Quantum Computing and Information - Problem Set 1 Solutions **Due Wed, Jan 19, 2011**

Exercise 1. **Polarization rotation** *A polarizer (or polarizing filter) is understood classically to permit one polarization of light (say, horizontal polarization) to pass through, while blocking orthogonally polarized light. For single photons, a polarizer performs a measurement, and either absorbs or transmits the photon depending on the outcome. More concretely, define*

$$|P_\theta\rangle = \cos(\theta)|0\rangle + \sin(\theta)|1\rangle .$$

One can check that $\{|P_\theta\rangle, |P_{\theta+\pi/2}\rangle\}$ *forms an orthonormal basis for* $\mathbb{C}^2$. *A linear polarizer at angle* θ *acts on a photon by measuring in the* $\{|P_\theta\rangle, |P_{\theta+\pi/2}\rangle\}$ *basis and either transmitting the photon (upon outcome* $|P_\theta\rangle$*) or absorbing it (upon outcome* $|P_{\theta+\pi/2}\rangle$*).*

a) *Suppose a photon is prepared in state* $|P_{\theta_1}\rangle$ *and is sent through a polarizer at angle* θ_2. *What is the probability that it is transmitted (i.e. not absorbed)?*

b) *Now suppose we insert a polarizer at angle* θ_3 *between the photon source and the polarizer at angle* θ_2. *Thus, the photon will first encounter the polarizer at angle* θ_3 *and then, if it is not absorbed, it will attempt to pass through the polarizer at angle* θ_2. *What is the probabilitity that it is successfully transmitted by both polarizers? Are there any choices of* $\theta_1, \theta_2, \theta_3$ *such that this is ever larger than the probability in part (a)?*

c) *Consider a photon initially in state* $|0\rangle$ *that passes through* N *polarizers. The* j^{th} *polarizer will be at angle* $\frac{\pi}{2}\frac{j}{N}$. *Show that the probability of being transmitted through all the polarizers is* $\geq 1 - c/N$ *for some constant* c.

a) The probability of being transmitted is $|\langle P_{\theta_1}|P_{\theta_2}\rangle|^2 = (\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2))^2 = \cos^2(\theta_2 - \theta_1)$.

b) The photon is transmitted through the first polarizer with probability $\cos^2(\theta_3 - \theta_1)$. Assuming it is not absorbed, it emerges with the state $|P_{\theta_3}\rangle$. In this case, it is transmitted through the second polarizer with probability $\cos^2(\theta_3 - \theta_2)$, so the total probability is

$$\cos^2(\theta_1 - \theta_3)\cos^2(\theta_3 - \theta_2).$$

If $\theta_1 = 0, \theta_2 = \pi/2, \theta_3 = \pi/4$, then the probability in part (a) is 0 and the probability in part (b) is 1/4.

c) This probability is $\cos^{2N}\left(\frac{\pi}{2N}\right)$. Since $\cos(x) \geq 1 - \frac{x^2}{2}$, we have

$$\cos^{2N}\left(\frac{\pi}{2N}\right) \geq \left(1 - \frac{\pi^2}{8N^2}\right)^{2N} \geq 1 - \frac{\pi^2}{4N}.$$

The second inequality here is known as the union bound and can be stated as $\prod_{i=1}^N (1-x_i) \geq 1 - \sum_{i=1}^N x_i$. It holds whenever $0 \leq x_i \leq 1$ and can be proved by induction on N.

Exercise 2. **Qubit states and operators**
The purpose of this exercise is to connect single-qubit states and unitaries to physical rotations of spin-1/2 particles.
The Pauli operators $\sigma_0, \sigma_1, \sigma_2, \sigma_3$ *on* $\mathbb{C}^2$ *are defined by*

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

a) *Up to an overall phase, any state* $|\psi\rangle \in \mathbb{C}^2$ *can be written as*

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right) e^{-i\frac{\phi}{2}} |0\rangle + \sin\left(\frac{\theta}{2}\right) e^{i\frac{\phi}{2}} |1\rangle . \tag{1}$$

Calculate $\langle\psi| \sigma_x |\psi\rangle$, $\langle\psi| \sigma_y |\psi\rangle$ *and* $\langle\psi| \sigma_z |\psi\rangle$.

b) *Show that* $\sigma_i^2 = I$ *for* $i = 0, 1, 2, 3$.

c) *For* $j, k, l \in \{1, 2, 3\}$, *define the antisymmetric tensor* ϵ_{jkl} *by* $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$, $\epsilon_{213} = \epsilon_{321} = \epsilon_{132} = -1$, *and* $\epsilon_{jkl} = 0$ *whenever two of* j, k, l *are equal (i.e. all other cases). Prove that for* $j, k \in \{1, 2, 3\}$

$$\sigma_j \sigma_k = \delta_{jk}\sigma_0 + i \sum_{l=1}^{3} \epsilon_{jkl}\sigma_l. \tag{2}$$

For example $\sigma_1\sigma_2 = i\sigma_3$, $\sigma_2\sigma_3 = i\sigma_2$, $\sigma_2\sigma_1 = -i\sigma_3, \ldots$. Hint: *Use (b) to reduce the number of calculations. The antisymmetric tensor also appears in cross products: if* $\vec{v}, \vec{w} \in \mathbb{R}^3$ *then* $(\vec{v} \times \vec{w})_i = \sum_{j,k} \epsilon_{ijk} v_j w_k$.

d) *For a vector* $\vec{v} \in \mathbb{R}^3$ *define* $\vec{v} \cdot \vec{\sigma} := v_1\sigma_1 + v_2\sigma_2 + v_3\sigma_3$. *For operators* A, B, *define* $[A, B] := AB - BA$. *Show that*

$$(\vec{v} \cdot \vec{\sigma})^2 = \|\vec{v}\|^2 \sigma_0 \tag{3}$$

$$[\vec{v} \cdot \vec{\sigma}, \vec{w} \cdot \vec{\sigma}] = 2i(\vec{v} \times \vec{w}) \cdot \vec{\sigma}. \tag{4}$$

e) *Let* $\vec{v}$ *be a unit vector and* α *a real number. Prove that*

$$e^{i\alpha\vec{v}\cdot\vec{\sigma}} = \cos(\alpha)\sigma_0 + i\sin(\alpha)\vec{v} \cdot \vec{\sigma}. \tag{5}$$

f) *Again let* $\vec{v}$ *be a unit vector and* α *a real number. Prove that*

$$e^{i\frac{\alpha}{2}\vec{v}\cdot\vec{\sigma}}(\vec{w} \cdot \vec{\sigma})e^{-i\frac{\alpha}{2}\vec{v}\cdot\vec{\sigma}} = (\cos(\alpha)\vec{w} + \sin(\alpha)\vec{w} \times \vec{v} + (1 - \cos(\alpha))(\vec{w} \cdot \vec{v})\vec{v}) \cdot \vec{\sigma}. \tag{6}$$

This is the formula for rotating the vector $\vec{w}$ *an angle* α *about the axis* $\vec{v}$.

g) *Let* $w_1 = \sin(\theta)\cos(\phi)$, $w_2 = \sin(\theta)\sin(\phi)$, $w_3 = \cos(\theta)$ *and define* $|\psi\rangle$ *as in Eq. (1). Show that* $\vec{w} \cdot \vec{\sigma} = 2|\psi\rangle\langle\psi| - I$. *Use this fact and Eq. (6) to interpret*

$$e^{i\frac{\alpha}{2}\vec{v}\cdot\vec{\sigma}} |\psi\rangle$$

as a 3-dimensional rotation.

a) First note that $\cos\left(\frac{\theta}{2}\right) e^{-i\frac{\phi}{2}} \langle 0| + \sin\left(\frac{\theta}{2}\right) e^{i\frac{\phi}{2}} \langle 1|$. Then

$$\begin{aligned}
\langle\psi| \sigma_x |\psi\rangle &= \left(\cos\left(\frac{\theta}{2}\right) e^{-i\frac{\phi}{2}} \langle 0| + \sin\left(\frac{\theta}{2}\right) e^{i\frac{\phi}{2}} \langle 1|\right) \sigma_x \left(\cos\left(\frac{\theta}{2}\right) e^{i\frac{\phi}{2}} |0\rangle + \sin\left(\frac{\theta}{2}\right) e^{-i\frac{\phi}{2}} |1\rangle\right) \\
&= \left(\cos\left(\frac{\theta}{2}\right) e^{-i\frac{\phi}{2}} \langle 0| + \sin\left(\frac{\theta}{2}\right) e^{i\frac{\phi}{2}} \langle 1|\right) \left(\cos\left(\frac{\theta}{2}\right) e^{i\frac{\phi}{2}} |1\rangle + \sin\left(\frac{\theta}{2}\right) e^{-i\frac{\phi}{2}} |0\rangle\right) \\
&= \cos(\theta/2)\sin(\theta/2)e^{-i\phi} + \sin(\theta/2)\cos(\theta/2)e^{i\phi} \\
&= \cos(\theta/2)\sin(\theta/2)2\cos(\phi) \\
&= \sin(\theta)\cos(\phi).
\end{aligned}$$

Next

$$\begin{aligned}
\langle\psi|\,\sigma_y\,|\psi\rangle &= \left(\cos\left(\frac{\theta}{2}\right)e^{-i\frac{\phi}{2}}\langle 0| + \sin\left(\frac{\theta}{2}\right)e^{i\frac{\phi}{2}}\langle 1|\right)\left(i\cos\left(\frac{\theta}{2}\right)e^{i\frac{\phi}{2}}|1\rangle - i\sin\left(\frac{\theta}{2}\right)e^{-i\frac{\phi}{2}}|0\rangle\right)\\
&= \sin(\theta/2)\cos(\theta/2)(ie^{i\phi} - ie^{-i\phi})\\
&= \sin(\theta/2)\cos(\theta/2)(-2\sin(\phi))\\
&= -\sin(\theta)\sin(\phi)
\end{aligned}$$

Finally $\langle\psi|\,\sigma_z\,|\psi\rangle = \cos^2(\theta/2) - \sin^2(\theta/2) = \cos(\theta)$.

Together, these correspond to the (x, y, z) coordinates of a point on the sphere with latitude θ and longtude $-\phi$.

b) This involves three direct calculations:

$$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\cdot\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \qquad \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\cdot\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \qquad \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\cdot\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

Alternatively, you can use the fact that $\sigma_1\sigma_2\sigma_3 = i\sigma_0$, that $\sigma_i^2 = 1$ and that each $\sigma_i = \sigma_i^\dagger$ to construct all of the desired relations.

c) Given the result from (b), we still need to show

$$\sigma_1\sigma_2 = i\sigma_3 \tag{7a}$$
$$\sigma_2\sigma_3 = i\sigma_1 \tag{7b}$$
$$\sigma_3\sigma_1 = i\sigma_2 \tag{7c}$$
$$\sigma_2\sigma_1 = -i\sigma_3 \tag{7d}$$
$$\sigma_3\sigma_2 = -i\sigma_1 \tag{7e}$$
$$\sigma_1\sigma_3 = -i\sigma_2 \tag{7f}$$

One possibility is to do six matrix multiplications. Here is a slightly easier method. Start by proving Eq. (7a) with a direct calculation

$$\sigma_1\sigma_2 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\cdot\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = \begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix} = i\sigma_3.$$

Now we can leverage this using (b). Multiply both sides by σ_1 on the left and obtain $\sigma_1^2\sigma_2 = i\sigma_1\sigma_3$. Replacing $\sigma_1^2\sigma_2 = \sigma_2$ and multiplying both sides by $-i$ proves Eq. (7f). Now right-multiply Eq. (7f) by σ_3 to obtain $\sigma_1 = \sigma_1\sigma_3^2 = -i\sigma_2\sigma_3$. Multiplying by i proves Eq. (7b). Next, left-multiply Eq. (7b) by σ_2 to find $\sigma_3 = i\sigma_2\sigma_1$, implying Eq. (7d). Right-multiply Eq. (7d) by σ_1 to find $\sigma_3\sigma_1 = i\sigma_2$, proving Eq. (7c). Left-multiply Eq. (7c) by σ_3 to obtain $\sigma_1 = i\sigma_3\sigma_2$, implying Eq. (7e). This completes the proof.

d)

$$(\vec{v}\cdot\vec{\sigma})^2 = \sum_{i=1}^{3} v_i\sigma_i \sum_{j=1}^{3} v_j\sigma_j = \sum_{i=1}^{3}\sum_{j=1}^{3} v_iv_j\sigma_i\sigma_j = \sum_{i=1}^{3}\sum_{j=1}^{3} v_iv_j(\delta_{i,j}\sigma_0 + \sum_{k=1}^{3} i\epsilon_{ijk}\sigma_k) = \sum_{i=1}^{3} v_i^2\sigma_0 + i\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\epsilon_{ijk}\sigma_k.$$

We claim this second term is zero. This is because

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\epsilon_{ijk}\sigma_k = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} -\epsilon_{jik}\sigma_k. = -\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\epsilon_{ijk}\sigma_k.$$

Thus $(\vec{v}\cdot\vec{\sigma})^2 = \sum_{i=1}^{3} v_i^2\sigma_0 = \|\vec{v}\|^2\sigma_0$.

Next

$$\begin{aligned}[\vec{v}\cdot\vec{\sigma},\vec{w}\cdot\vec{\sigma}] &= \sum_{i=1}^{3}\sum_{j=1}^{3} v_i w_j(\sigma_i\sigma_j - \sigma_j\sigma_i)\\ &= \sum_{i=1}^{3}\sum_{j=1}^{3} v_i w_j\left((\delta_{ij}-\delta_{ji})\sigma_0 + i\sum_{k=1}^{3}(\epsilon_{ijk}-\epsilon_{jik})\sigma_k\right)\\ &= 2i\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} v_i w_j \epsilon_{ijk}\sigma_k\\ &= 2i\sum_{k=1}^{3}(\vec{v}\times\vec{w})_k\sigma_k\\ &= 2i(\vec{v}\times\vec{w})\cdot\vec{\sigma}\end{aligned}$$

e) Since $(\vec{v}\cdot\vec{\sigma})^2 = \|\vec{v}\|^2\sigma_0 = \sigma_0$, then for any nonnegative integer k, we have $(\vec{v}\cdot\vec{\sigma})^{2k} = \sigma_0$ and $(\vec{v}\cdot\vec{\sigma})^{2k+1} = \vec{v}\cdot\vec{\sigma}$. Thus

$$e^{i\alpha\vec{v}\cdot\vec{\sigma}} = \sum_{n\geq 0}\frac{(i\alpha\vec{v}\cdot\vec{\sigma})^n}{n!} = \sum_{k\geq 0}\frac{(-1)^k\alpha^{2k}}{(2k)!}\sigma_0 + i\sum_{k\geq 0}\frac{(-1)^k\alpha^{2k+1}}{(2k+1)!}\vec{v}\cdot\vec{\sigma} = \cos(\alpha)\sigma_0 + i\sin(\alpha)\vec{v}\cdot\vec{\sigma},$$

where we have used the Taylor expansions of $e^x = \sum_{n\geq 0} x^n/n!$, $\cos(x) = \sum_{n\geq 0}(-1)^n x^{2n}/(2n)!$ and $\sin(x) = \sum_{n\geq 0}(-1)^n x^{2n+1}/(2n+1)!$.

f) First expand out the exponentials using Eq. (5).

$$e^{i\frac{\alpha}{2}\vec{v}\cdot\vec{\sigma}}(\vec{w}\cdot\vec{\sigma})e^{-i\frac{\alpha}{2}\vec{v}\cdot\vec{\sigma}} = \left(\cos\left(\frac{\alpha}{2}\right)\sigma_0 + i\sin\left(\frac{\alpha}{2}\right)\vec{v}\cdot\vec{\sigma}\right)(\vec{w}\cdot\vec{\sigma})\left(\cos\left(\frac{\alpha}{2}\right)\sigma_0 - i\sin\left(\frac{\alpha}{2}\right)\vec{v}\cdot\vec{\sigma}\right) \tag{8}$$

$$= \cos^2\left(\frac{\alpha}{2}\right)\vec{w}\cdot\vec{\sigma} \tag{9}$$

$$+ i\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)((\vec{v}\cdot\vec{\sigma})(\vec{w}\cdot\vec{\sigma}) - (\vec{w}\cdot\vec{\sigma})(\vec{v}\cdot\vec{\sigma})) \tag{10}$$

$$- \sin^2\left(\frac{\alpha}{2}\right)(\vec{v}\cdot\vec{\sigma})(\vec{w}\cdot\vec{\sigma})(\vec{v}\cdot\vec{\sigma}). \tag{11}$$

To analyse Eq. (10), we note that the term in brackets is $[(\vec{v}\cdot\vec{\sigma}),(\vec{w}\cdot\vec{\sigma})]$ and use Eq. (4). For Eq. (11), we use the fact that

$$(\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma}) = [\vec{a}\cdot\vec{\sigma},\vec{b}\cdot\vec{\sigma}] + (\vec{b}\cdot\vec{\sigma})(\vec{a}\cdot\vec{\sigma}) \tag{12}$$

to simplify

$$\begin{aligned}(\vec{v}\cdot\vec{\sigma})(\vec{w}\cdot\vec{\sigma})(\vec{v}\cdot\vec{\sigma}) &= ([(\vec{v}\cdot\vec{\sigma}),(\vec{w}\cdot\vec{\sigma})] + (\vec{w}\cdot\vec{\sigma})(\vec{v}\cdot\vec{\sigma}))(\vec{v}\cdot\vec{\sigma}) && \text{using Eq. (12)}\\ &= [(\vec{v}\cdot\vec{\sigma}),(\vec{w}\cdot\vec{\sigma})](\vec{v}\cdot\vec{\sigma}) + (\vec{w}\cdot\vec{\sigma})(\vec{v}\cdot\vec{\sigma})(\vec{v}\cdot\vec{\sigma})\\ &= [(\vec{v}\cdot\vec{\sigma}),(\vec{w}\cdot\vec{\sigma})](\vec{v}\cdot\vec{\sigma}) + (\vec{w}\cdot\vec{\sigma}) && \text{using Eq. (3)}\\ &= [[(\vec{v}\cdot\vec{\sigma}),(\vec{w}\cdot\vec{\sigma})],(\vec{v}\cdot\vec{\sigma})] + (\vec{v}\cdot\vec{\sigma})[(\vec{v}\cdot\vec{\sigma}),(\vec{w}\cdot\vec{\sigma})] + (\vec{w}\cdot\vec{\sigma}) && \text{using Eq. (12)}\\ &= [[(\vec{v}\cdot\vec{\sigma}),(\vec{w}\cdot\vec{\sigma})],(\vec{v}\cdot\vec{\sigma})] + (\vec{v}\cdot\vec{\sigma})(\vec{v}\cdot\vec{\sigma})(\vec{w}\cdot\vec{\sigma}) - (\vec{v}\cdot\vec{\sigma})(\vec{w}\cdot\vec{\sigma})(\vec{v}\cdot\vec{\sigma}) + (\vec{w}\cdot\vec{\sigma})\\ &= [[(\vec{v}\cdot\vec{\sigma}),(\vec{w}\cdot\vec{\sigma})],(\vec{v}\cdot\vec{\sigma})] - (\vec{v}\cdot\vec{\sigma})(\vec{w}\cdot\vec{\sigma})(\vec{v}\cdot\vec{\sigma}) + 2(\vec{w}\cdot\vec{\sigma}) && \text{using Eq. (3)}\end{aligned}$$

Rearranging, we find that

$$\begin{aligned}(\vec{v}\cdot\vec{\sigma})(\vec{w}\cdot\vec{\sigma})(\vec{v}\cdot\vec{\sigma}) &= \frac{1}{2}[[(\vec{v}\cdot\vec{\sigma}),(\vec{w}\cdot\vec{\sigma})],(\vec{v}\cdot\vec{\sigma})] + \vec{w}\cdot\vec{\sigma}\\ &= i[(\vec{v}\times\vec{w})\cdot\vec{\sigma},\vec{v}\cdot\vec{\sigma}] + \vec{w}\cdot\vec{\sigma}\\ &= -2((\vec{v}\times\vec{w})\times\vec{v})\cdot\vec{\sigma} + \vec{w}\cdot\vec{\sigma}\\ &= -2(\vec{w} - (\vec{w}\cdot\vec{v})\vec{v})\cdot\vec{\sigma} + \vec{w}\cdot\vec{\sigma}\\ &= 2(\vec{w}\cdot\vec{v})\vec{v}\cdot\vec{\sigma} - \vec{w}\cdot\vec{\sigma}\end{aligned}$$

Putting this together, we find that

$$e^{i\frac{\alpha}{2}\vec{v}\cdot\vec{\sigma}}(\vec{w}\cdot\vec{\sigma})e^{-i\frac{\alpha}{2}\vec{v}\cdot\vec{\sigma}} = (\cos(\alpha)\vec{w} + \sin(\alpha)\vec{w}\times\vec{v} + (1-\cos(\alpha))(\vec{w}\cdot\vec{v})\vec{v})\cdot\vec{\sigma}.$$

This is the formula for rotating the vector $\vec{w}$ an angle α about the axis $\vec{v}$.

g)

$$\begin{aligned}
|\psi\rangle\langle\psi| &= \cos^2\left(\frac{\theta}{2}\right)|0\rangle\langle 0| + \sin^2\left(\frac{\theta}{2}\right)|1\rangle\langle 1| + \cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\left(e^{-i\phi}|1\rangle\langle 0| + e^{-i\phi}|0\rangle\langle 1|\right)\\
&= \frac{I}{2} + \frac{\cos(\theta)}{2}\sigma_3 + \cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\left(e^{-i\phi}|1\rangle\langle 0| + e^{-i\phi}|0\rangle\langle 1|\right)\\
&= \frac{I}{2} + \frac{\cos(\theta)}{2}\sigma_3 + \cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\left(\cos(\phi)(|1\rangle\langle 0| + |0\rangle\langle 1|) + i\sin(\phi)(|1\rangle\langle 0| - |0\rangle\langle 1|)\right)\\
&= \frac{I}{2} + \frac{\cos(\theta)}{2}\sigma_3 + \frac{\sin(\theta)}{2}\left(\cos(\phi)\sigma_1 + \sin(\phi)\sigma_2\right)\\
&= \frac{I + \vec{w}\cdot\vec{\sigma}}{2}.
\end{aligned}$$

Rearranging yields $\vec{w}\cdot\vec{\sigma} = 2|\psi\rangle\langle\psi| - I$.

Next, let $|\varphi\rangle = e^{i\frac{\alpha}{2}\vec{v}\cdot\vec{\sigma}}|\psi\rangle$. Then

$$|\varphi\rangle\langle\varphi| = e^{i\frac{\alpha}{2}\vec{v}\cdot\vec{\sigma}}|\psi\rangle\langle\psi|e^{-i\frac{\alpha}{2}\vec{v}\cdot\vec{\sigma}} = e^{i\frac{\alpha}{2}\vec{v}\cdot\vec{\sigma}}\frac{\vec{w}\cdot\vec{\sigma}}{2}e^{-i\frac{\alpha}{2}\vec{v}\cdot\vec{\sigma}} + \frac{I}{2},$$

which, by the results of (f), is the state corresponding to rotating $\vec{w}$ by an angle α about the axis $\vec{v}$.

Exercise 3. **Entanglement**

a) *Prove that the state* $\frac{|0,0\rangle+|1,1\rangle}{\sqrt{2}}$ *is not equal to* $|\alpha\rangle\otimes|\beta\rangle$ *for any* $|\alpha\rangle, |\beta\rangle \in \mathbb{C}^2$. *Here,* $|0,0\rangle$ *is shorthand for* $|0\rangle\otimes|0\rangle$ *and similarly for* $|1,1\rangle$.

b) *Let* $\mathcal{U}(d)$ *denote the set of* $d\times d$ *unitary matrices. The singular value decomposition states that for any* $d_1\times d_2$ *matrix* A *there exists a* $X\in\mathcal{U}(d_1)$, $Y\in\mathcal{U}(d_2)$ *and a* $d_1\times d_2$ *diagonal matrix* Λ *such that* $A = X\Lambda Y$. *The entries of* Λ *are real, nonnegative, and unique up to reordering.*

   Use this to prove that for any $|\psi\rangle\in\mathbb{C}^{d_1d_2}$ *there exists* $U\in\mathcal{U}(d_1), V\in\mathcal{U}(d_2)$ *and nonnegative real numbers* $\lambda_1,\ldots,\lambda_d$ *(with* $d=\min(d_1,d_2)$*) such that*

$$(U\otimes V)|\psi\rangle = \sum_{i=1}^{d}\lambda_i|i\rangle\otimes|i\rangle \tag{13}$$

c) *Show that for any* $|\psi\rangle\in\mathbb{C}^{d_1d_2}$ *there exist nonnegative real numbers* $\lambda_1,\ldots,\lambda_d$ *and orthonormal sets* $|\alpha_1\rangle,\ldots,|\alpha_d\rangle\in\mathbb{C}^{d_1}$ *and* $|\beta_1\rangle,\ldots,|\beta_d\rangle\in\mathbb{C}^{d_2}$ *such that*

$$|\psi\rangle = \sum_{i=1}^{d}\lambda_i|\alpha_i\rangle\otimes|\beta_i\rangle \tag{14}$$

a) This follows from part (c), but here is a direct proof. Let $|\alpha\rangle = a_0|0\rangle + a_1|1\rangle$ and $|\beta\rangle = b_0|0\rangle + b_1|1\rangle$. Let $|\Phi\rangle := \frac{|00\rangle+|11\rangle}{\sqrt{2}}$. If $|\Phi\rangle = |\alpha\rangle\otimes|\beta\rangle$, then we must have $a_0b_0 = a_1b_1 = 1/\sqrt{2}$ and $a_0b_1 = a_1b_0 = 0$. Thus, either $a_0 = 0$ or $b_1 = 0$. This contradicts either the claim that $a_0b_0 \neq 0$ or the claim that $a_1b_1 \neq 0$.

b) We can always expand $|\psi\rangle$ as

$$|\psi\rangle = \sum_{i=1}^{d_1} \sum_{j=1}^{d_2} A_{i,j} |i\rangle \otimes |j\rangle \, ,$$

for some coefficients $\{A_{i,j}\}$. Let A denote the matrix $\sum_{i,j} A_{i,j} |i\rangle \langle j|$. If we apply $U \otimes V$ to $|\psi\rangle$ then the resulting state is

$$(U \otimes V) |\psi\rangle = \sum_{i=1}^{d_1} \sum_{j=1}^{d_2} A_{i,j} U |i\rangle \otimes V |j\rangle \tag{15}$$

$$= \sum_{i,k=1}^{d_1} \sum_{j,l=1}^{d_2} A_{i,j} U_{k,i} V_{l,j} |k\rangle \otimes |l\rangle \tag{16}$$

$$= \sum_{k=1}^{d_1} \sum_{l=1}^{d_2} (UAV^T)_{k,l} |k\rangle \otimes |l\rangle \tag{17}$$

Now use the SVD (singular value decomposition) theorem to find X, Λ, Y such that $A = X\Lambda Y$, with $\Lambda \geq 0$ diagonal, $X \in \mathcal{U}(d_1), Y \in \mathcal{U}(d_2)$. Set $U = X^\dagger$ and $V = \bar{Y}$ (defined to be the complex conjugate of Y, so that $V^T = Y^\dagger$. Then $UAV^T = \Lambda$ and $(U \otimes V) |\psi\rangle$ is of the desired form. This decomposition is known as the *Schmidt decomposition.* One feature that it inherits from the singular value decomposition is that the λ_i are unique up to reordering and the unitaries U, V are also unique if the λ_i are distinct. If some of the λ_i are the same, then the only freedom of U, V corresponds to rotations on those subspaces.

c) Using U, V from part (b), define $|\alpha_i\rangle := U^\dagger |i\rangle$ and $|\beta_i\rangle := V^\dagger |i\rangle$.