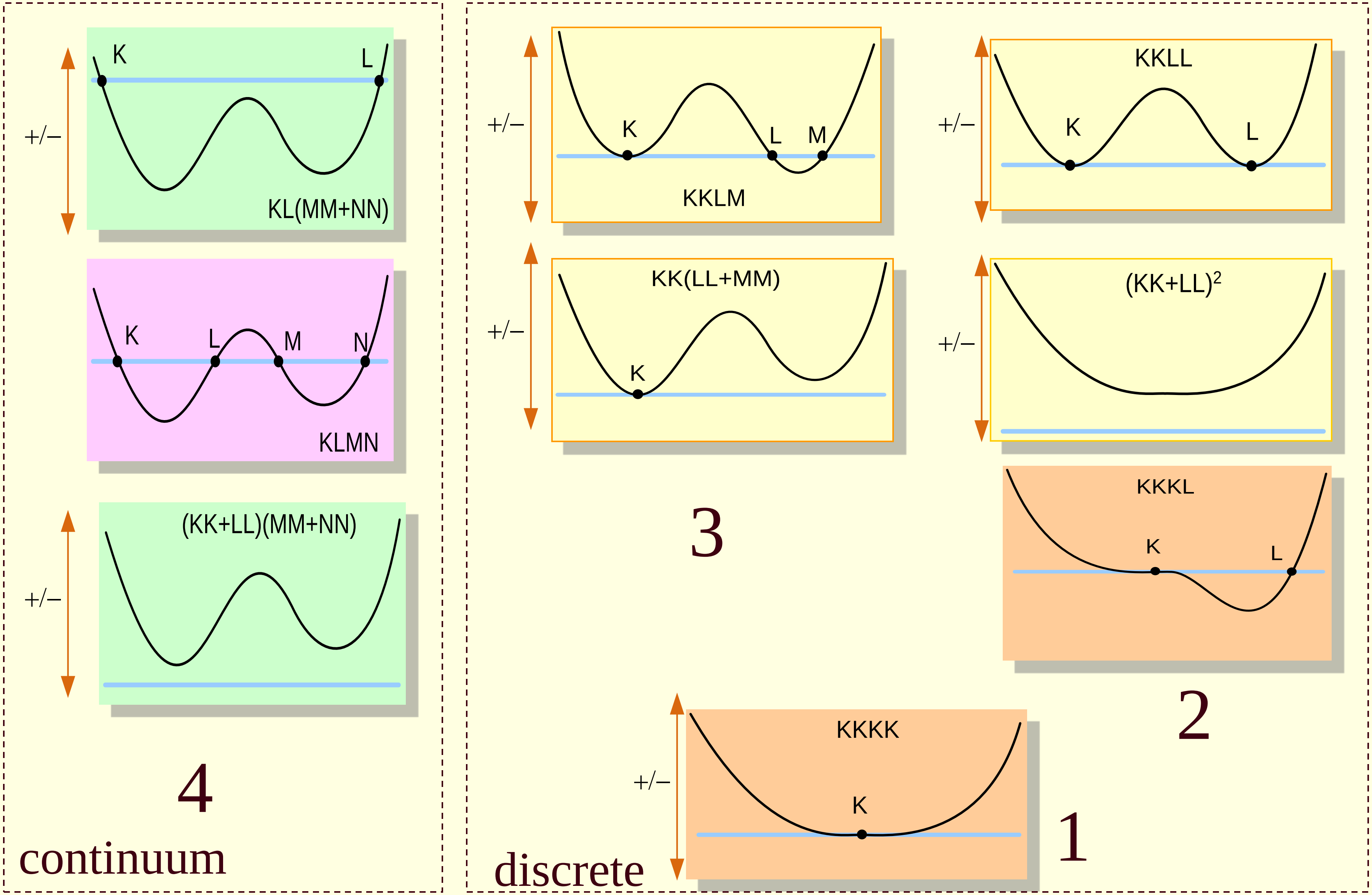
discriminant



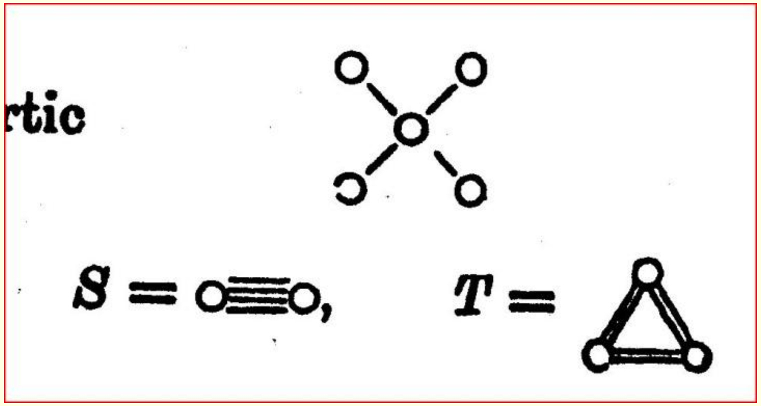
	$\frac{1}{1}-$	2-	11	2+	$\frac{1}{1}+$
	−	0	+	0	−
	+	+		−	−

Note: sign not invariant

Types of Quartic Root Structure

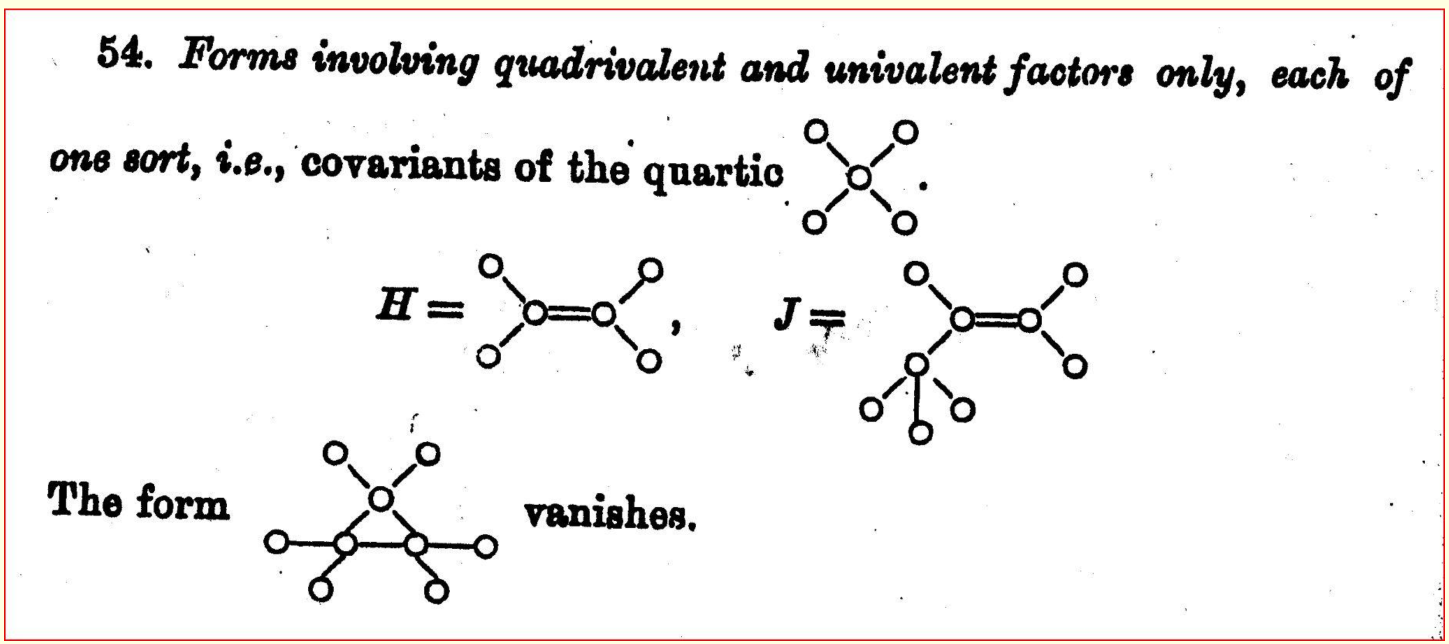


Kempe (1885)



rtic

$S =$, $T =$



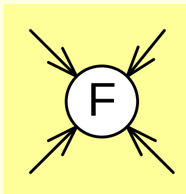
54. *Forms involving quadrivalent and univalent factors only, each of one sort, i.e.,* covariants of the quartic .

$H =$, $J =$

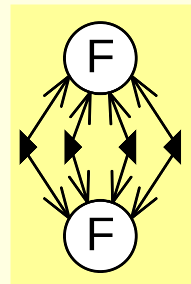
The form vanishes.

Diagrams

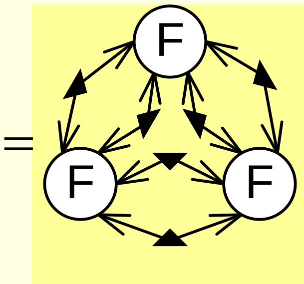
$\mathbf{F} =$



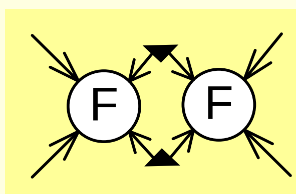
$S = \mathrm{F}_2 =$



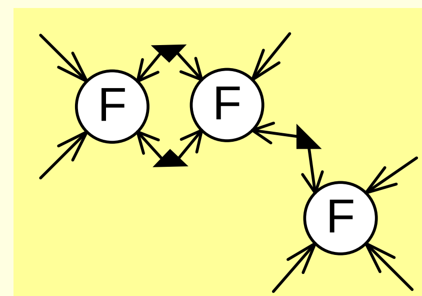
$T = \mathrm{F}_3 =$



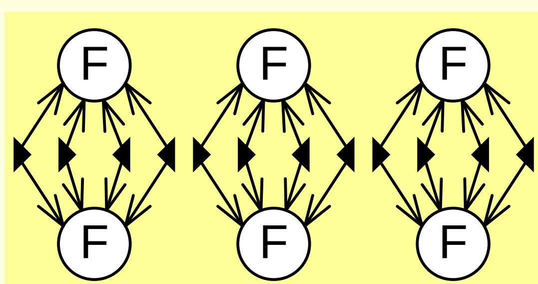
$H(\mathbf{F}) =$



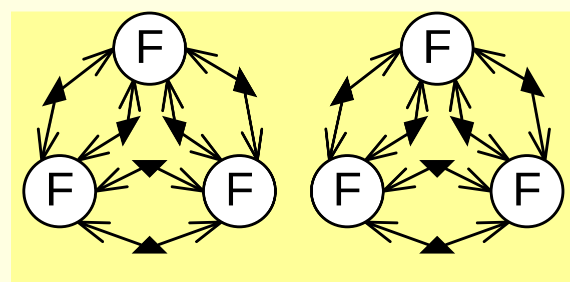
$J(\mathbf{F}) =$



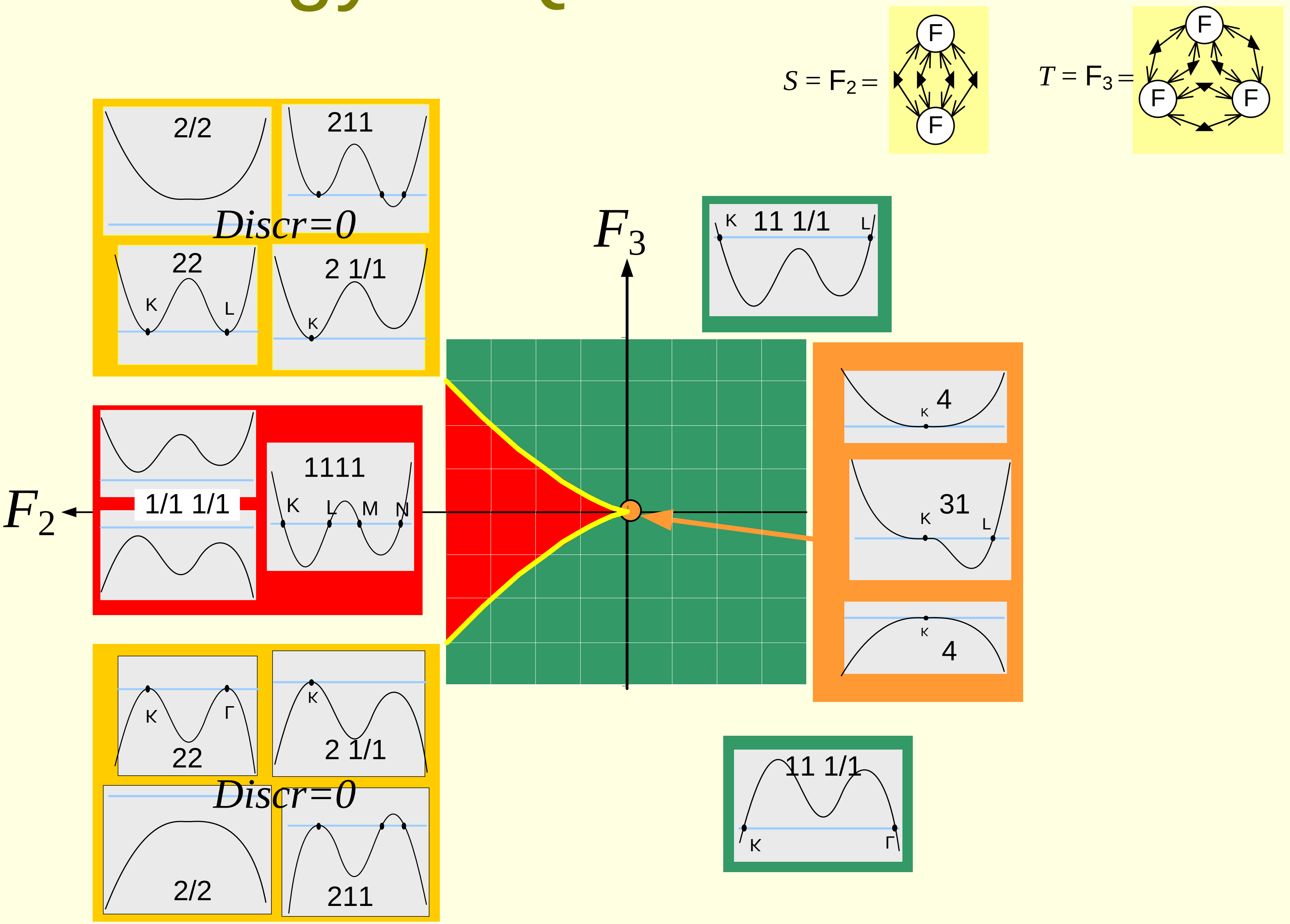
$\Delta(\mathbf{F}) =$



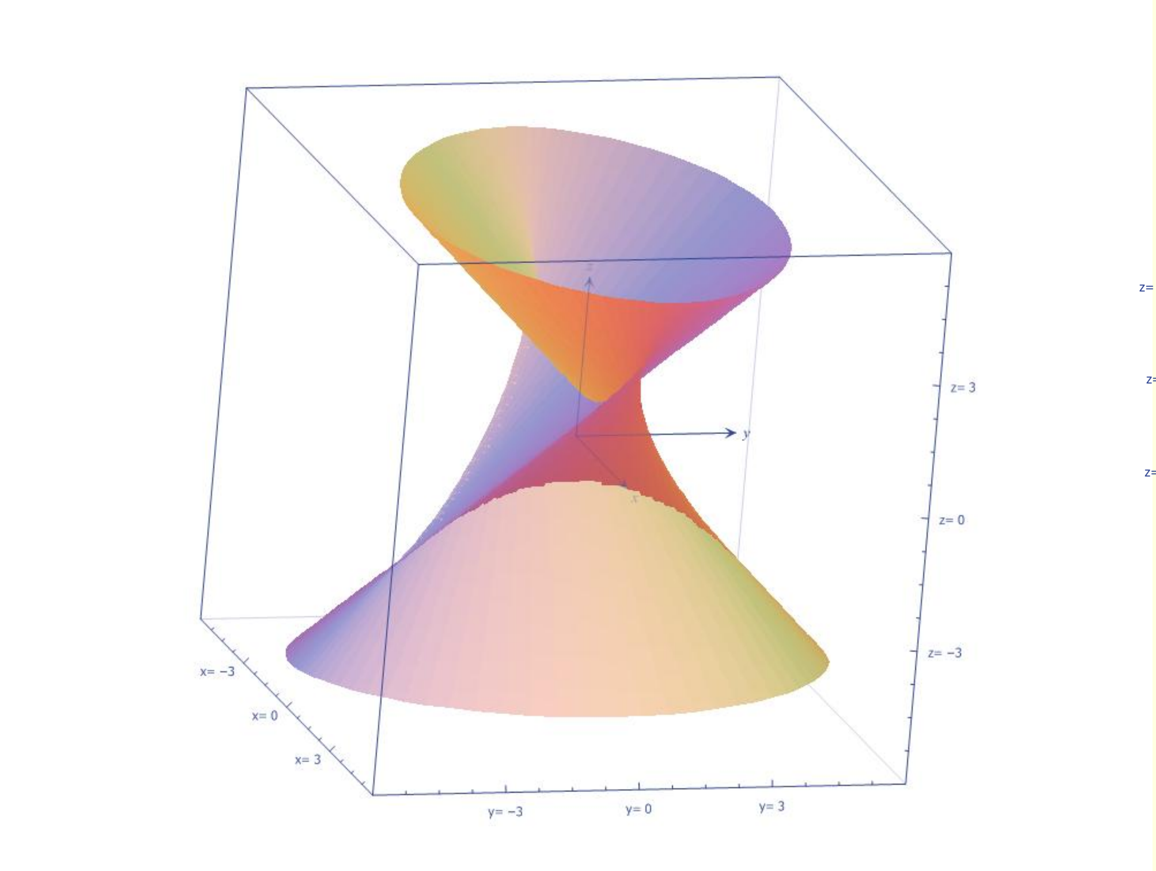
- 6

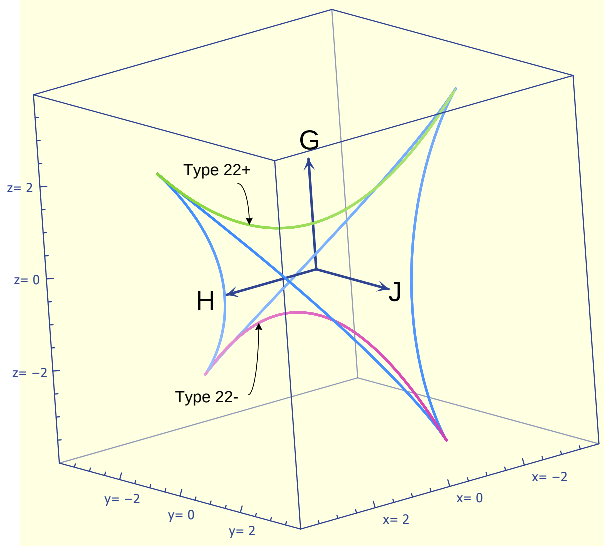


The Ecology of Quartics



Poston/Stewart plot of discriminant





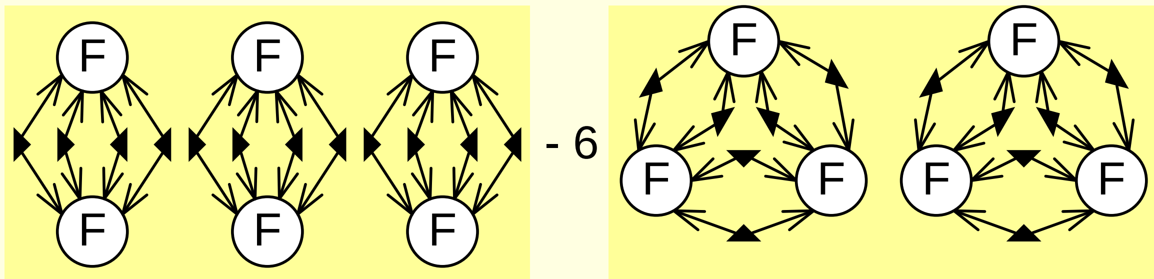
T. Poston and I Stewart

THE CROSS-RATIO FOLIATION OF BINARY QUARTIC FORMS

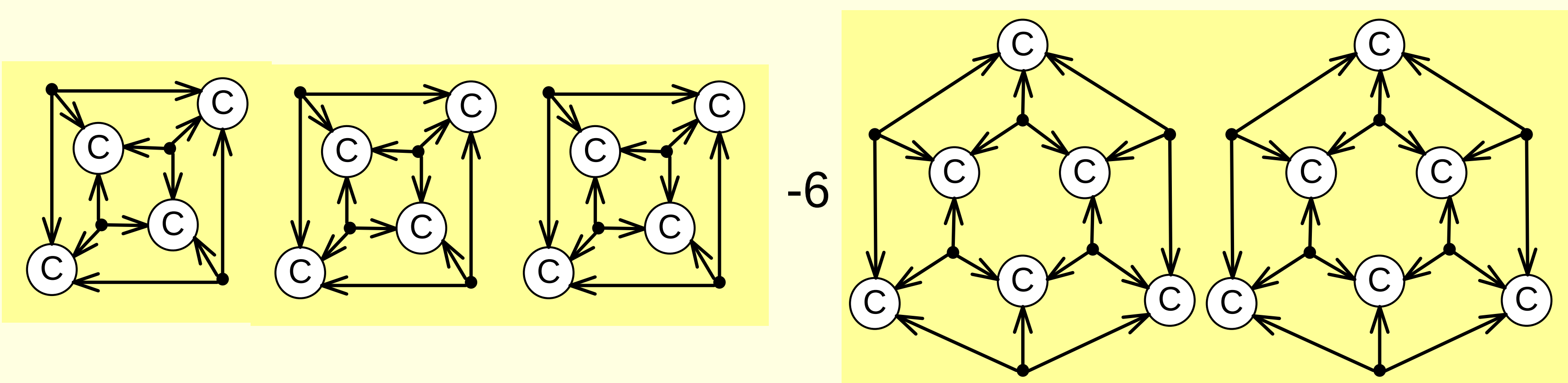
Geom. Dedicata **27** (1988), no 3, 263-280

A Puzzle about Discriminants

Order 4, Dimension 2 = P^1

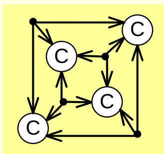


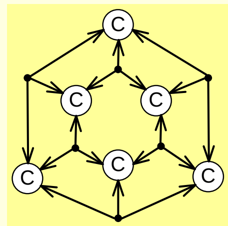
Order 3, Dimension 3 = P^2

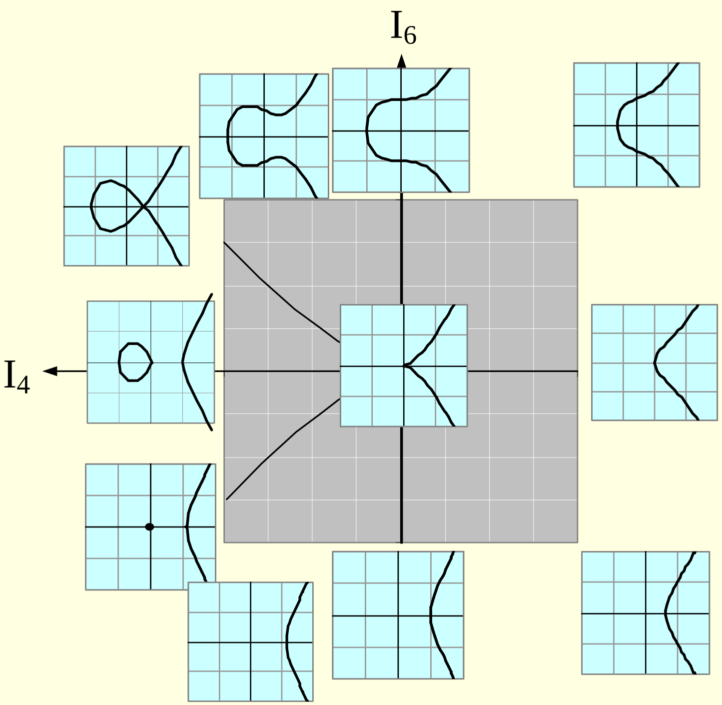


Comparison of Invariant space

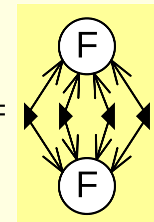
P^2 Cubic Curves

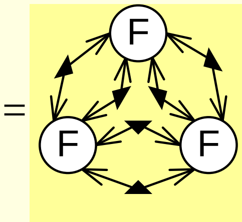
$I_4 =$ 

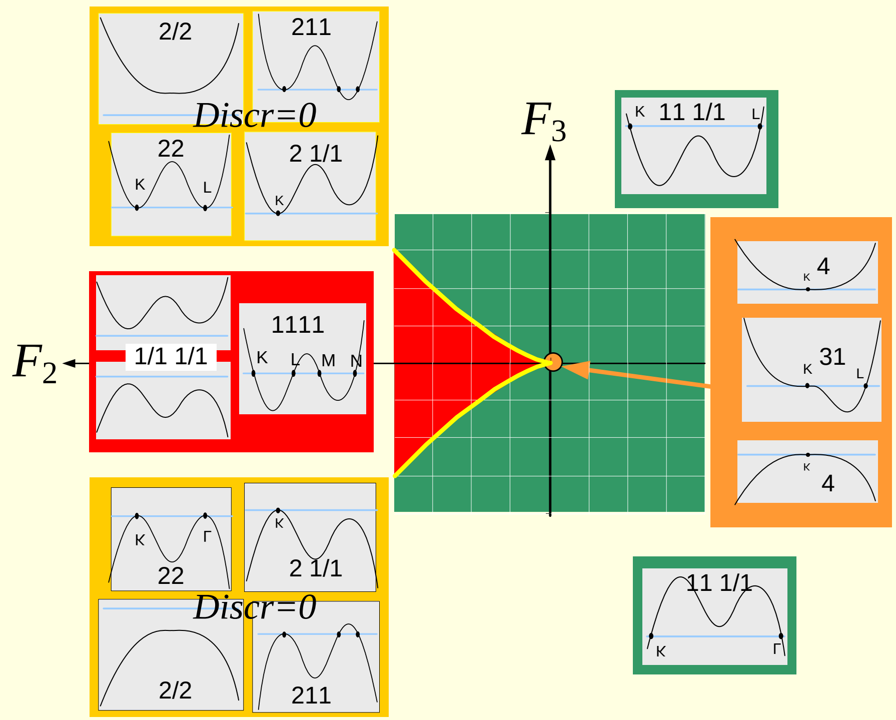
$I_6 =$ 



P^1 Quartic Polynomials

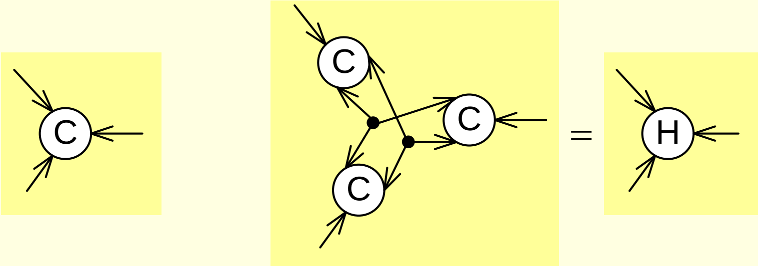
$F_2 =$ 

$F_3 =$ 

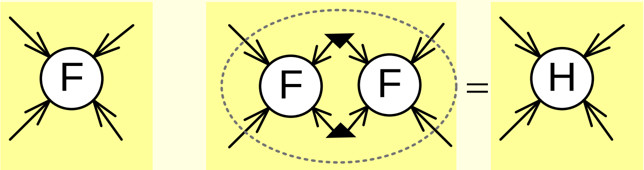


Comparison of Hessians

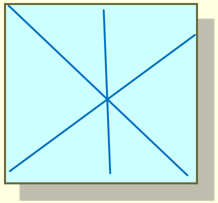
P^2 Cubic Curves



P^1 Quartic Polynomials

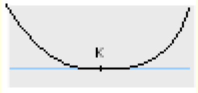


$$H = 0$$

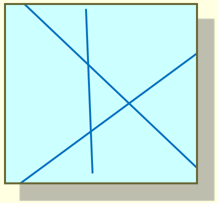


C is three lin-dep lines

F is single 4-tuple root

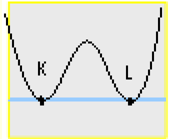


$$H = \kappa F$$



C is three lin indep lines

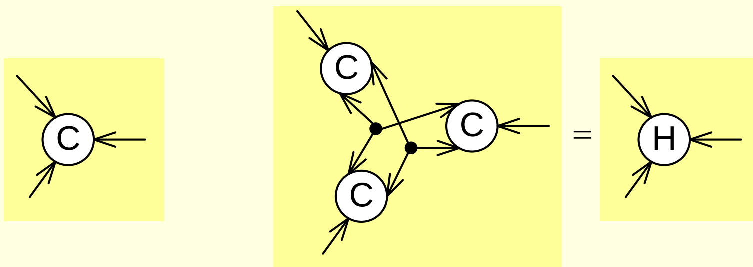
F has two double roots

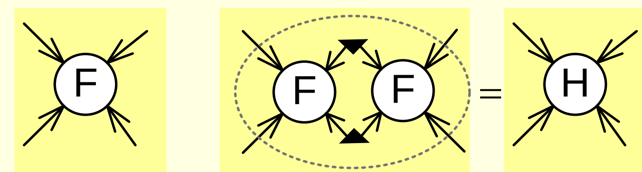


Comparison of Hessians

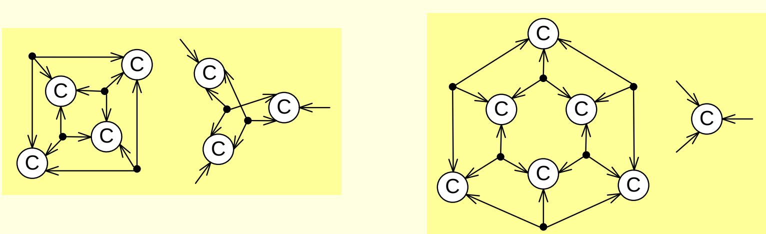
P^2 Cubic Curves

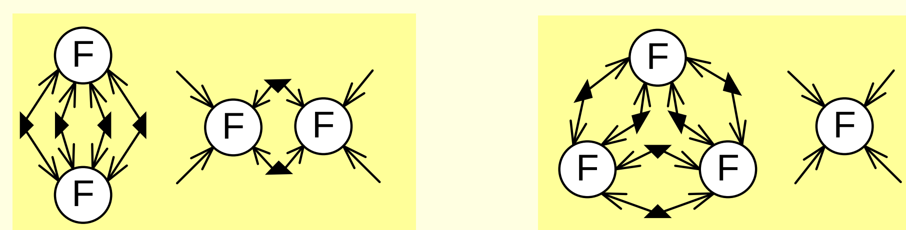
P^1 Quartic Polynomials





$H = \kappa F$ implies $\alpha F + \beta H = 0$





$I_4 H - I_6 C = 0$

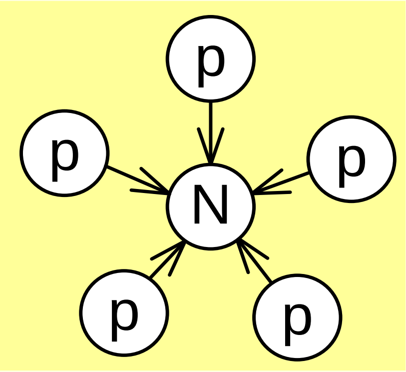
$F_2 H + F_3 F = 0$

C is three indep lines

F has two double roots

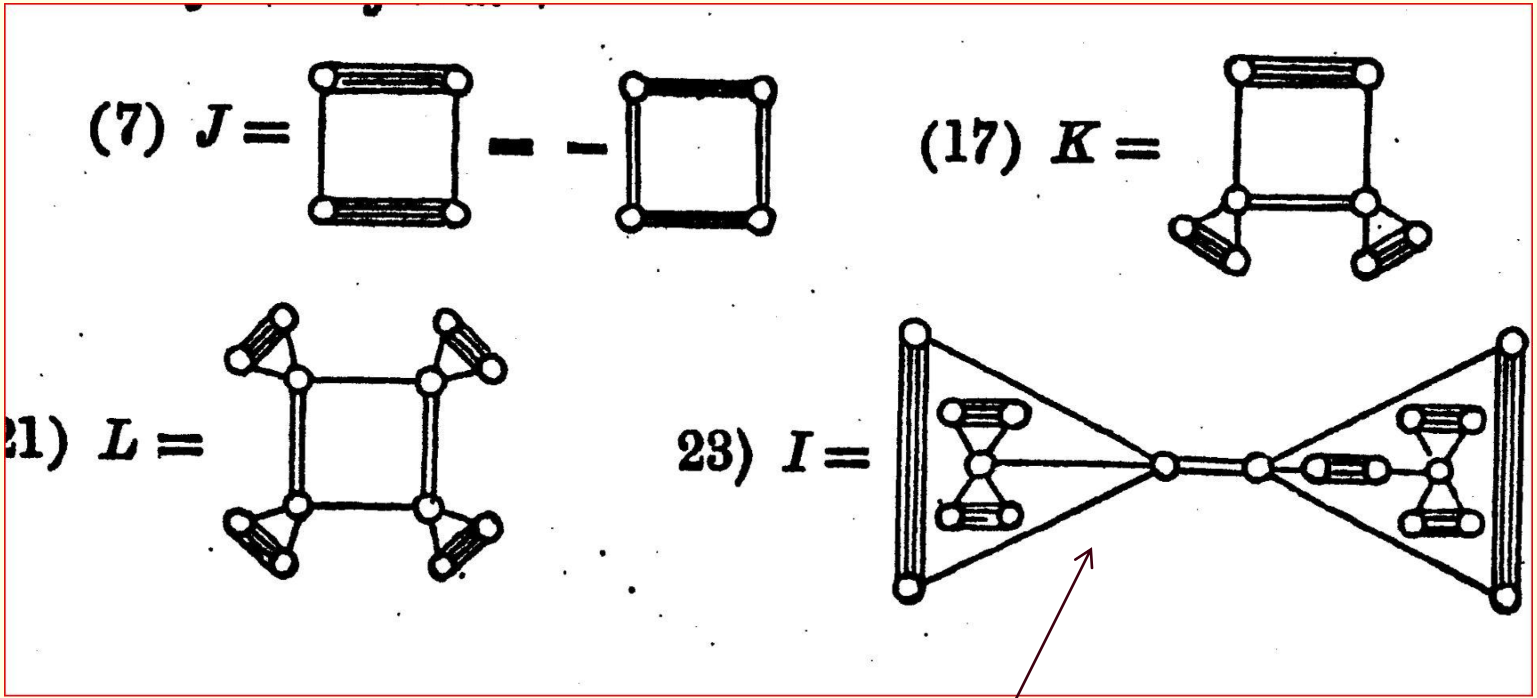
2D Quintic Polynomial

$$Ax^5 + 4Bx^4w + 6Cx^3w^2 + 4Dx^2w^3 + Exw^4 + Fw^5 = 0$$



$= 0$

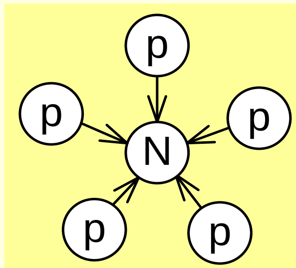
Kempe (1885)

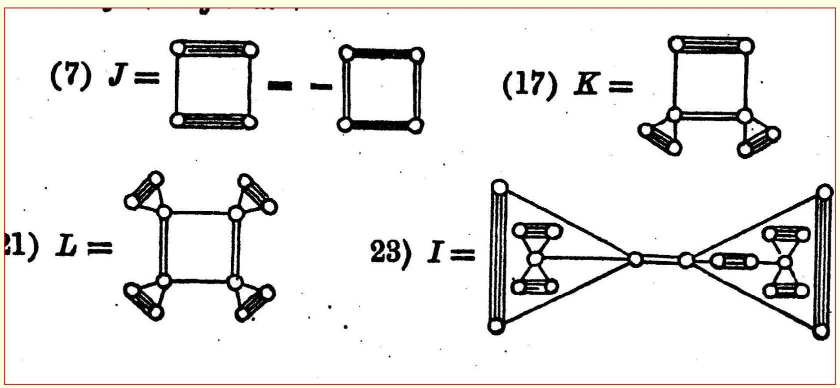


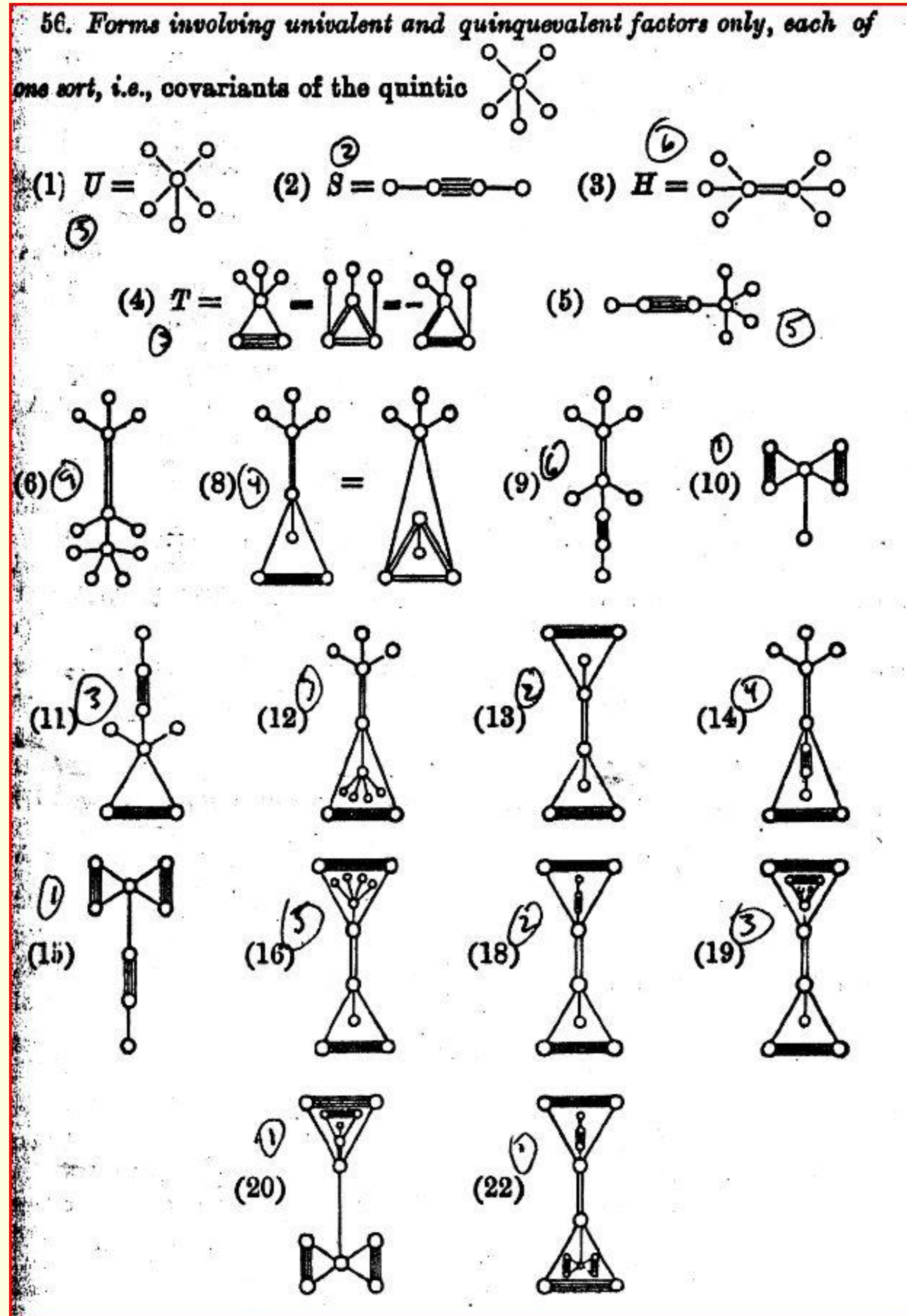
Odd number of arcs (epsilons)

I^2 = fcn of (J,K,L)

Kempe (1885)

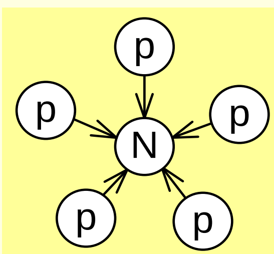
$U =$ 
p
p
p
N
p
p


(7) J =
(17) K =
21) L =
23) I =


56. Forms involving univalent and quinquevalent factors only, each of one sort, i.e., covariants of the quintic
(1) U =
(2) S =
(3) H =
(4) T =
(5)
(6)
(8)
=
(9)
(10)
(11)
(12)
(13)
(14)
(15)
(16)
(18)
(19)
(20)
(22)

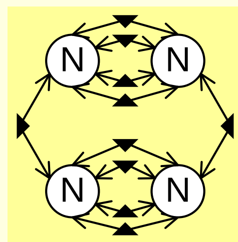
2D Quintic Polynomial

$$Ax^5 + 4Bx^4w + 6Cx^3w^2 + 4Dx^2w^3 + Exw^4 + Fw^5 = 0$$

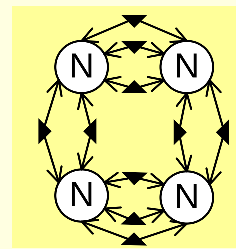


$= 0$

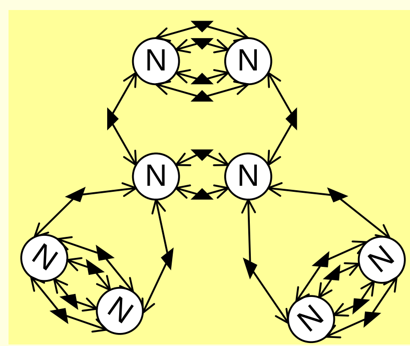
J =



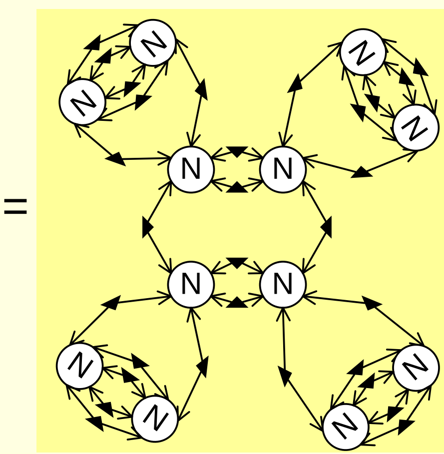
= -



K =



L =

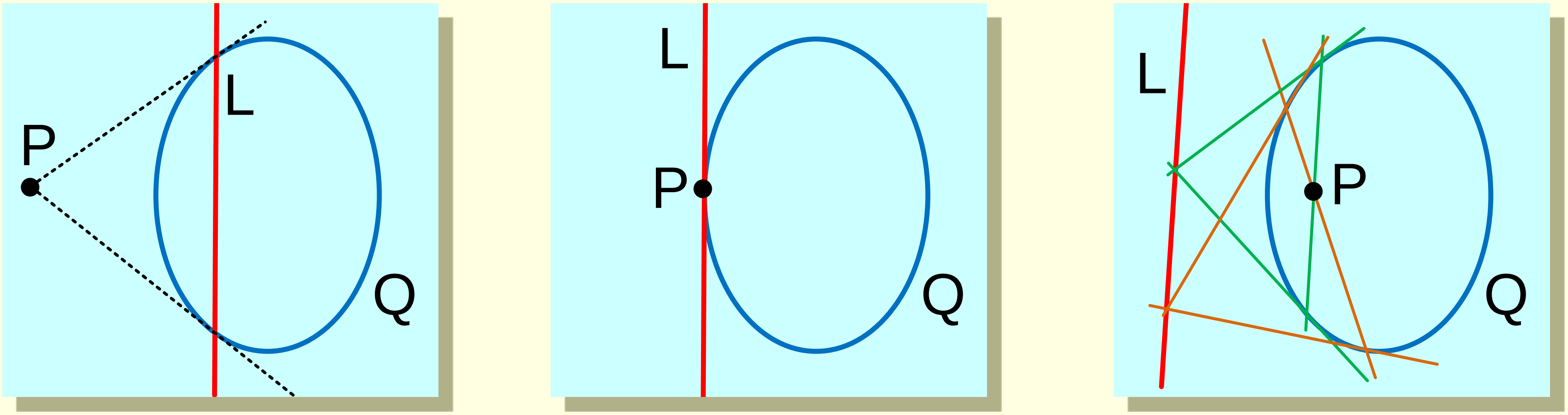


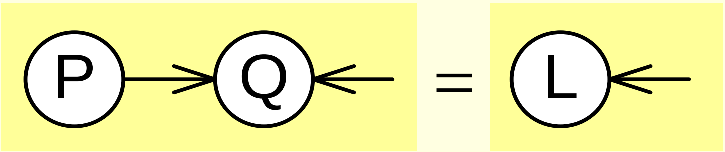
$$\text{discriminant} = J^2 - s\,K$$

?

$$L = 0 \Rightarrow \text{solvable?}$$

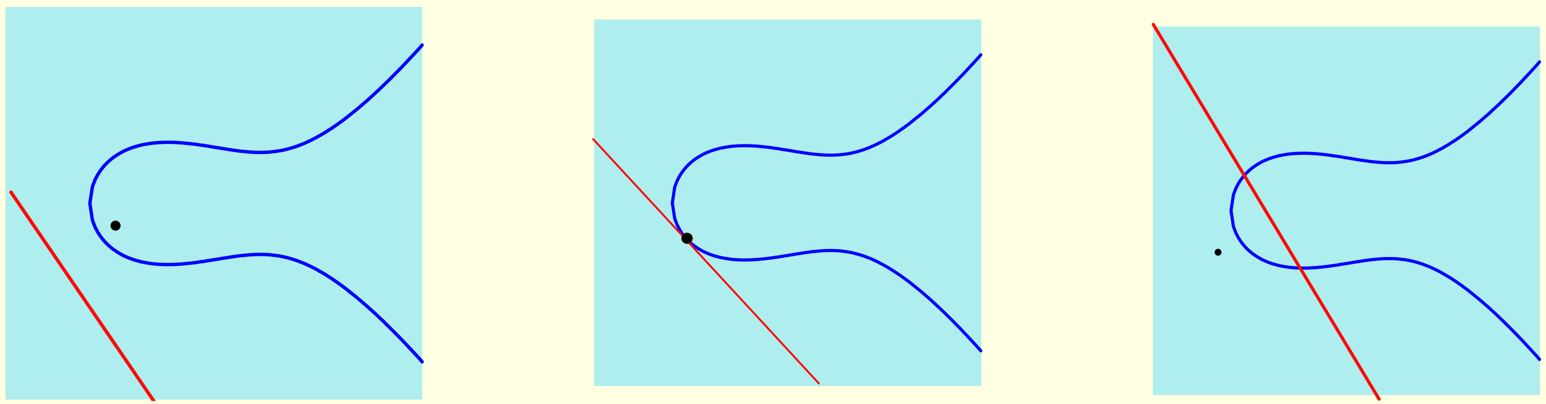
Polar Line of Quadratic

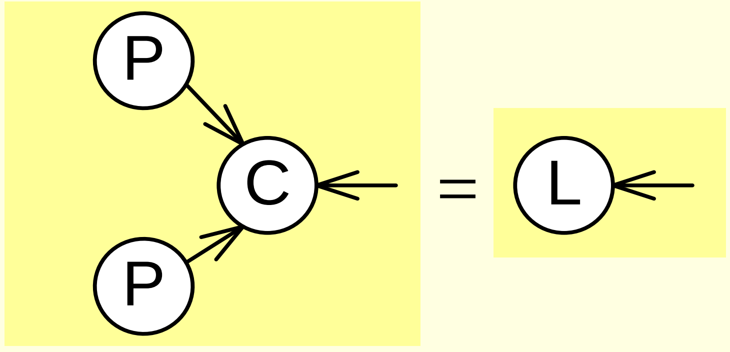




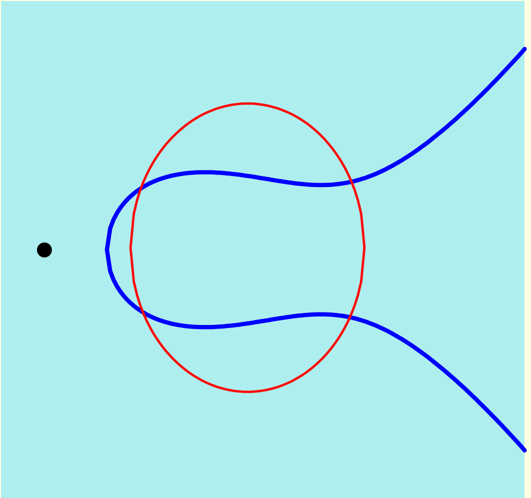
Generalize to 3D

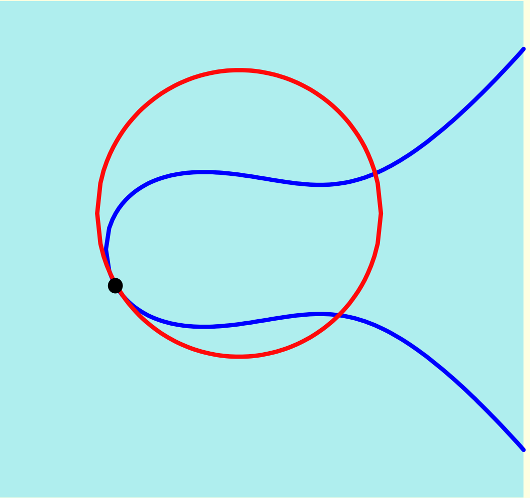
Polar Line of Cubic

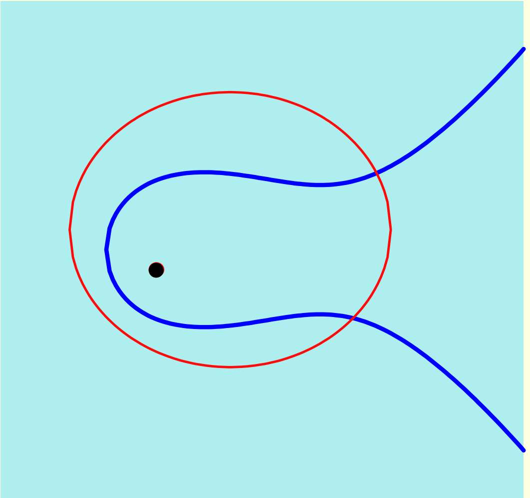


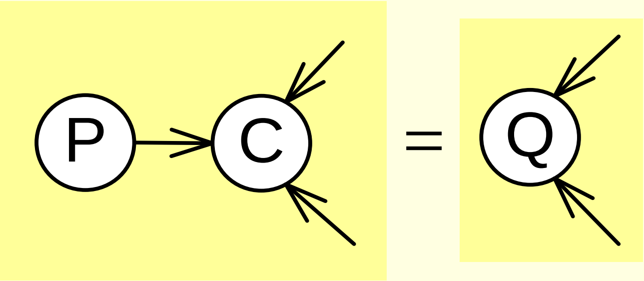


Polar Quadratic of Cubic

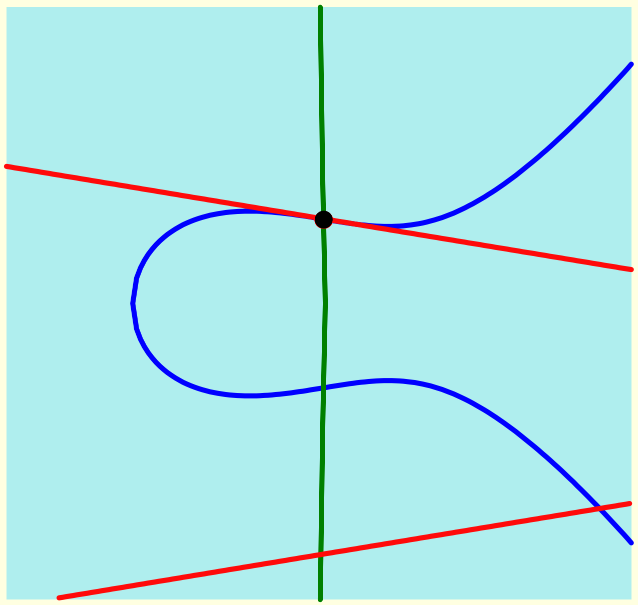




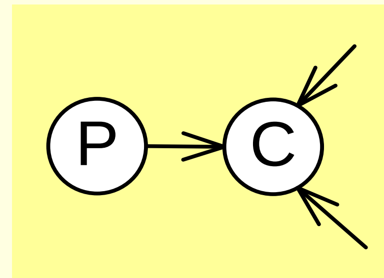




If Polar Quadratic is Singular

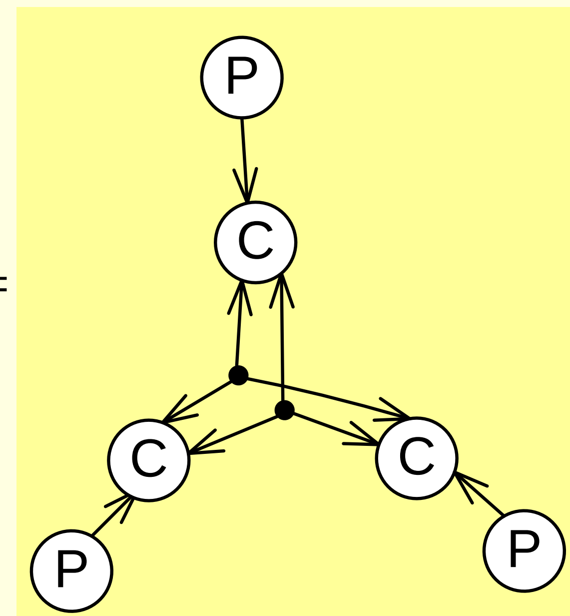


det

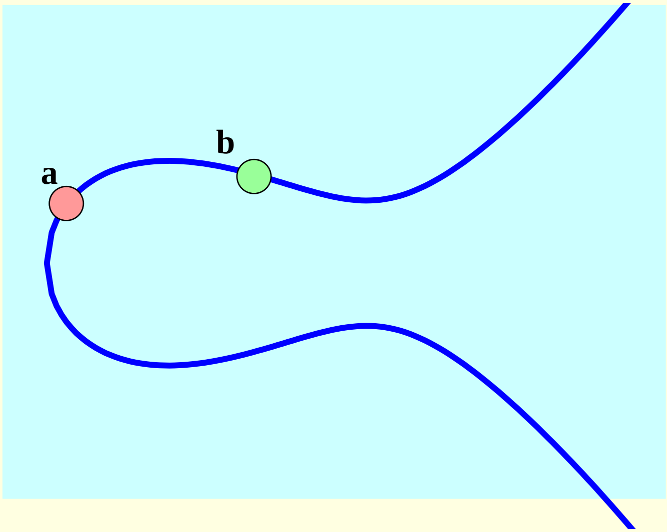


$= 0$

$=$

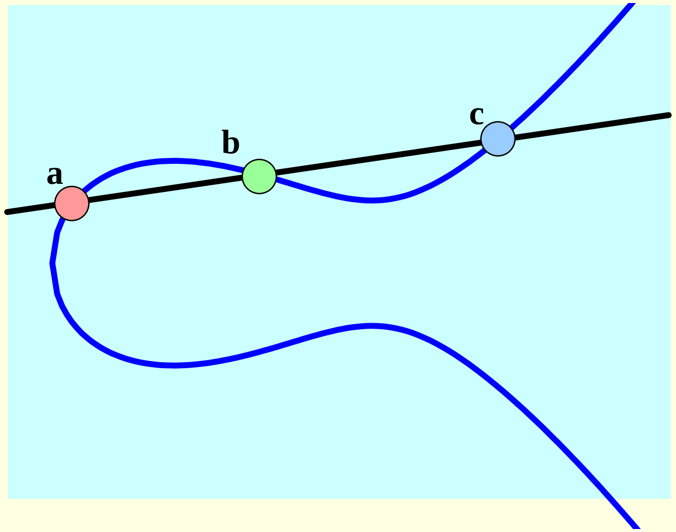


Group Structure of Cubic



$$\mathbf{a} + \mathbf{b} = ?$$

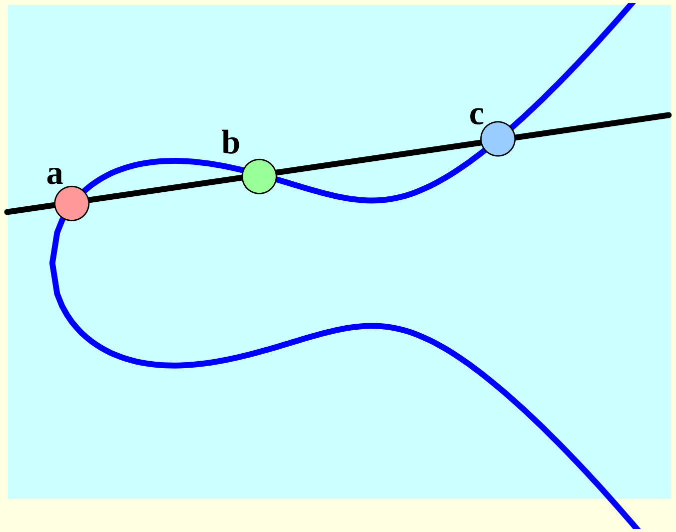
Group Structure of Cubic



$$\mathbf{a} + \mathbf{b} = \mathbf{c}?$$

$$\mathbf{a} + \mathbf{c} = \mathbf{b}?$$

Group Structure of Cubic



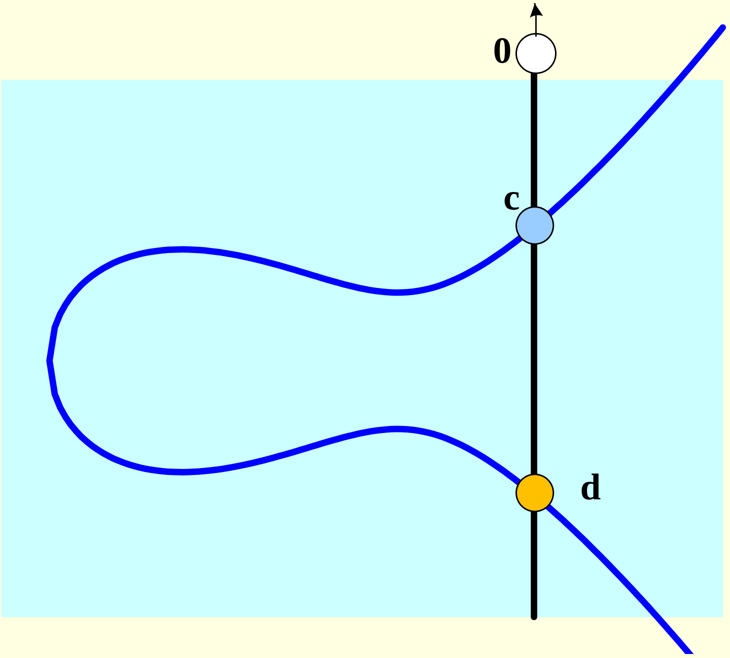
$$\mathbf{a}+\mathbf{b}+\mathbf{c}=0$$

$$\mathbf{a}+\mathbf{b}=-\mathbf{c}$$

$$\mathbf{a}+\mathbf{c}=-\mathbf{b}$$

$$\mathbf{b}+\mathbf{c}=-\mathbf{a}$$

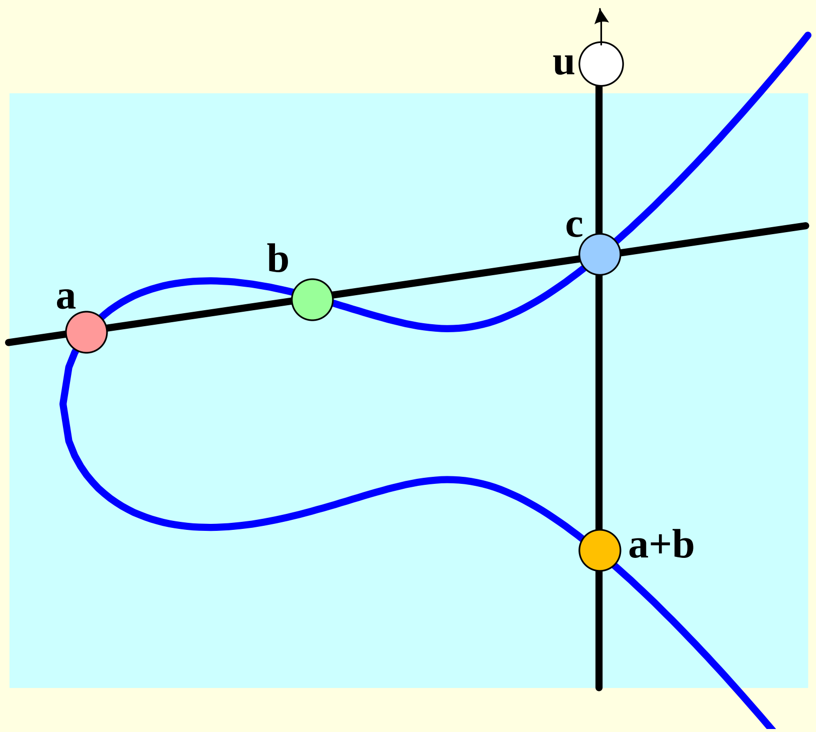
Group Structure of Cubic



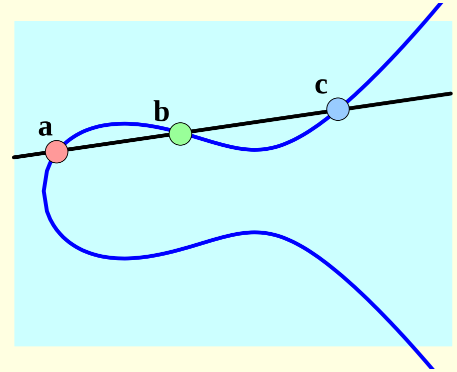
$$0 + \mathbf{c} + \mathbf{d} = 0$$

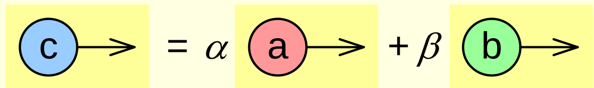
$$\mathbf{d} = -\mathbf{c}$$

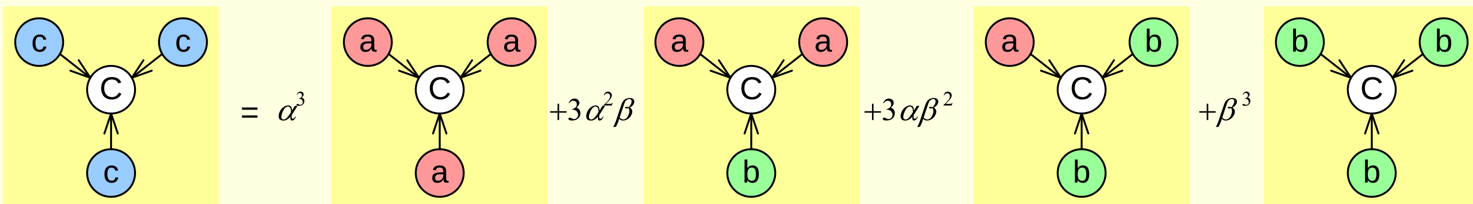
Group Structure of Cubic

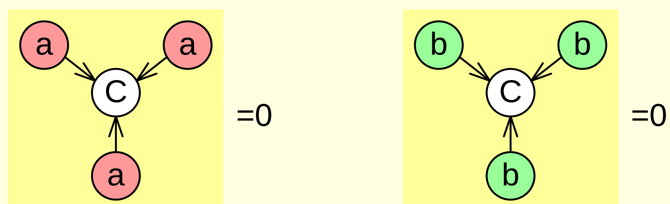


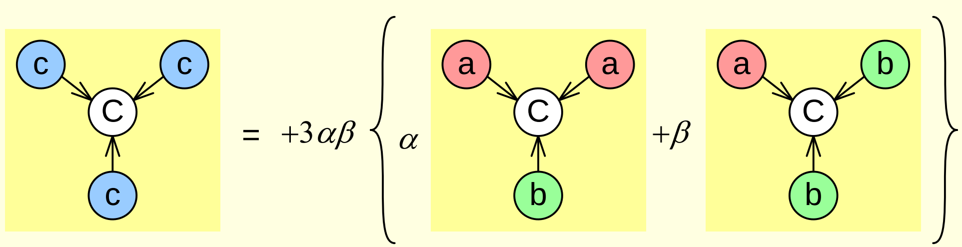
Finding c



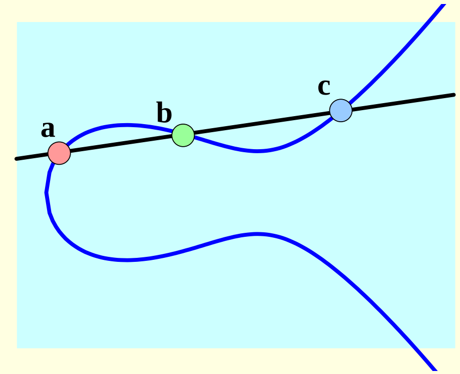


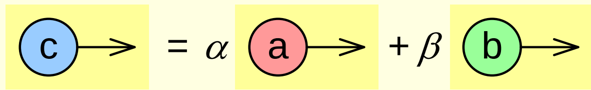


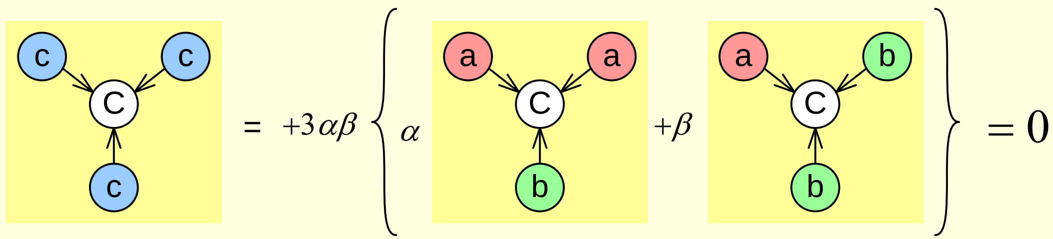


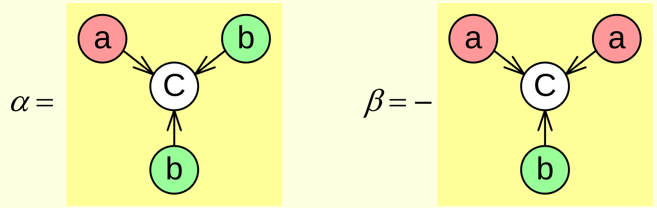


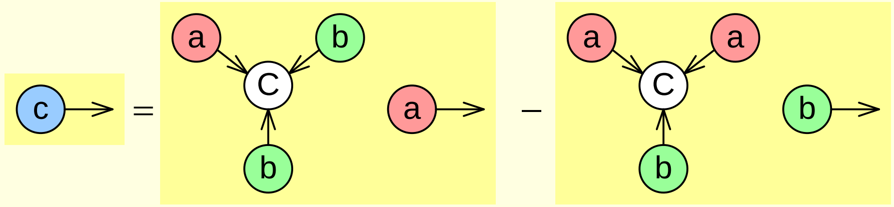
Finding c



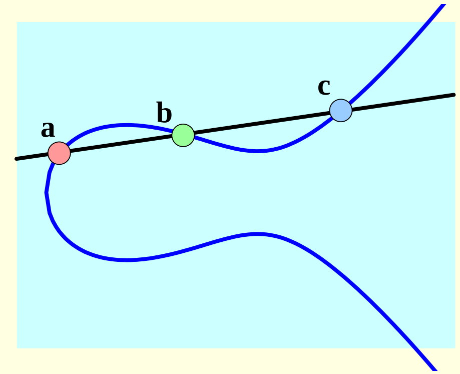


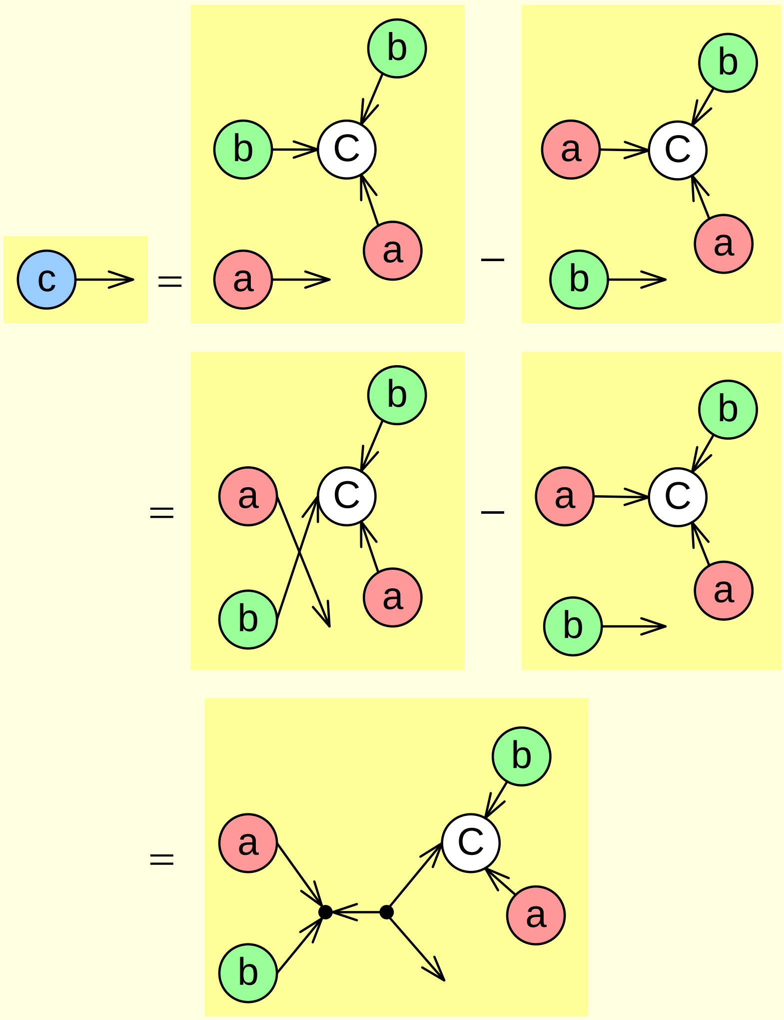


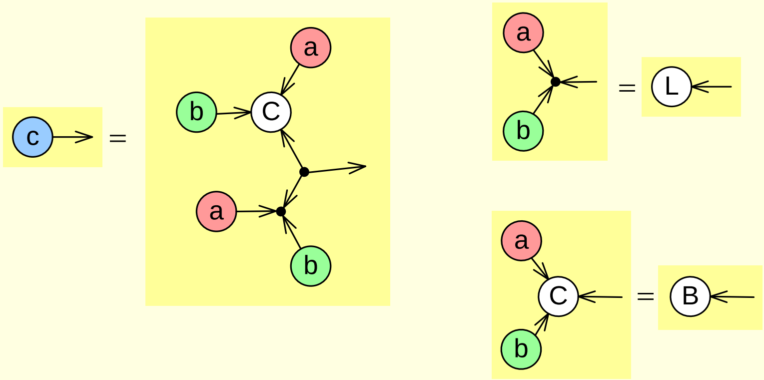


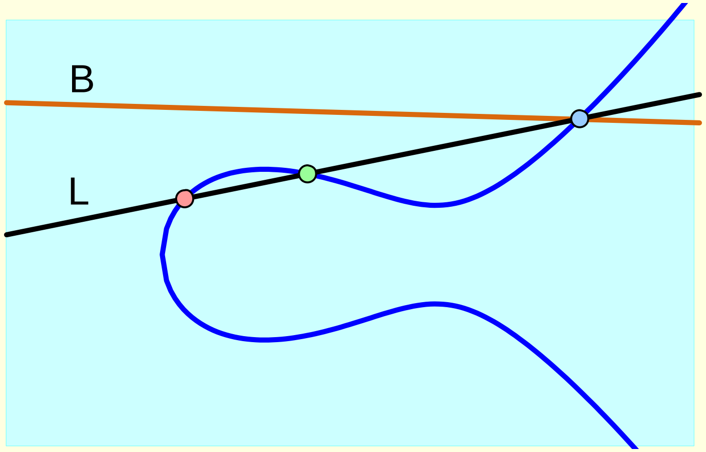


Finding c

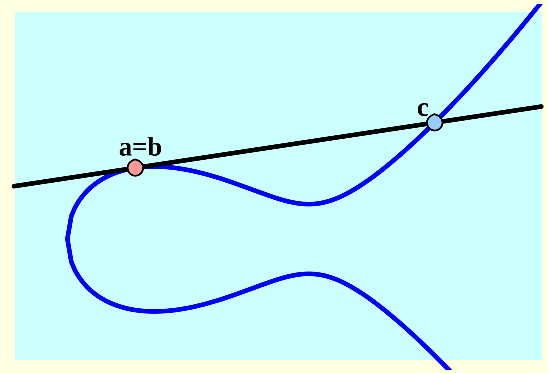


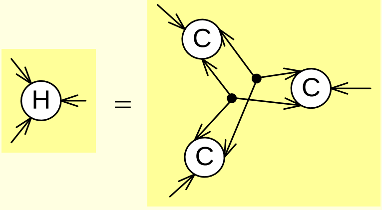


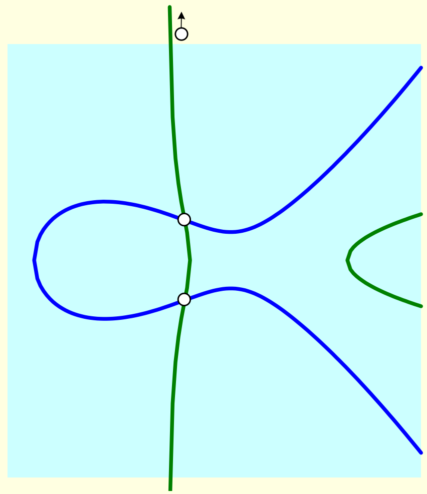


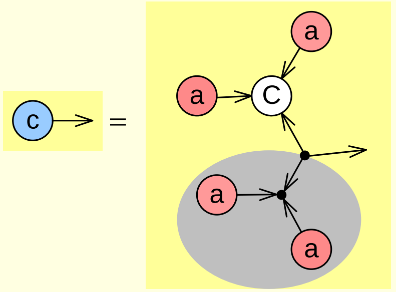


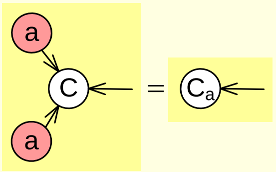
Same a,b

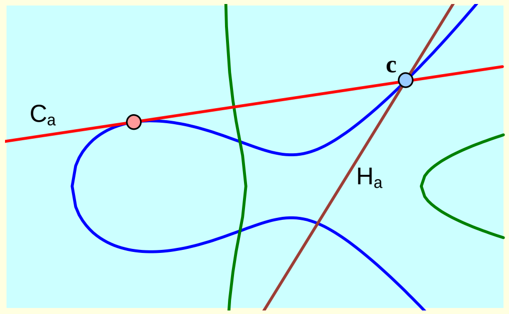


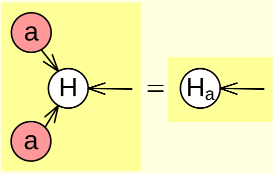


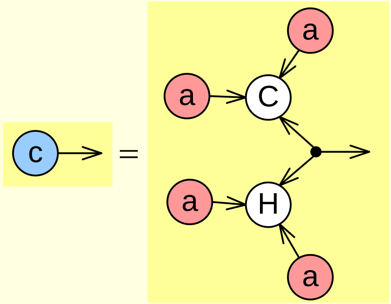




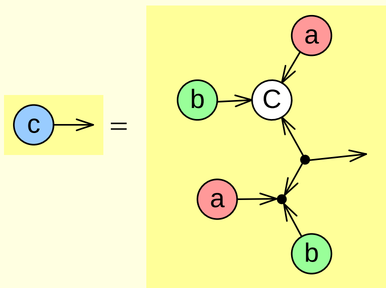


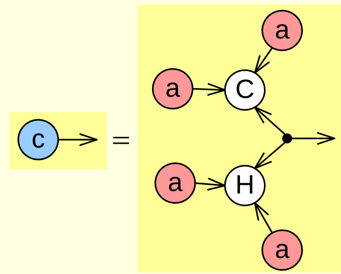


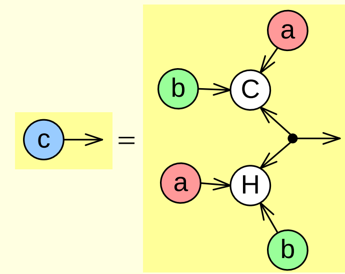


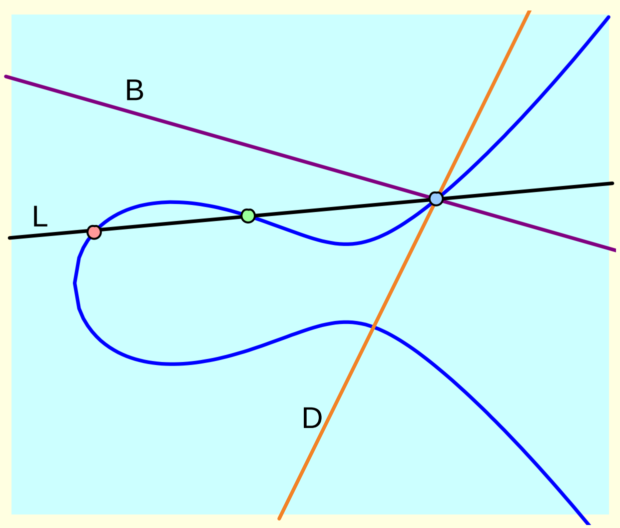


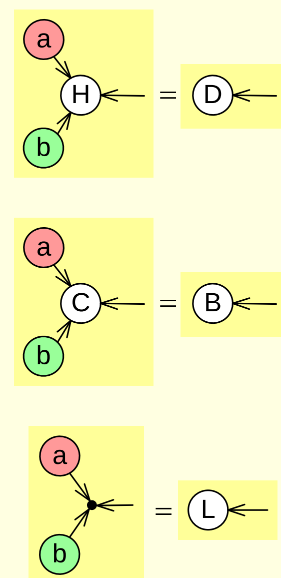
Common Formula











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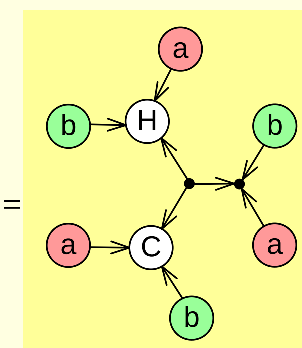
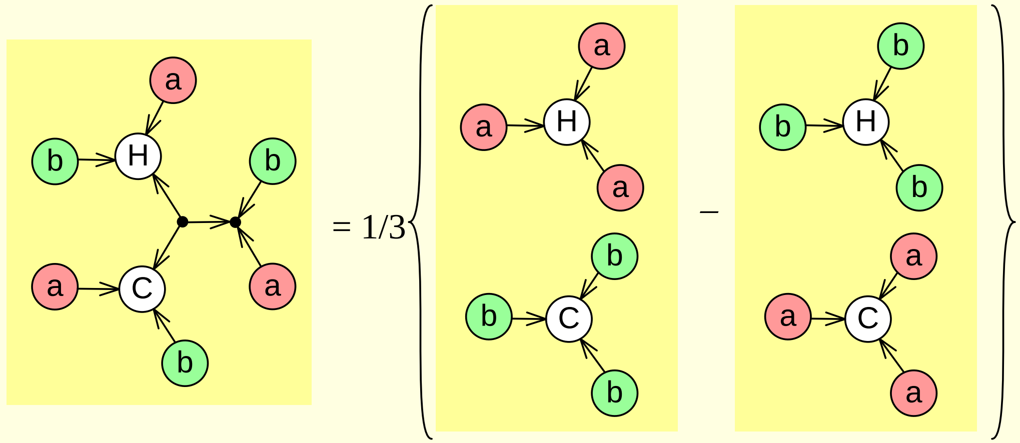
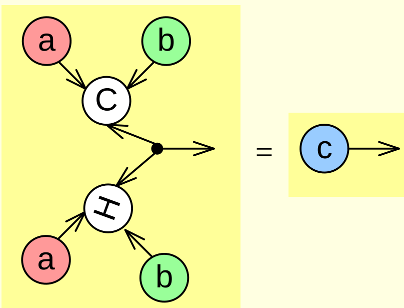
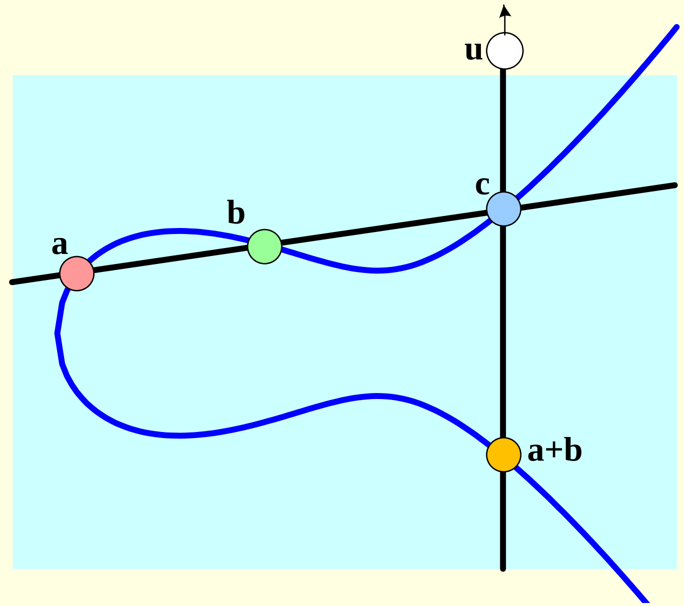


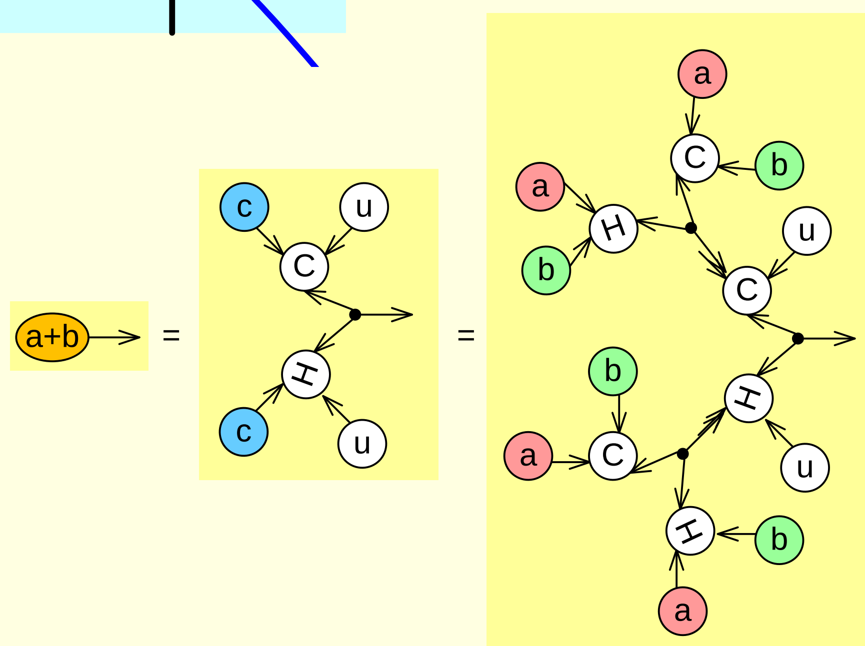
After some arc swapping



Group Structure of Cubic







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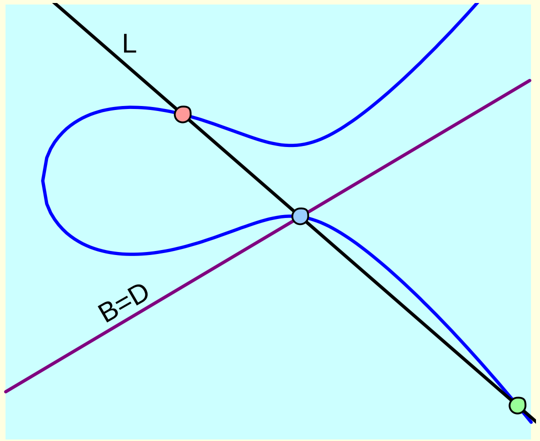
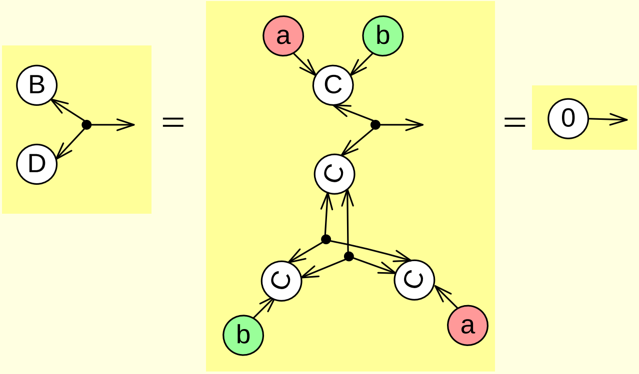
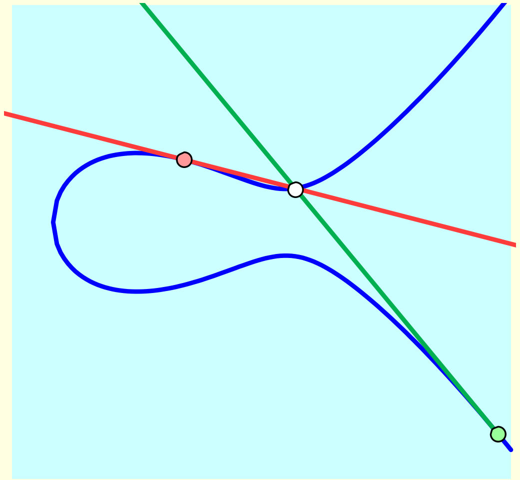
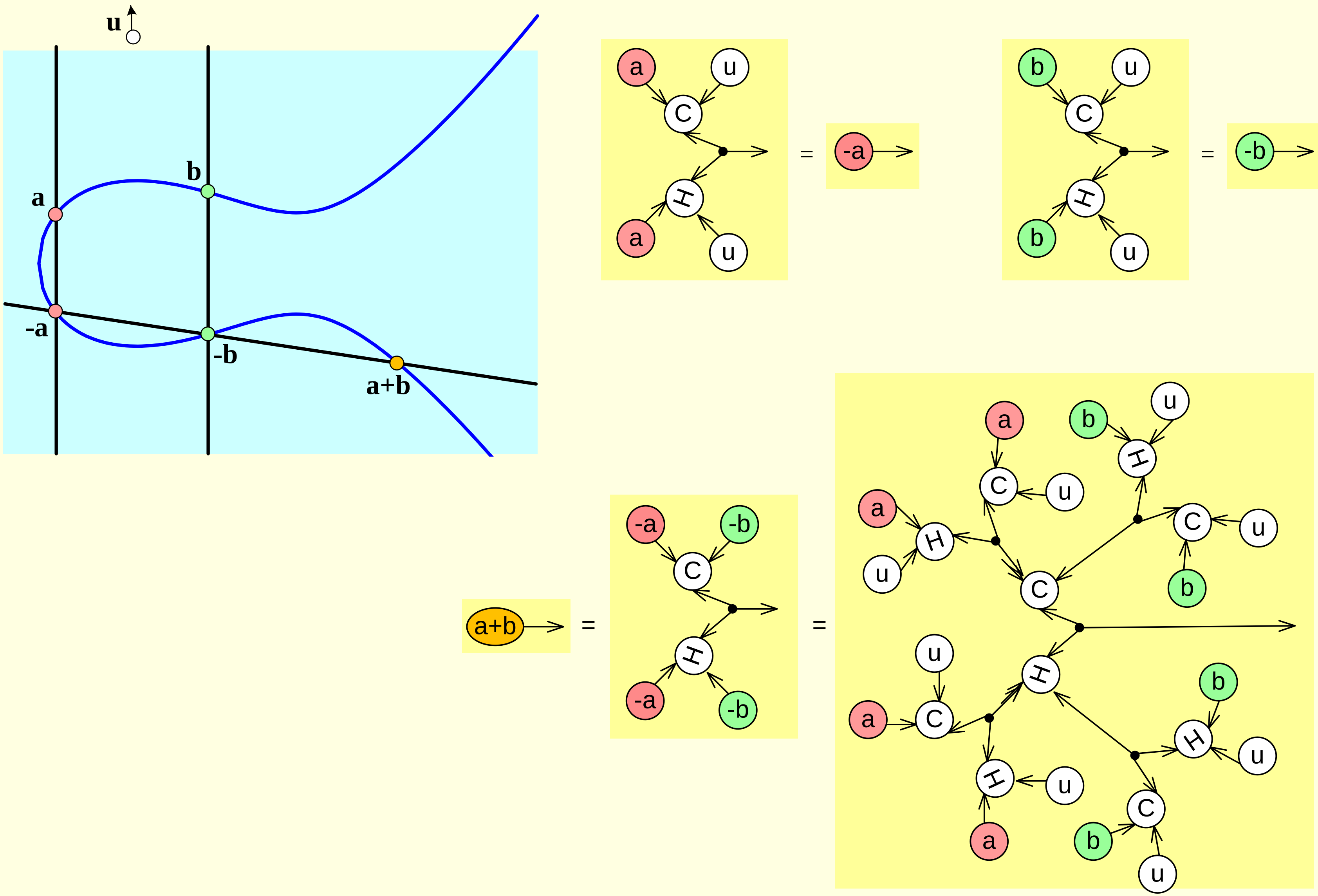




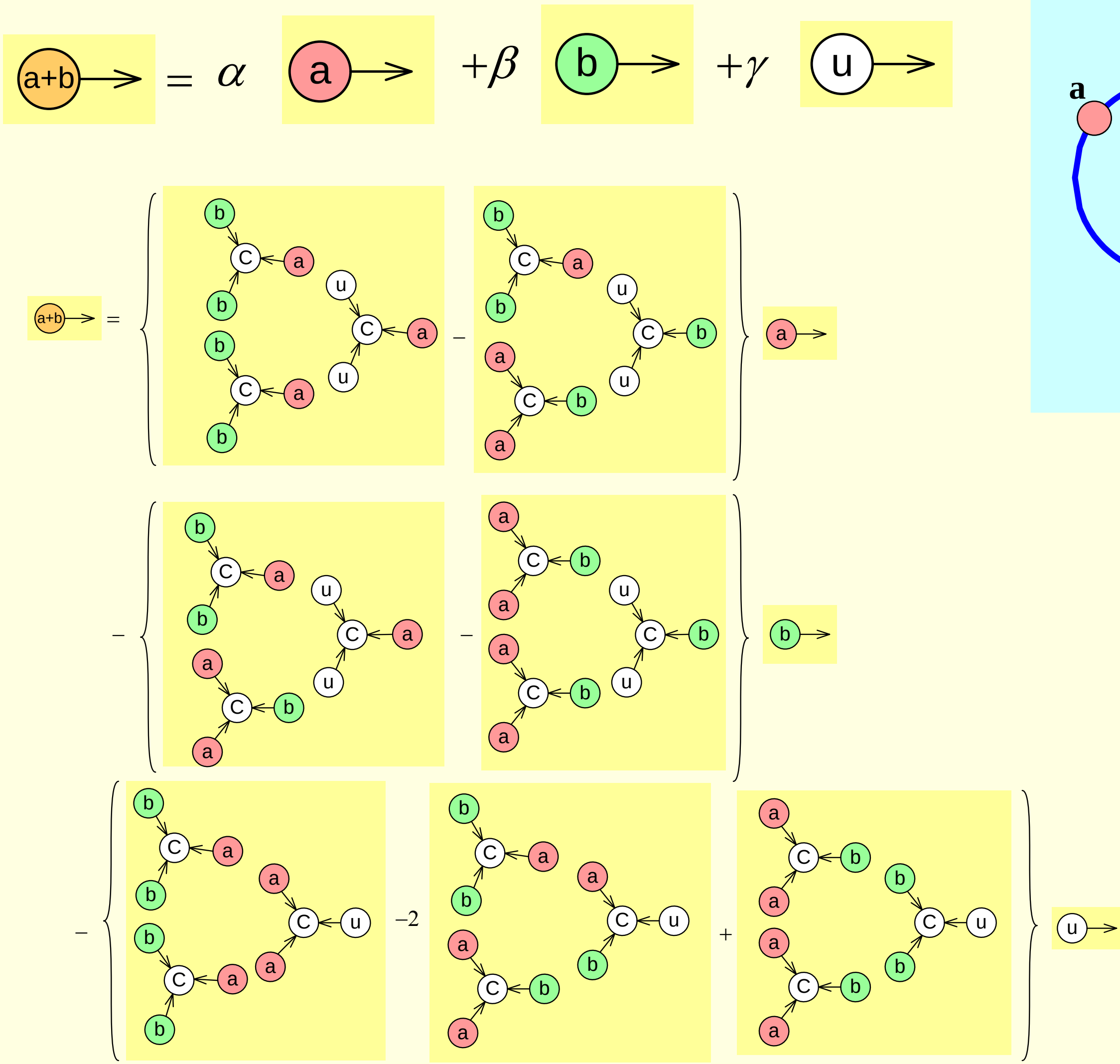
Possible condition

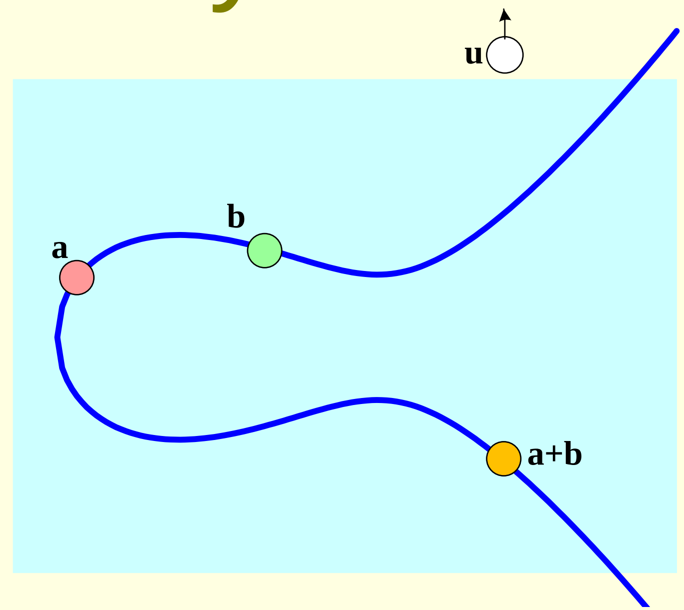


Another way



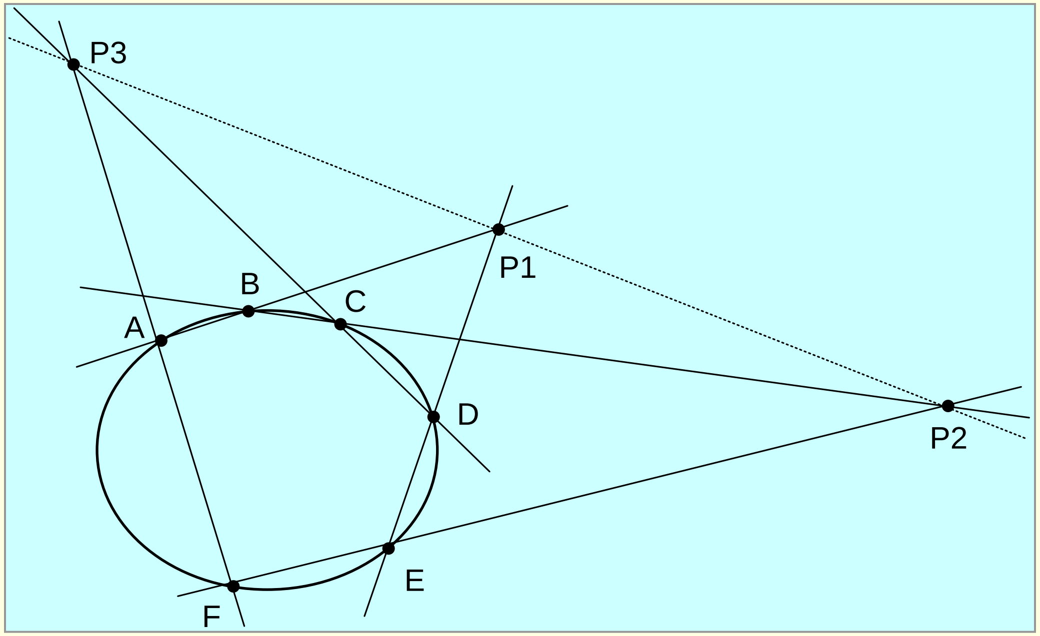
Another Another way

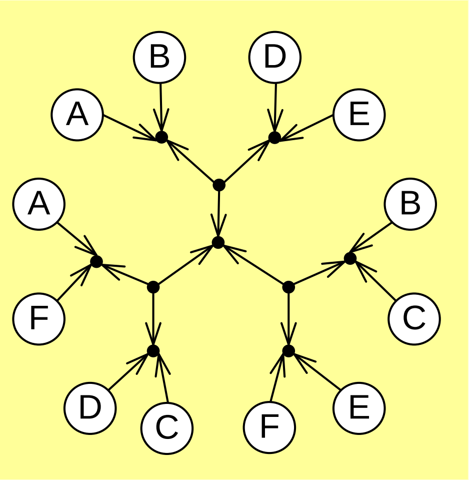




Possible Future Topics

Theorem of Pascal





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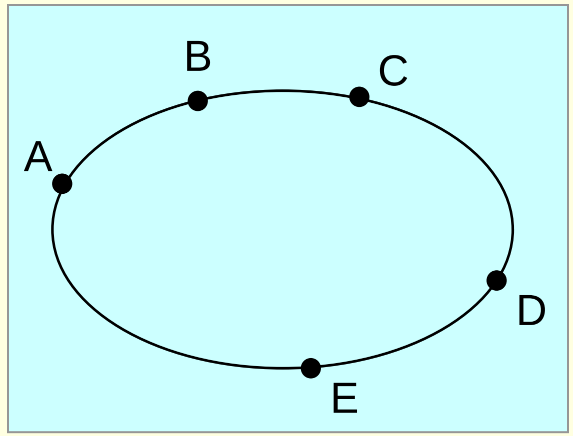
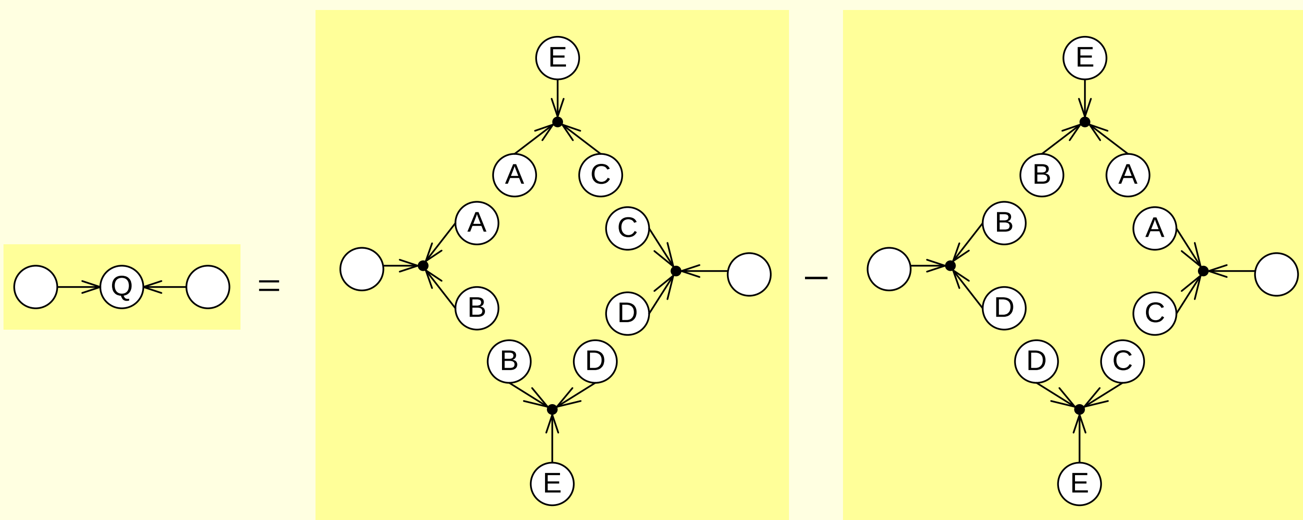
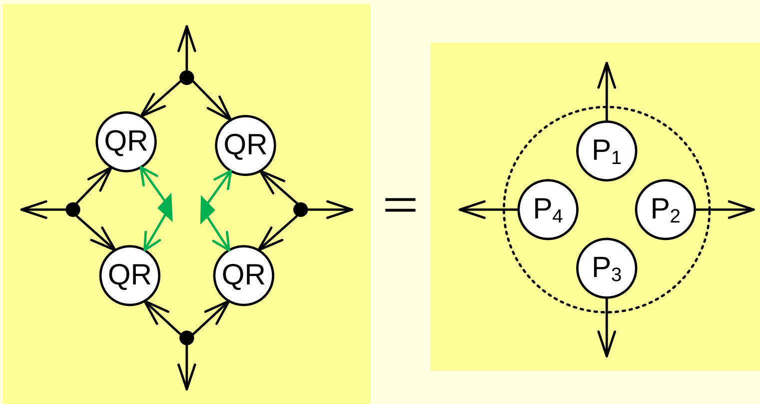
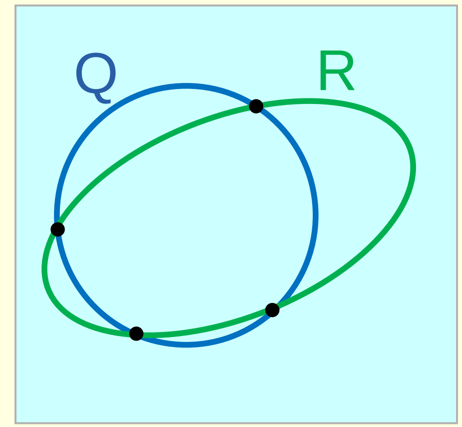
5 Points Determine a Quadratic

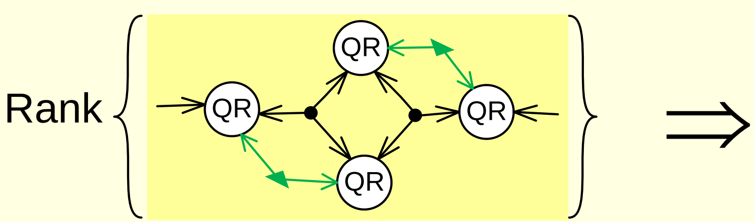


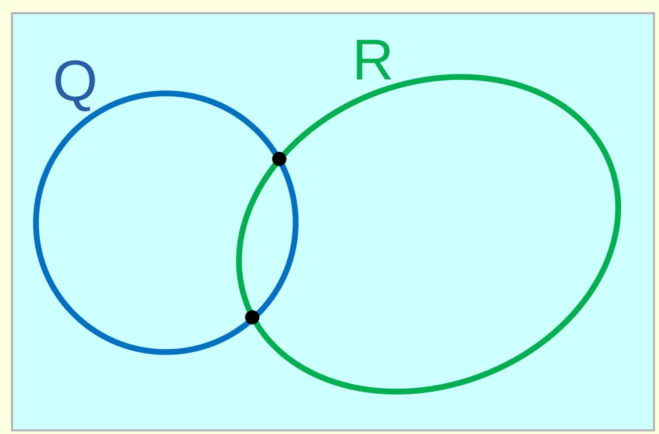


Intersecting Two Quadratic Curves



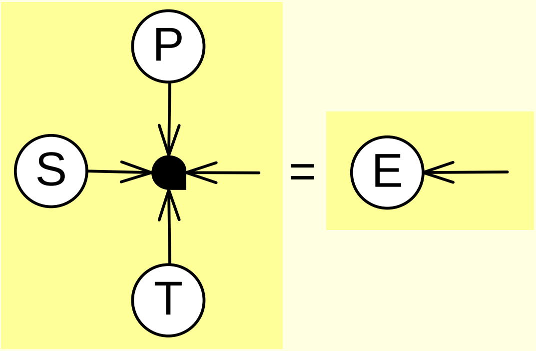




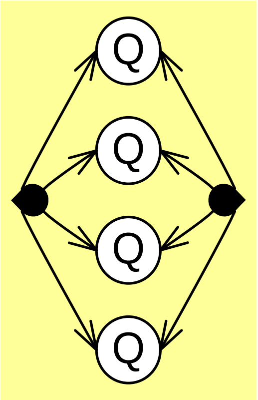


Three Dimensional Projective Geometry

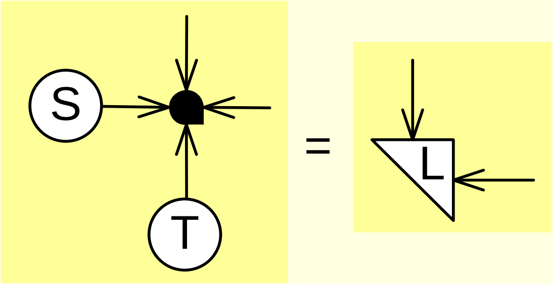
3 Points = A Plane



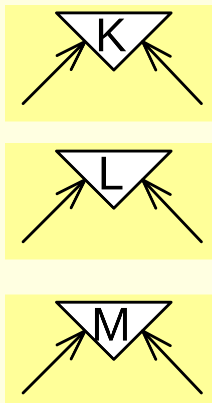
Discriminant of Quadric

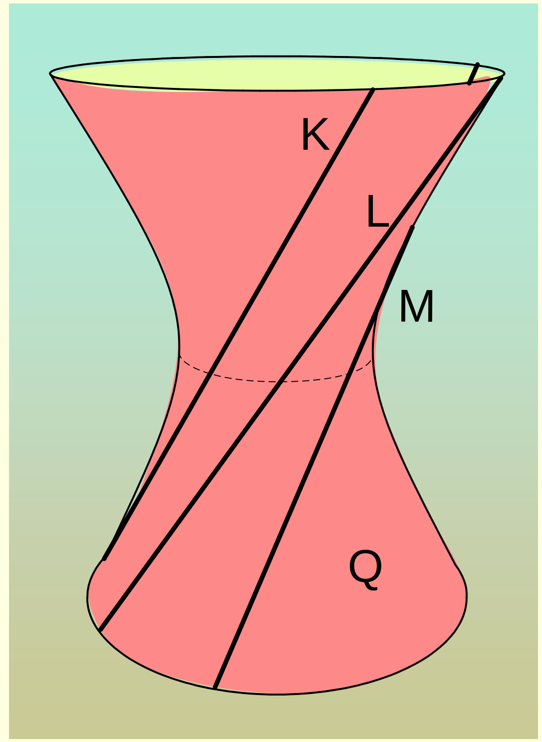


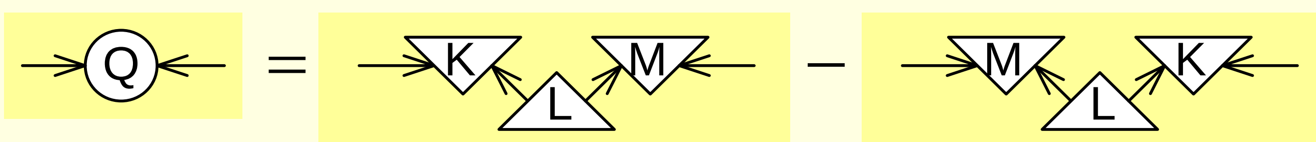
2 Points = A Line



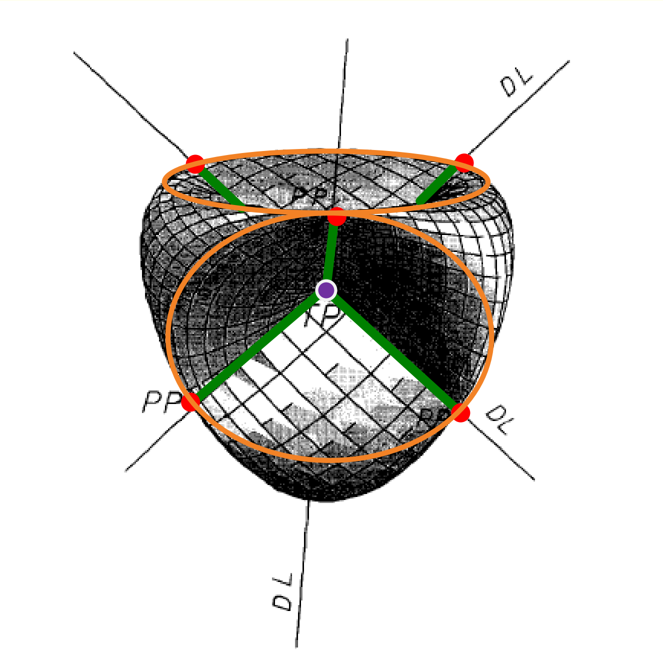
Three Skew Lines



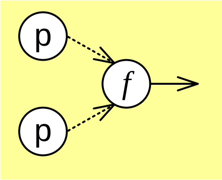




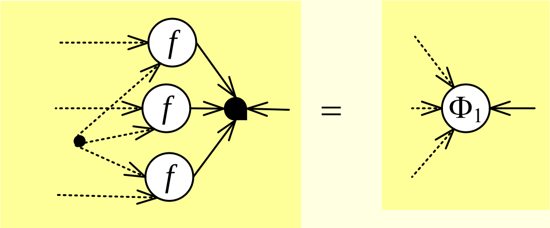
Steiner Surfaces



Parametric



Tangent



Implicit