Quantum Computing and Information - Problem Set 1 **Due Wed, Jan 19, 2011**

Exercise 1. **Polarization rotation** A polarizer (or polarizing filter) is understood classically to permit one polarization of light (say, horizontal polarization) to pass through, while blocking orthogonally polarized light. For single photons, a polarizer performs a measurement, and either absorbs or transmits the photon depending on the outcome. More concretely, define

$$|P_\theta\rangle = \cos(\theta)\,|0\rangle + \sin(\theta)\,|1\rangle\,.$$

One can check that $\{|P_\theta\rangle, |P_{\theta+\pi/2}\rangle\}$ forms an orthonormal basis for $\mathbb{C}^2$. A linear polarizer at angle θ acts on a photon by measuring in the $\{|P_\theta\rangle, |P_{\theta+\pi/2}\rangle\}$ basis and either transmitting the photon (upon outcome $|P_\theta\rangle$) or absorbing it (upon outcome $|P_{\theta+\pi/2}\rangle$).

a) Suppose a photon is prepared in state $|P_{\theta_1}\rangle$ and is sent through a polarizer at angle θ_2. What is the probability that it is transmitted (i.e. not absorbed)?

b) Now suppose we insert a polarizer at angle θ_3 between the photon source and the polarizer at angle θ_2. Thus, the photon will first encounter the polarizer at angle θ_3 and then, if it is not absorbed, it will attempt to pass through the polarizer at angle θ_2. What is the probabilitity that it is successfully transmitted by both polarizers? Are there any choices of $\theta_1, \theta_2, \theta_3$ such that this is ever larger than the probability in part (a)?

c) Consider a photon initially in state $|0\rangle$ that passes through N polarizers. The j^{th} polarizer will be at angle $\frac{\pi}{2}\frac{j}{N}$. Show that the probability of being transmitted through all the polarizers is $\geq 1 - c/N$ for some constant c.

Exercise 2. **Qubit states and operators**
The purpose of this exercise is to connect single-qubit states and unitaries to physical rotations of spin-1/2 particles.
The Pauli operators $\sigma_0, \sigma_1, \sigma_2, \sigma_3$ on $\mathbb{C}^2$ are defined by

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

a) Up to an overall phase, any state $|\psi\rangle \in \mathbb{C}^2$ can be written as

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right) e^{-i\frac{\phi}{2}}\,|0\rangle + \sin\left(\frac{\theta}{2}\right) e^{i\frac{\phi}{2}}\,|1\rangle\,. \tag{1}$$

Calculate $\langle\psi|\,\sigma_x\,|\psi\rangle$, $\langle\psi|\,\sigma_y\,|\psi\rangle$ and $\langle\psi|\,\sigma_z\,|\psi\rangle$.

b) Show that $\sigma_i^2 = I$ for $i = 0, 1, 2, 3$.

c) For $j, k, l \in \{1, 2, 3\}$, define the antisymmetric tensor ϵ_{jkl} by $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$, $\epsilon_{213} = \epsilon_{321} = \epsilon_{132} = -1$, and $\epsilon_{jkl} = 0$ whenever two of j, k, l are equal (i.e. all other cases). Prove that for $j, k \in \{1, 2, 3\}$

$$\sigma_j\sigma_k = \delta_{jk}\sigma_0 + i\sum_{l=1}^{3} \epsilon_{jkl}\sigma_l. \tag{2}$$

For example $\sigma_1\sigma_2 = i\sigma_3$, $\sigma_2\sigma_3 = i\sigma_2$, $\sigma_2\sigma_1 = -i\sigma_3, \ldots$. *Hint:* Use (b) to reduce the number of calculations. The antisymmetric tensor also appears in cross products: if $\vec{v}, \vec{w} \in \mathbb{R}^3$ then $(\vec{v} \times \vec{w})_i = \sum_{j,k} \epsilon_{ijk} v_j w_k$.

d) For a vector $\vec{v} \in \mathbb{R}^3$ define $\vec{v} \cdot \vec{\sigma} := v_1\sigma_1 + v_2\sigma_2 + v_3\sigma_3$. For operators A, B, define $[A, B] := AB - BA$. Show that $(\vec{v} \cdot \vec{\sigma})^2 = \|\vec{v}\|^2\sigma_0$ and that $[\vec{v} \cdot \vec{\sigma}, \vec{w} \cdot \vec{\sigma}] = 2i(\vec{v} \times \vec{w}) \cdot \vec{\sigma}$.

e) Let $\vec{v}$ be a unit vector and α a real number. Prove that

$$e^{i\alpha\vec{v}\cdot\vec{\sigma}} = \cos(\alpha)\sigma_0 + i\sin(\alpha)\vec{v} \cdot \vec{\sigma}.$$

f) Again let $\vec{v}$ be a unit vector and α a real number. Prove that

$$e^{i\frac{\alpha}{2}\vec{v}\cdot\vec{\sigma}}(\vec{w}\cdot\vec{\sigma})e^{-i\frac{\alpha}{2}\vec{v}\cdot\vec{\sigma}} = (\cos(\alpha)\vec{w} + \sin(\alpha)\vec{w}\times\vec{v} + (1-\cos(\alpha))(\vec{w}\cdot\vec{v})\vec{v})\cdot\vec{\sigma}. \tag{3}$$

This is the formula for rotating the vector $\vec{w}$ an angle α about the axis $\vec{v}$.

g) Let $w_1 = \sin(\theta)\cos(\phi)$, $w_2 = \sin(\theta)\sin(\phi)$, $w_3 = \cos(\theta)$ and define $|\psi\rangle$ as in Eq. (1). Show that $\vec{w}\cdot\vec{\sigma} = 2|\psi\rangle\langle\psi| - I$. Use this fact and Eq. (3) to interpret

$$e^{i\frac{\alpha}{2}\vec{v}\cdot\vec{\sigma}}\,|\psi\rangle$$

as a 3-dimensional rotation.

Exercise 3. **Entanglement**

a) Prove that the state $\frac{|0,0\rangle+|1,1\rangle}{\sqrt{2}}$ is not equal to $|\alpha\rangle\otimes|\beta\rangle$ for any $|\alpha\rangle, |\beta\rangle \in \mathbb{C}^2$. Here, $|0,0\rangle$ is shorthand for $|0\rangle\otimes|0\rangle$ and similarly for $|1,1\rangle$.

b) Let $\mathcal{U}(d)$ denote the set of $d\times d$ unitary matrices. The singular value decomposition states that for any $d_1\times d_2$ matrix A there exists a $X\in\mathcal{U}(d_1)$, $Y\in\mathcal{U}(d_2)$ and a $d_1\times d_2$ diagonal matrix Λ such that $A = X\Lambda Y$. The entries of Λ are real, nonnegative, and unique up to reordering.Use this to prove that for any $|\psi\rangle\in\mathbb{C}^{d_1d_2}$ there exists $U\in\mathcal{U}(d_1), V\in\mathcal{U}(d_2)$ and nonnegative real numbers $\lambda_1,\ldots,\lambda_d$ (with $d=\min(d_1,d_2)$) such that

$$(U\otimes V)\,|\psi\rangle = \sum_{i=1}^{d}\lambda_i\,|i\rangle\otimes|i\rangle \tag{4}$$

c) Show that for any $|\psi\rangle\in\mathbb{C}^{d_1d_2}$ there exist nonnegative real numbers $\lambda_1,\ldots,\lambda_d$ and orthonormal sets $|\alpha_1\rangle,\ldots,|\alpha_d\rangle\in\mathbb{C}^{d_1}$ and $|\beta_1\rangle,\ldots,|\beta_d\rangle\in\mathbb{C}^{d_2}$ such that

$$|\psi\rangle = \sum_{i=1}^{d}\lambda_i\,|\alpha_i\rangle\otimes|\beta_i\rangle \tag{5}$$