Algorithms and Economics of Networks:

Convergence and Approximation in Games

Microsoft Research, Theory Group

Reference

- Convergence in Competitive Games.
  - Vetta, Mirrokni
- Sink equilibria and Convergence.
  - Goemans, Mirrokni, Vetta.
- Convergnec in Potential Games.
  - Christodoulou, Mirrokni, Sidiropoulos.

Outline

- Price of Anarchy.
- State Graph and Convergence on Best-response Walks.
- Sink equilibria and Price of Sinking.
- Weighted and Unweighted Network Congestion Games.
- Cut Games.
- Valid-utility Games (Submodular-utility Games).

Price of Anarchy

- Performance in lack of Coordination.
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- Price of anarchy: Approximation Factor of a Decentralized Mechanism for selfish agents.

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  1) Question: What do they converge to?
- Selfish behavior of players may converge to Nash equilibria, but it may take exponential time!
  2) Question: How fast players converge to approximate solutions?(and not to Nash equilibria)
  $\rightarrow$ Running Time of the Decentralized Mechanism

Related Work

- Price of Anarchy: Papadimitriou, Koutsoupias (1999), Papadimitriou (2000), Roughgarden, Tardos (2001), Vetta (2002).

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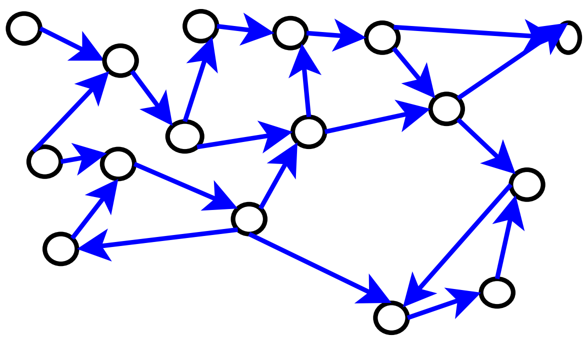
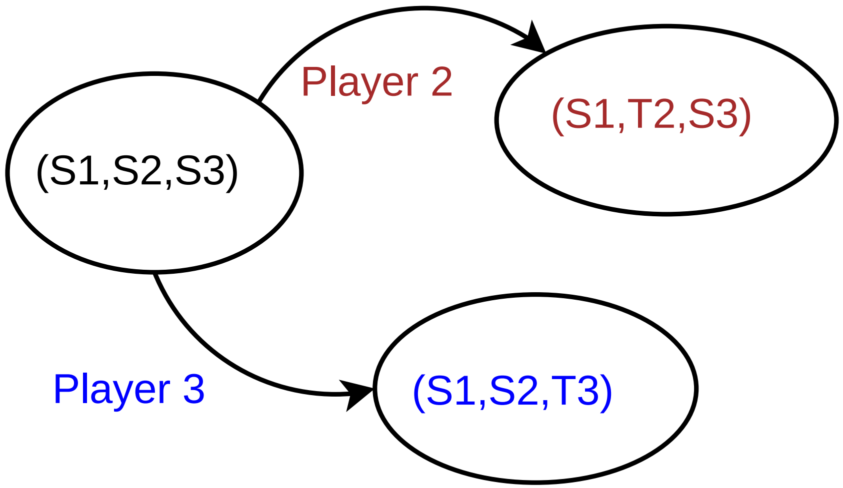
- Convergence to Equilibria in CS: Johnson, Papadimitriou, Yannakakis (1988), Schaffer, Yannakakis (1990), Even-dar, Kesselman and Mansour (2003), Fabrikant, Papadimitriou, Talwar (2004)).

The State Graph

We can model selfish behavior of players by a sequence of improvement moves by players in the state graph.

- The state graph, $\mathcal{D} = (\mathcal{V}, \mathcal{E})$, is a directed graph.





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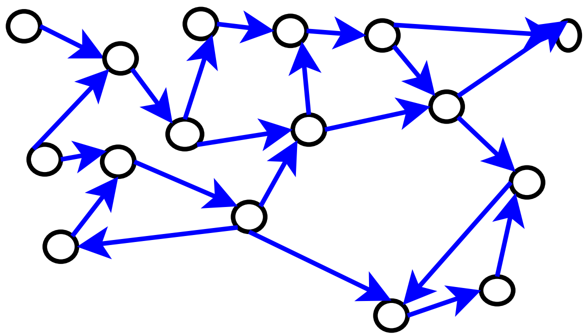
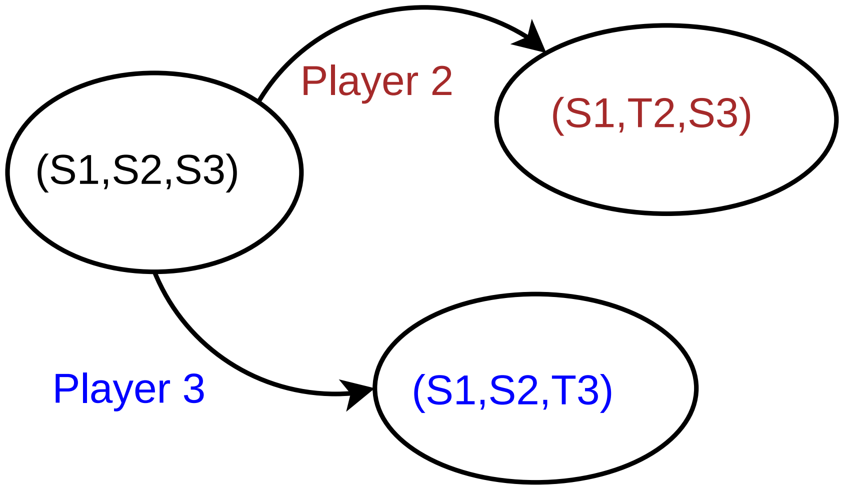
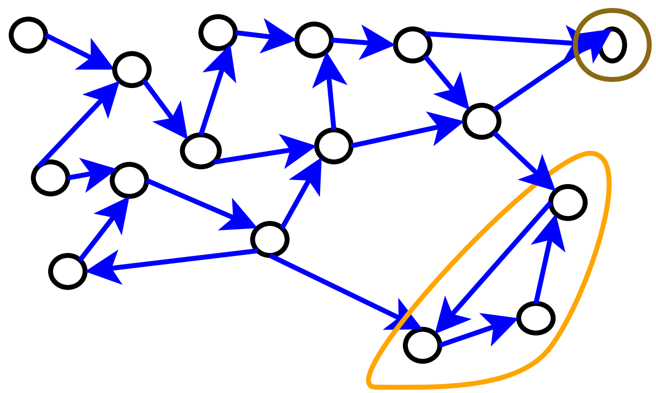




- Each vertex in $\mathcal{V}$: a strategy profile.
- an Arc from state S to state S' with label j: j improves his payoff from S to S'.

Sink Equilibria

- A sink equilibrium in the state graph is a strongly connected component without any outgoing edge in the state graph.



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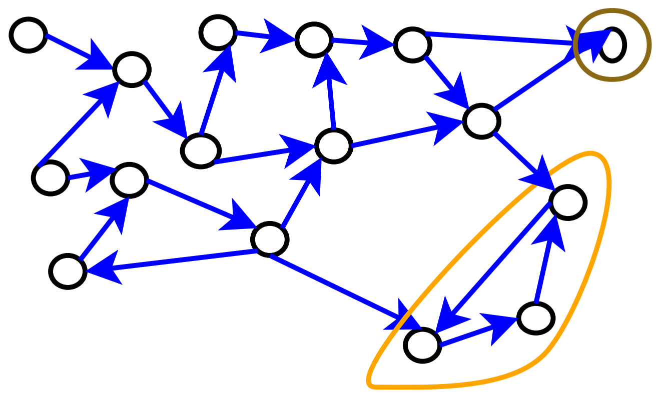

- It is a sink, i.e., when players arrive to a state in the sink equilibrium, they do not leave the sink equilibrium.

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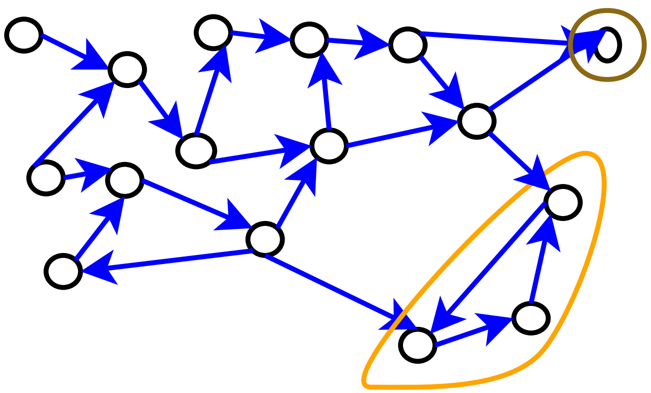

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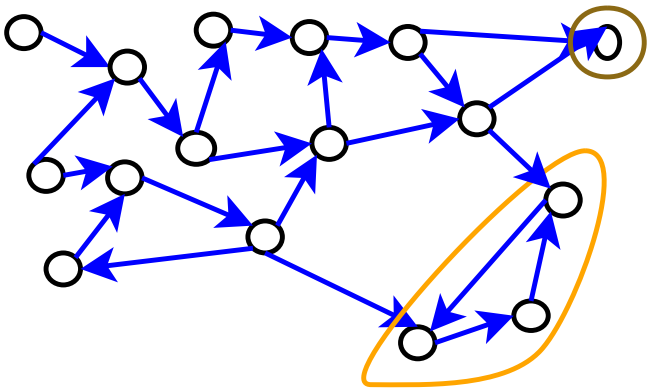

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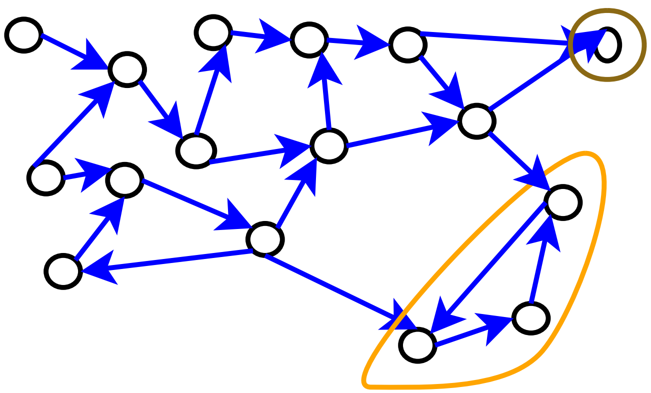

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- Social Value of a Sink equilibrium?
- Social Value of a Sink equilibrium = Average Social value of states on a random best-response walk.
- Random Best-response Walk: Choose a player uniformly at random at each step.

Price of Sinking

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- Price of Sinking = $\frac{\text{OPT}}{\min_{Q \in \mathcal{Q}} \Gamma(Q)}$.

Questions

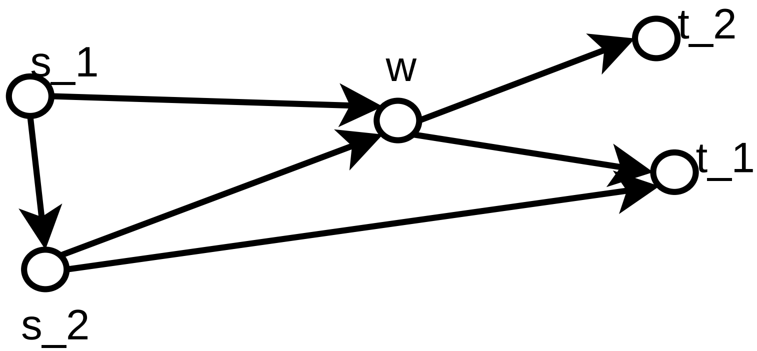
1. What do players converge to?
   - Find Potential Functions? Characterize Sink Equilibria?
2. Performance in Sink Equilibria?
   - Price of Sinking, Price of Anarchy?
3. Speed of Convergence to Sink Equilibria?
   - PLS-Complete?
4. Convergence to Approximate Solutions?
   - Deterministic and Random Walks?

Rest of the Talk

- Weighted congestion games:
  - Convergence to Equilibria.
  - Price of Sinking.
  - Speed of Convergence on Random Walks.
  - Speed of Convergence on Deterministic Walks.
- Cut Games.
- Valid-utility games

Weighted Congestion Games(WCG)

- Definition for Unsplittable Selfish Routing Games.
- Given a network $G(V, E)$.



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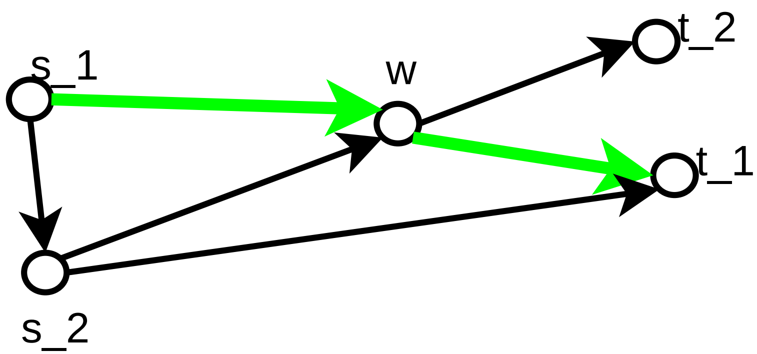

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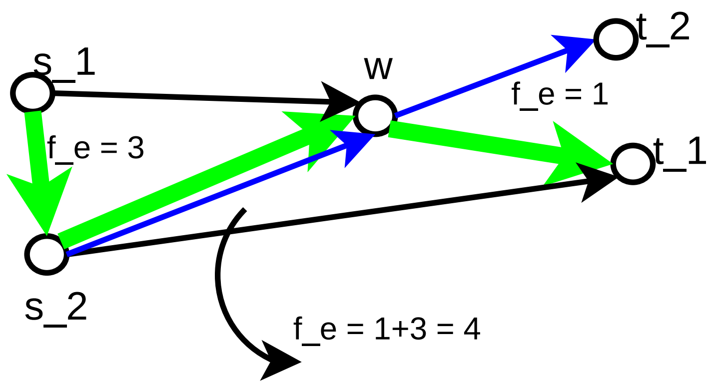
- Flow of an edge e: $f_e = \sum_{i:e \in P_i} r_i$.



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- Delay of path P: $l(P) = \sum_{e \in P} l_e(f_e)$.
- Delay of Player i: $l_i(f) = r_i l(P_i) = r_i \sum_{e \in P_i} l_e(f_e)$.

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- Social Function: Total Delay: $l(f) = \sum_i l_i(f)$.
- Assumption: latency function is a polynomial of degree d, $l_e(x) = \sum_{i=0}^{d} a_{e,i} x^i$.

WCG: Price of Anarchy

- Price of anarchy for Mixed Nash equilibria: for linear latency functions: 2.618 and for polynomials of degree d: $O(2^d(d+1)^{d+1})$ Awerbuch, Azar, and Epstien, 2005.
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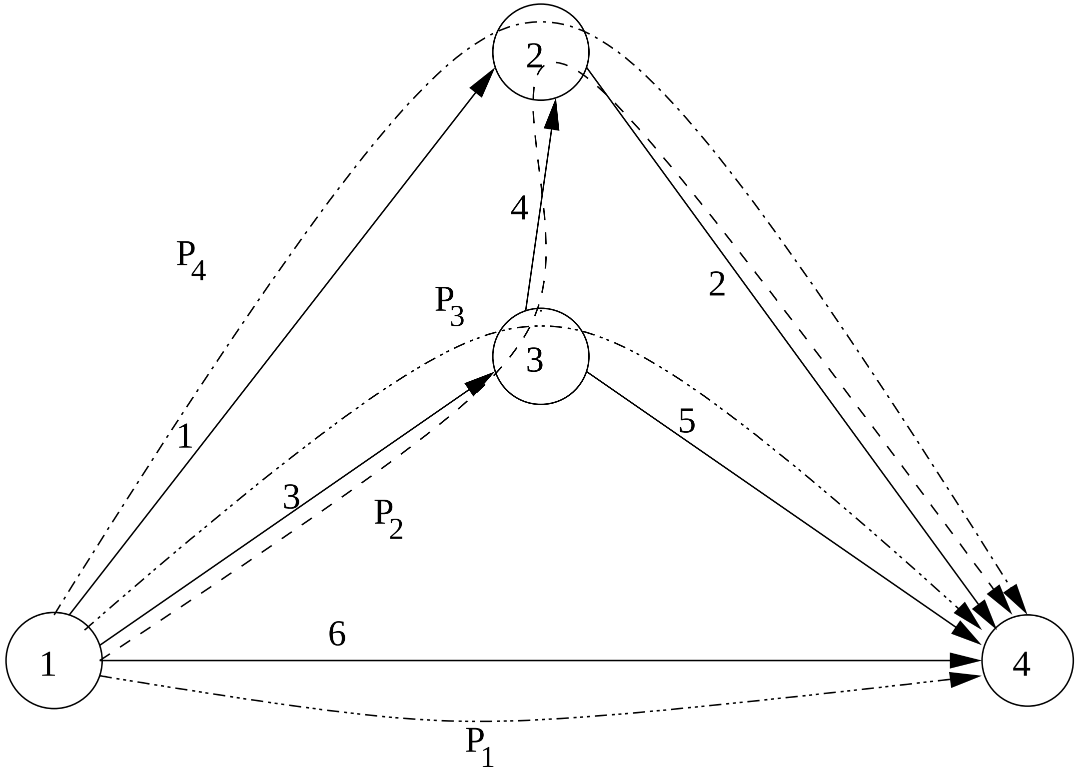
- For quadratic delay functions, Nash equilibria do not necessarily exist.

WCG: An Example



- Two agents: ($r_1 = 1$, $r_2 = 2$).
- $l_1(x) = x + 33$, $l_2(x) = 13x$, $l_3(x) = 3x^2$, $l_4(x) = 6x^2$, $l_5(x) = x^2 + 44$, and $l_6(x) = 47x$.

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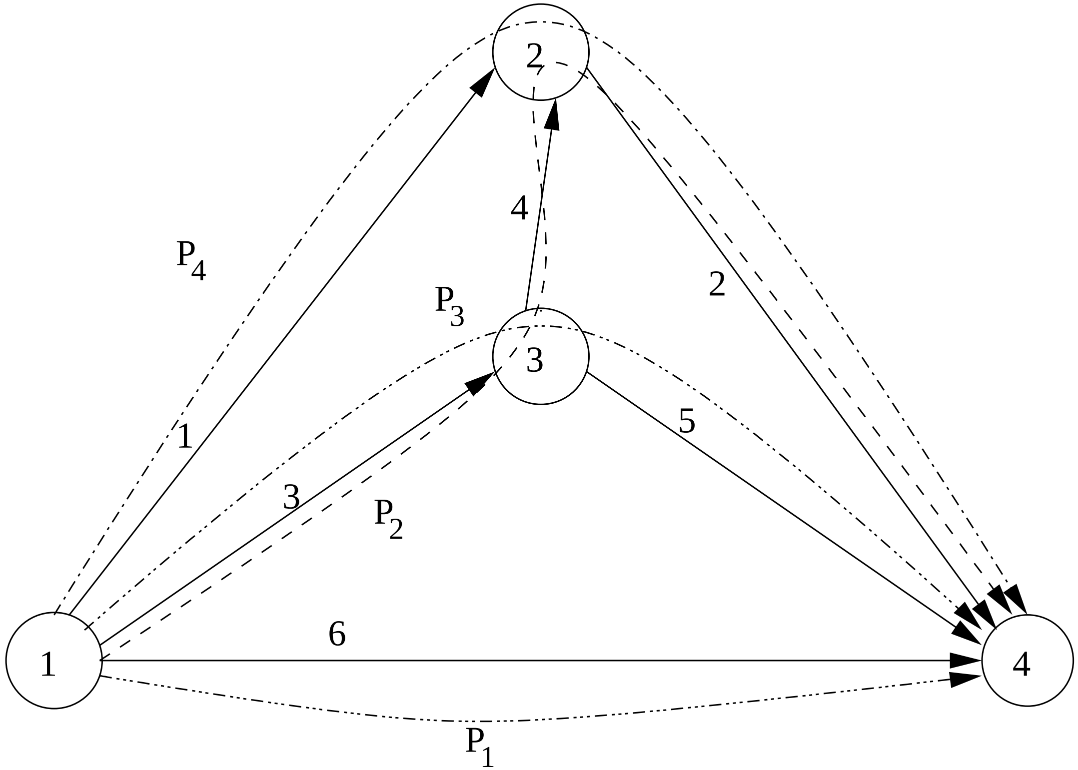

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- Only Sink equilibrium:
  $\{(P_1, P_2), (P_3, P_2), (P_3, P_4), (P_1, P_4)\}$.
- No Pure Nash equilibrium.

WCG: Price of Sinking

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- Proof Idea:

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- Proof Idea:
- Lemma 1: If player i plays his best response and change the flow from f to flow f_i', then $l(f_i') \leq l(f) + (d+1)l_i(f_i') - l_i(f) \leq l(f) + dl_i(f)$.

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- Let $f_0, f_1, f_2, \ldots, f_N$ be the sequence of flows on the random walk.
- So by induction $E[l(f_j)] \le O(2^{2d}(d+1)^{2d+3})$OPT.
- Thus, the price of sinking is $O(2^{2d}(d+1)^{2d+3})$.

WCG: Fast Convergence

- Fabrikant, Papadimitriou, and Talwar'04: Finding a pure NE is PLS-complete and there may be exponential best response walks to equilibria.
- Our result: Even though convergence to equilibria is bad, this game has a fast convergence to approximate solutions.

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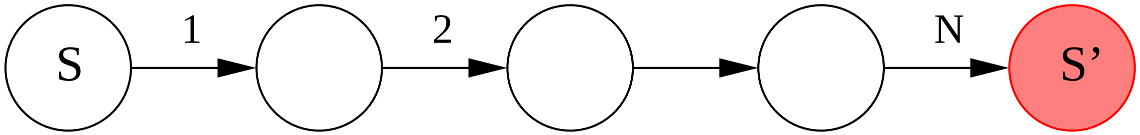
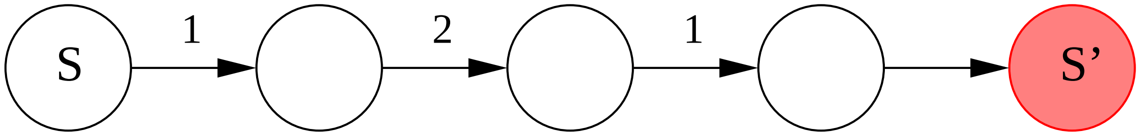
- Our result: Even though convergence to equilibria is bad, this game has a fast convergence to approximate solutions.
- Theorem: In the weighted unsplittable selfish routing game with polynomial latency functions of degree at most d, starting from any state with total latency C the expected latency of the flow after $O(n \log C)$ random best responses is at most $O(2^{2d}(d+1)^{2d+3})$OPT.

One-round walk



- arbitrary ordering $i_1, \ldots, i_N$
- j-th edge has label i_j
- **Random one-round walk**: the ordering is picked randomly

Covering walk



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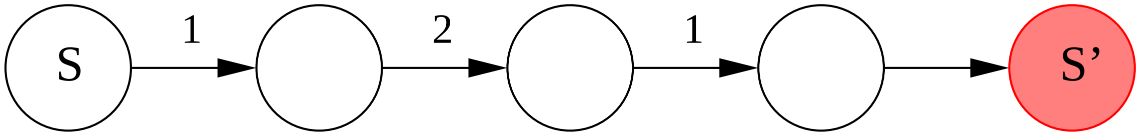

- $\mathcal{P}$:for each player i there exists an edge with label i.
- **k-Covering walk:**$\mathcal{P}_1, \ldots, \mathcal{P}_k$.

Linear Congestion Games - UBs

Theorem 1. Starting from an arbitrary initial state S^0, any one-round walk $\mathcal{P}$ leads to a state S^N that has approximation ratio $O(N)$.

Theorem 2. Starting from the empty state S^0, any one-round walk $\mathcal{P}$ leads to a state S^N that has approximation ratio of at most $\frac{(\phi+1)^2}{\phi} \approx 4.24$.

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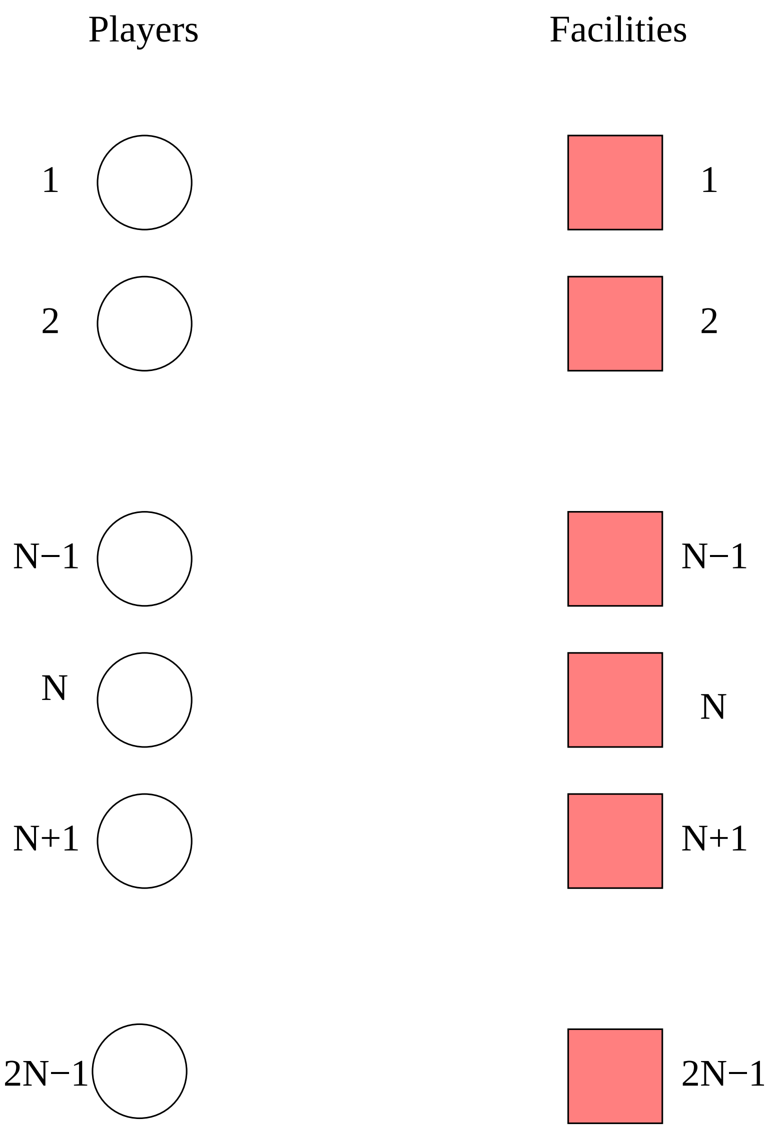
- Lower Bound 3.08 for scheduling.
  [Suri, Tóth, Zhou,2004]

Linear Network Congestion Games - LBs

Theorem 3. For any $N > 0$, there exists an N-player instance of the unweighted congestion game, and an initial state S^0 and a one-round walk $\mathcal{P}$ that results to an $\Omega(N)$-approximate solution.

Linear Congestion Games - LBs



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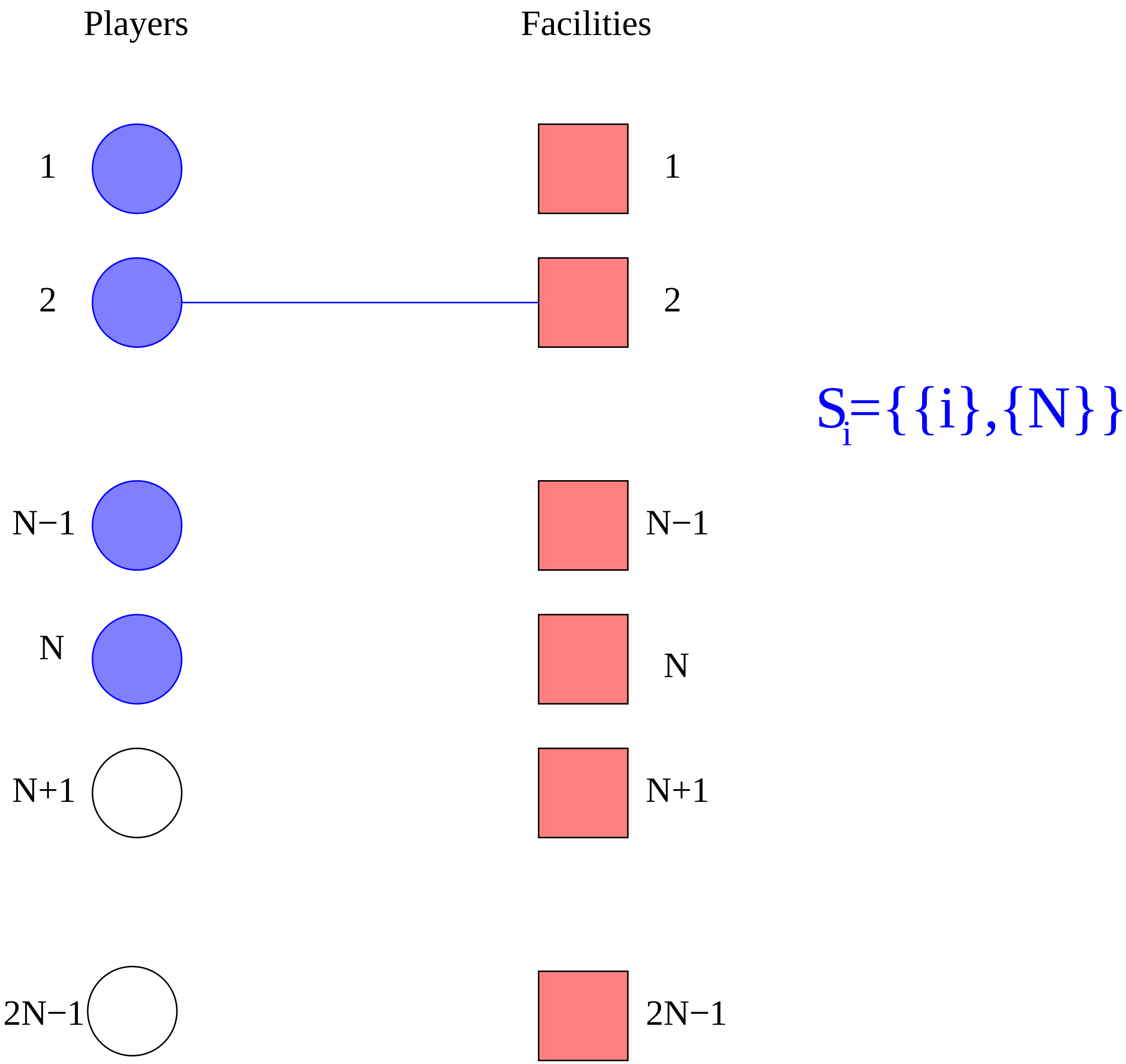

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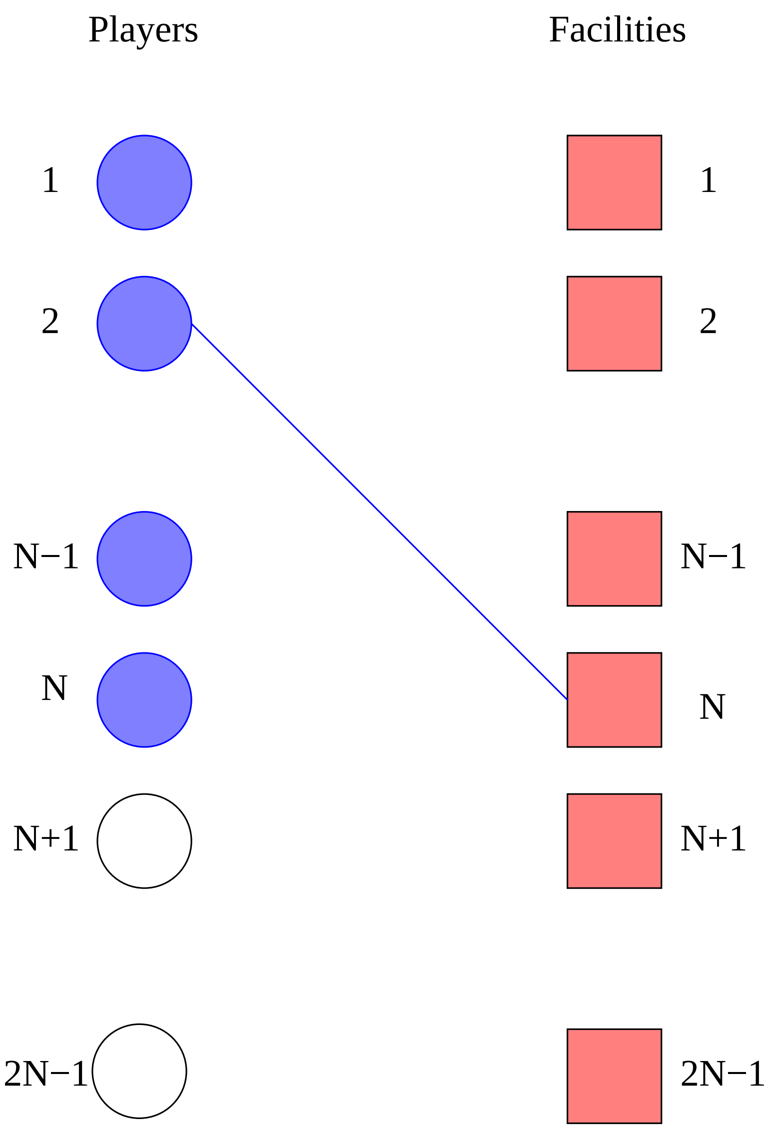

$S_i = \{\{i\}, \{N\}\}$

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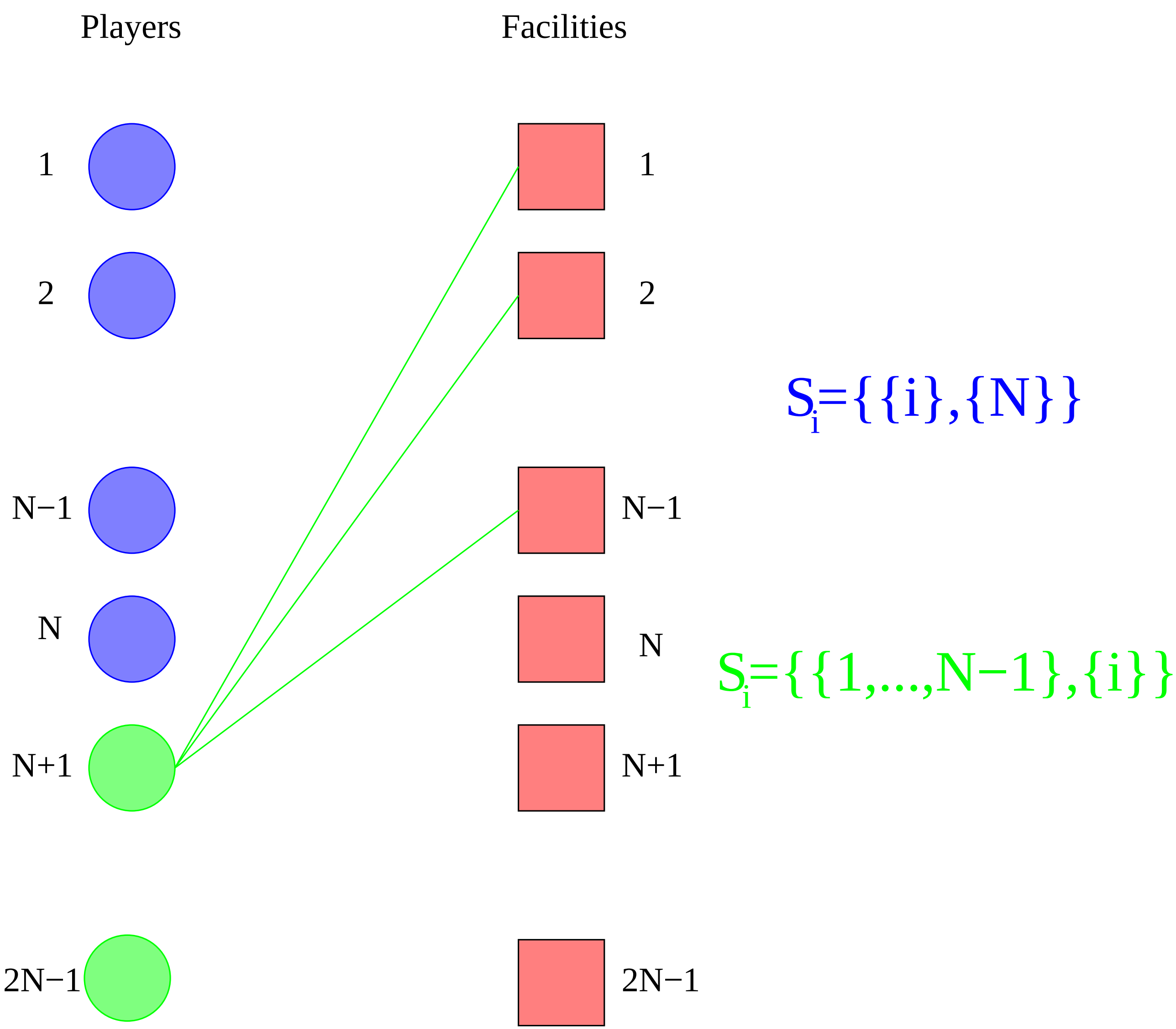

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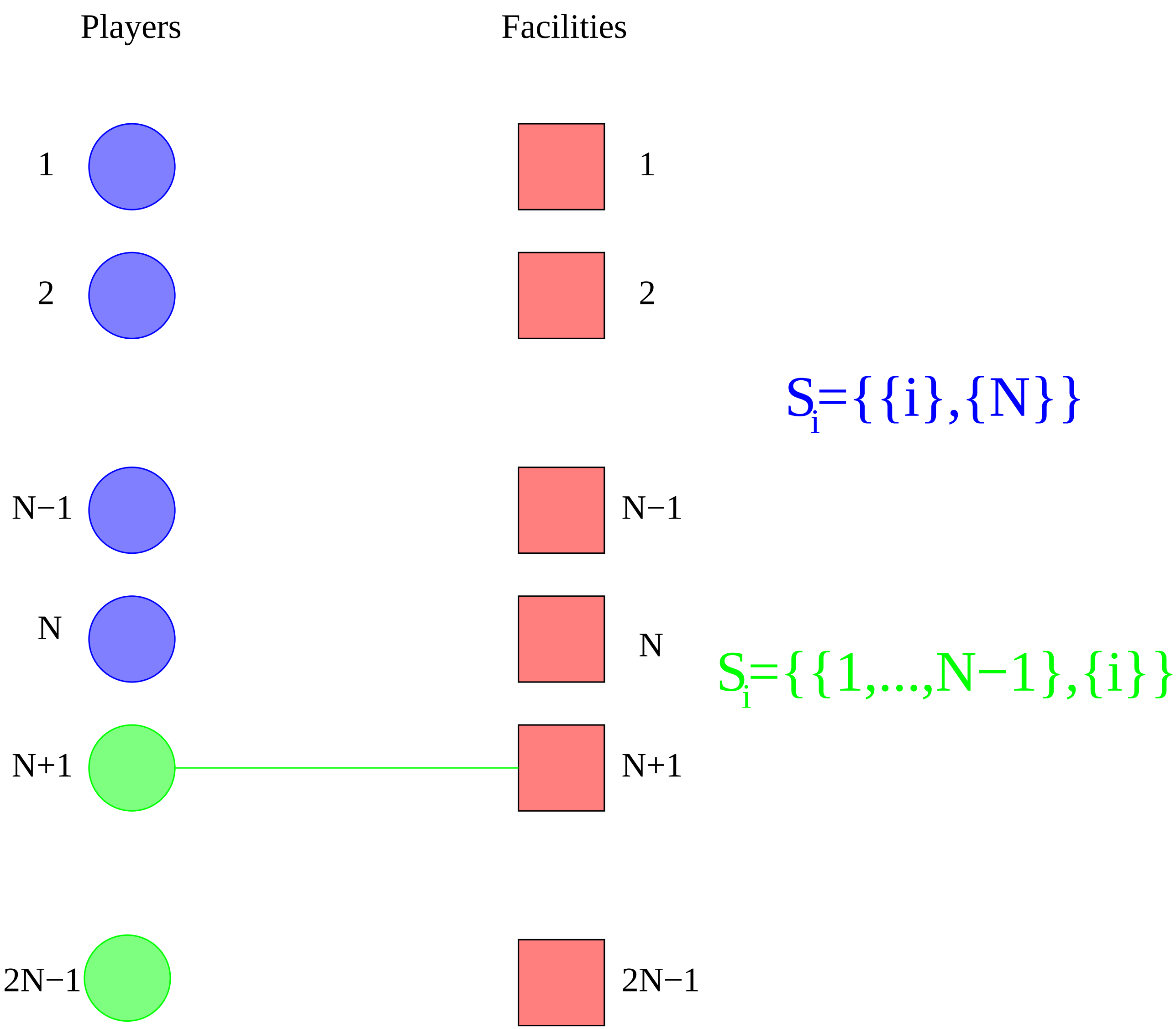

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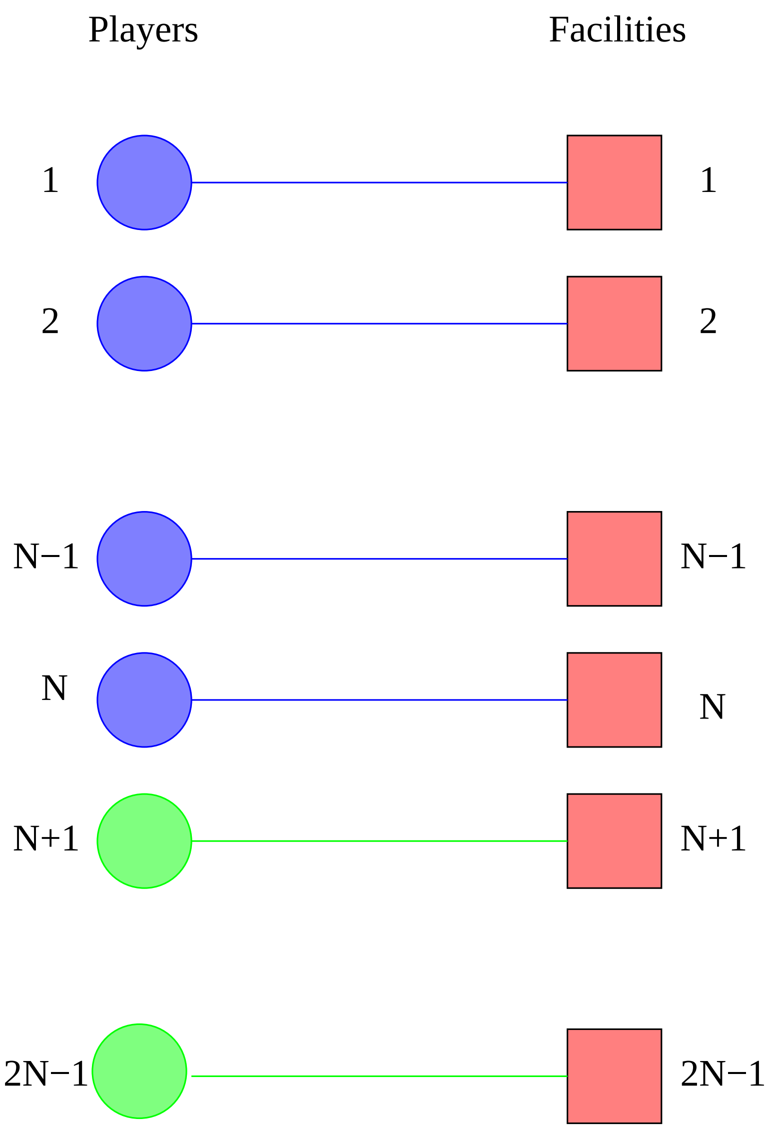

opt=2N−1

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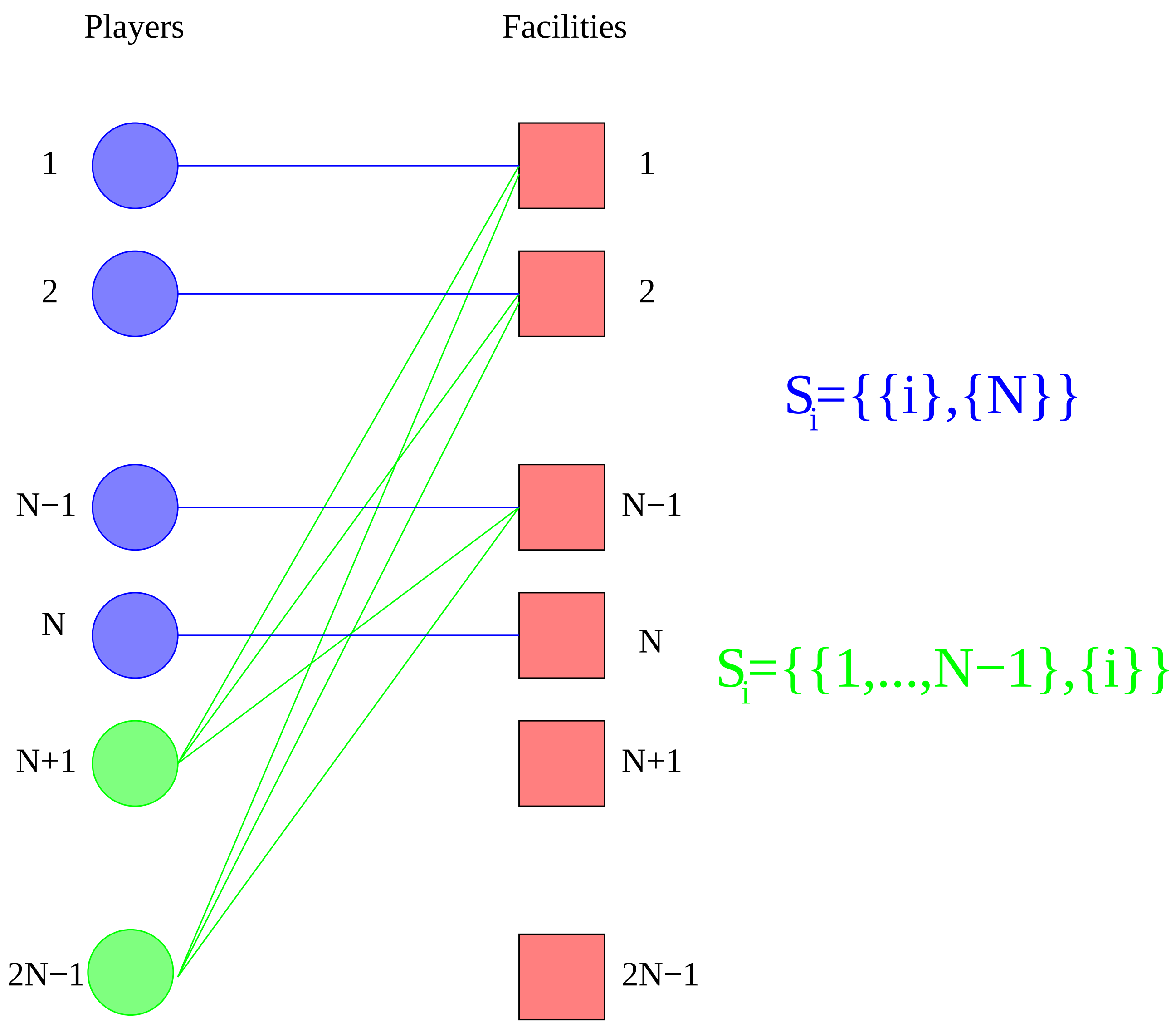

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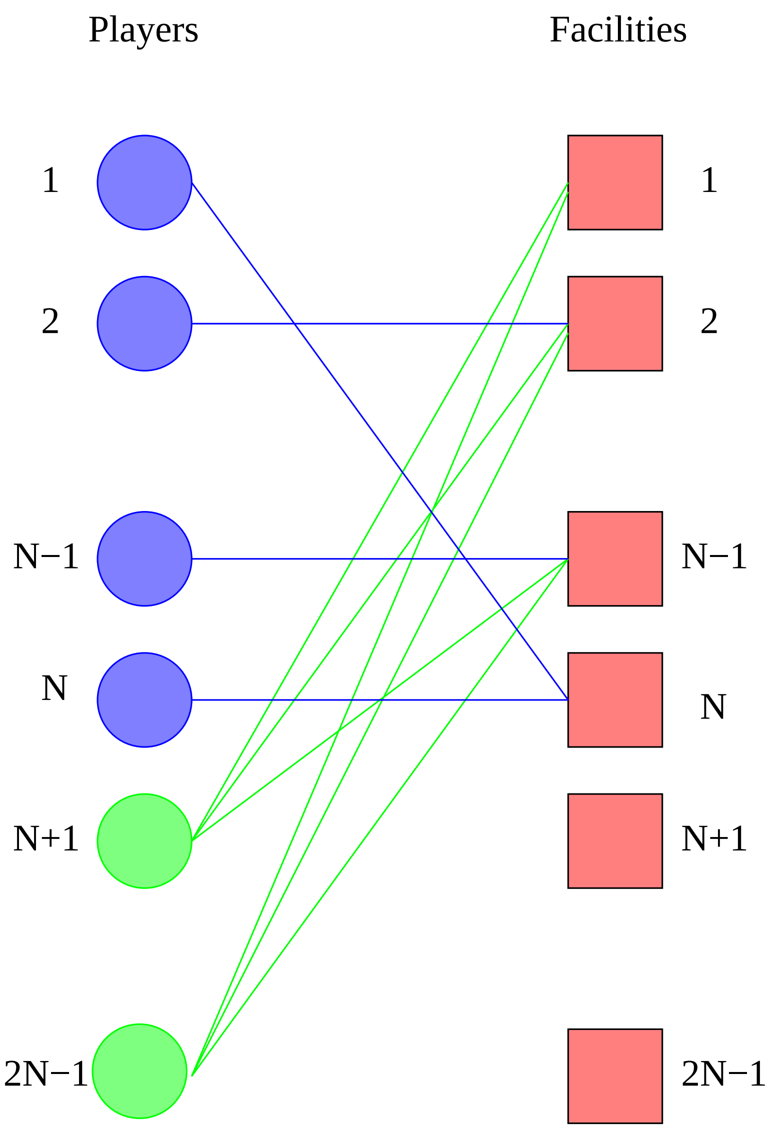

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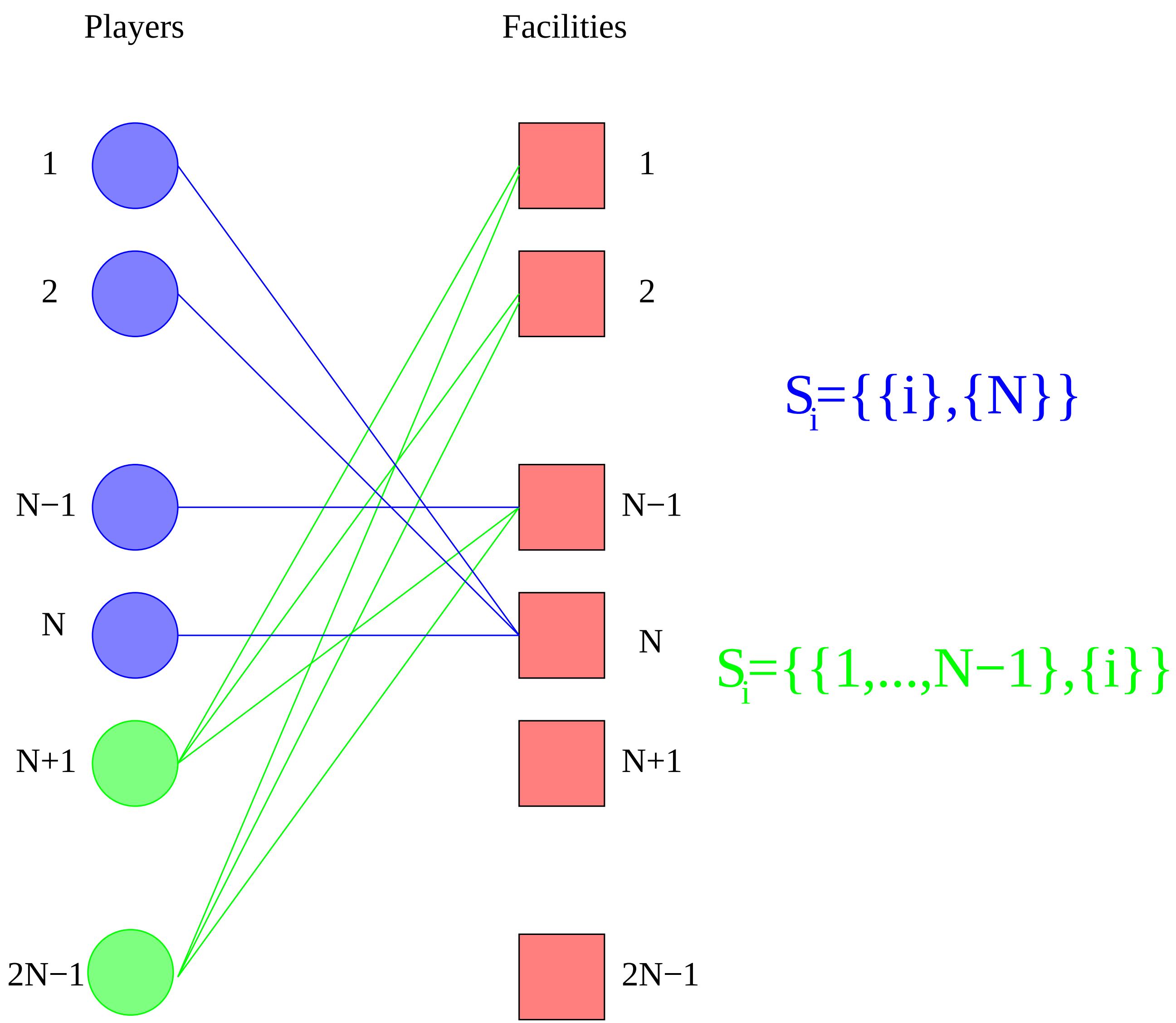
Linear Congestion Games - LBs



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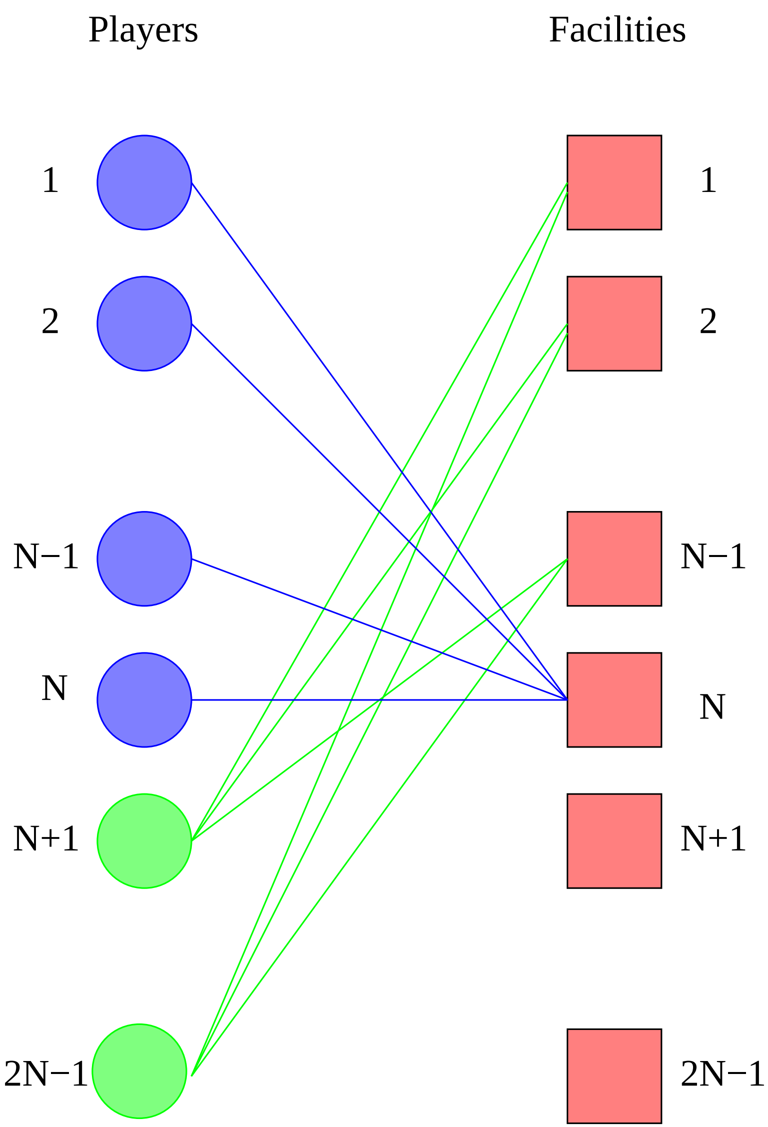

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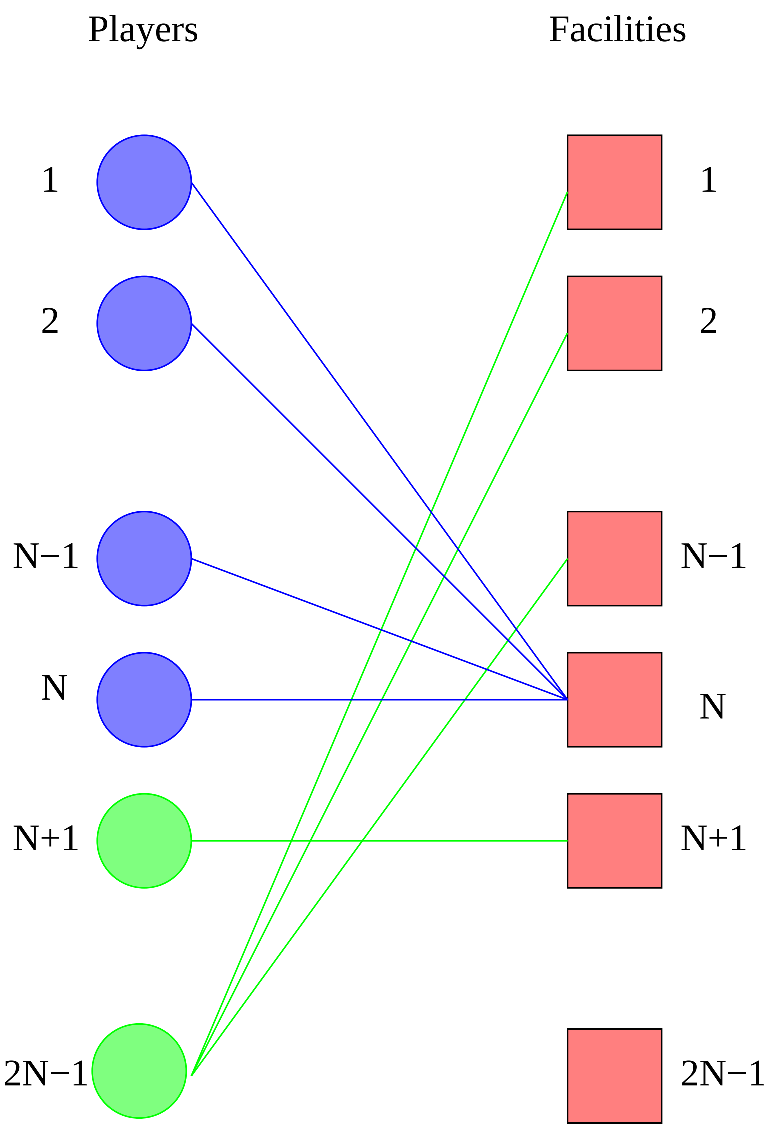

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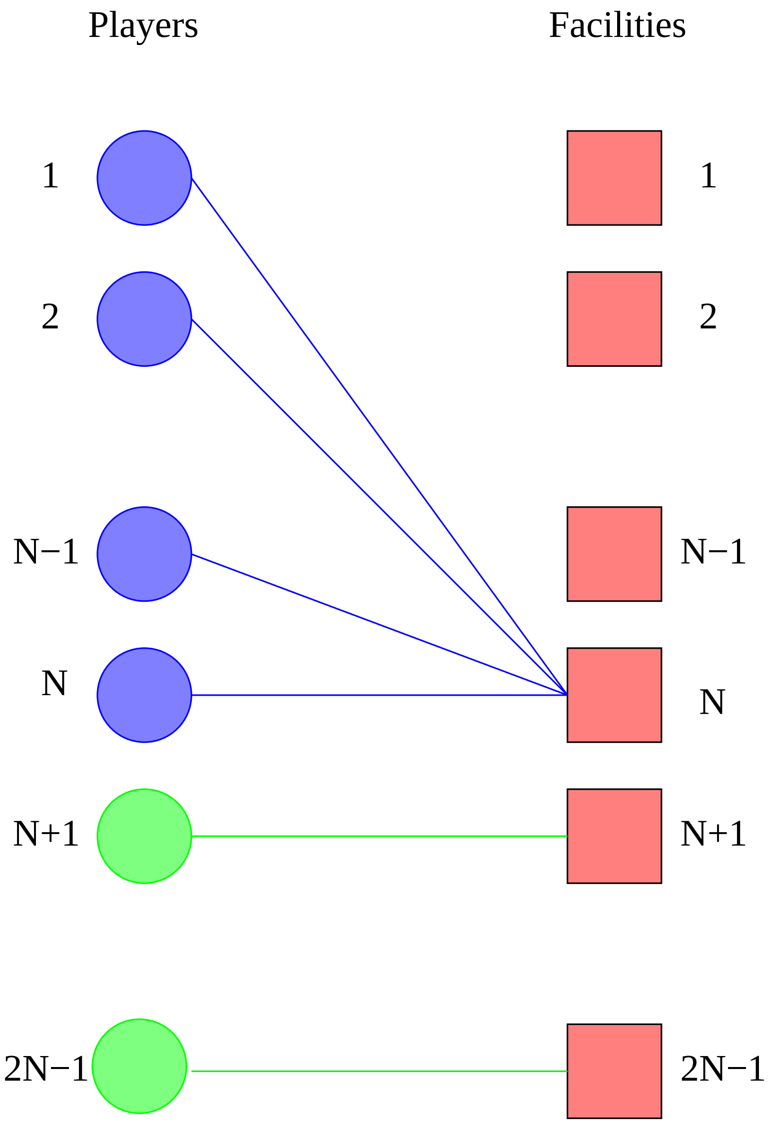


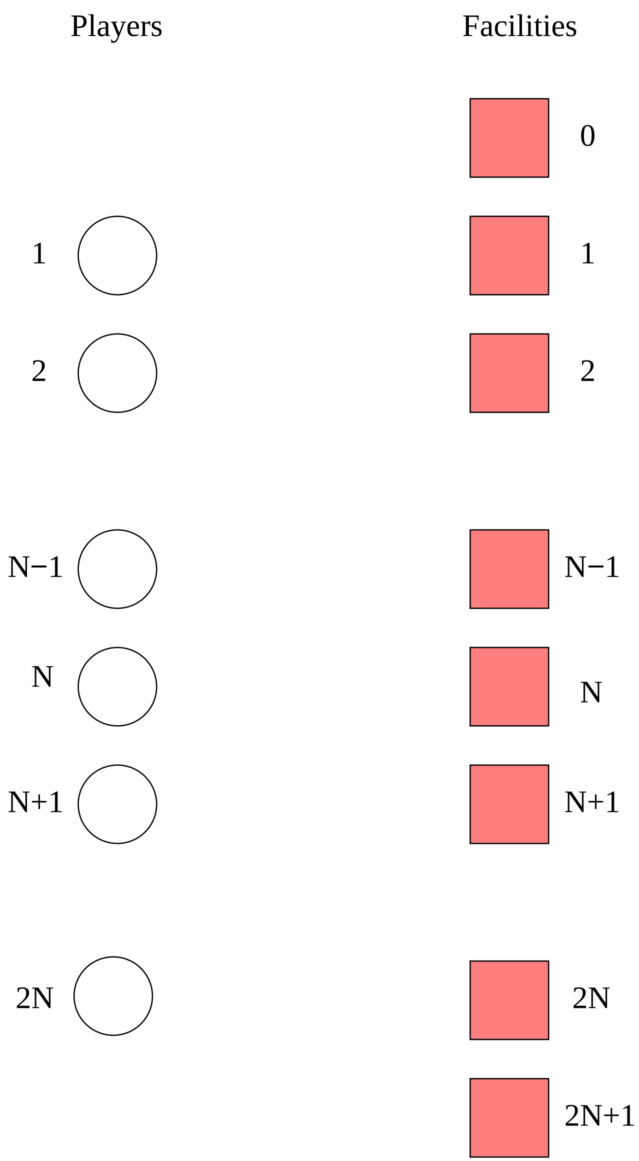
$\text{cost} = N^2 + N - 1$

$\text{opt} = 2N - 1$

Linear Congestion Games - LBs

- **Theorem.** For any $t > 0$, and for any sufficiently large $N > 0$, there exists an N-player instance of the unweighted congestion game, an initial state S^0, and an ordering σ of the players, such that starting from S^0, after t rounds where the players play according to σ, the cost of the resulting allocation is a $(N/t)^\epsilon$-approximation, where $\epsilon = 2^{-O(t)}$.
- **Theorem.** For any $N > 0$, there exists an N-player instance of the unweighted congestion game, and an initial state S^0 such that *for any* one-round walk $\mathcal{P}$ starting from S^0, the state at the end of $\mathcal{P}$ is an $\Omega(N)$-approximate solution.

Linear Congestion Games - LBs



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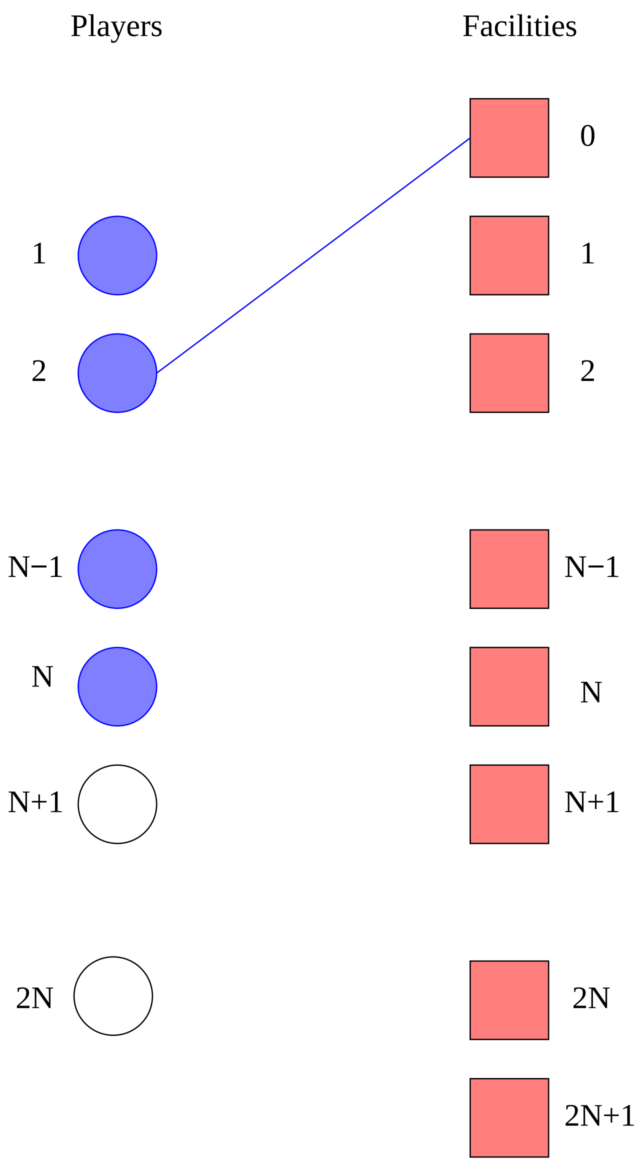
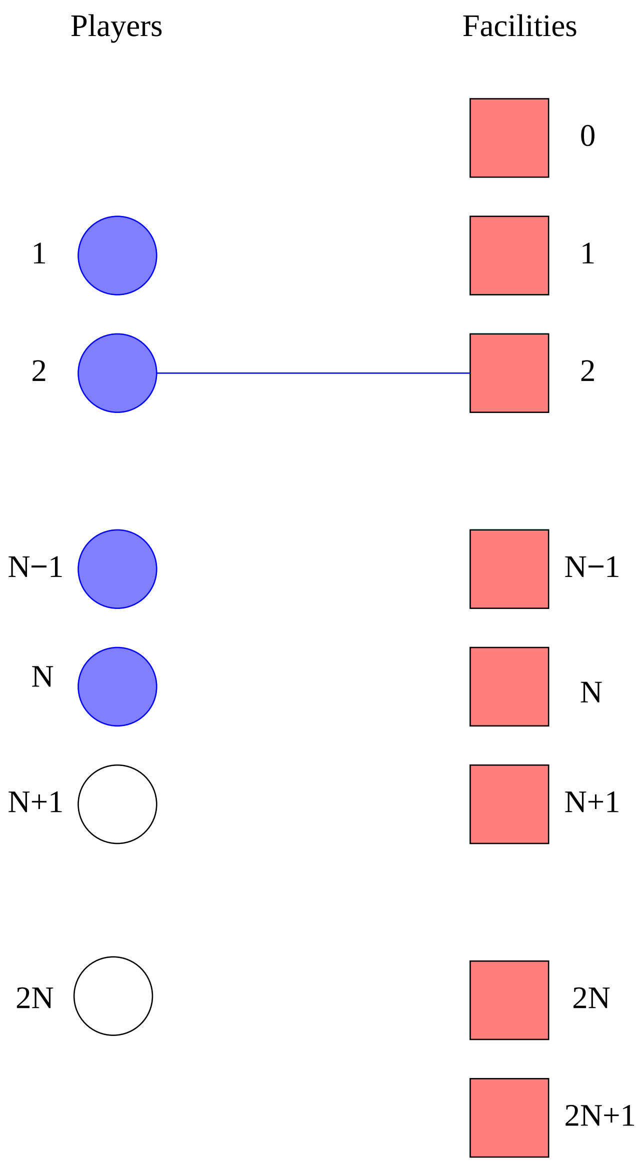


$S_i = \{\{0\},\{i\},\{N+1,\ldots,2N\}\}$

Linear Congestion Games - LBs



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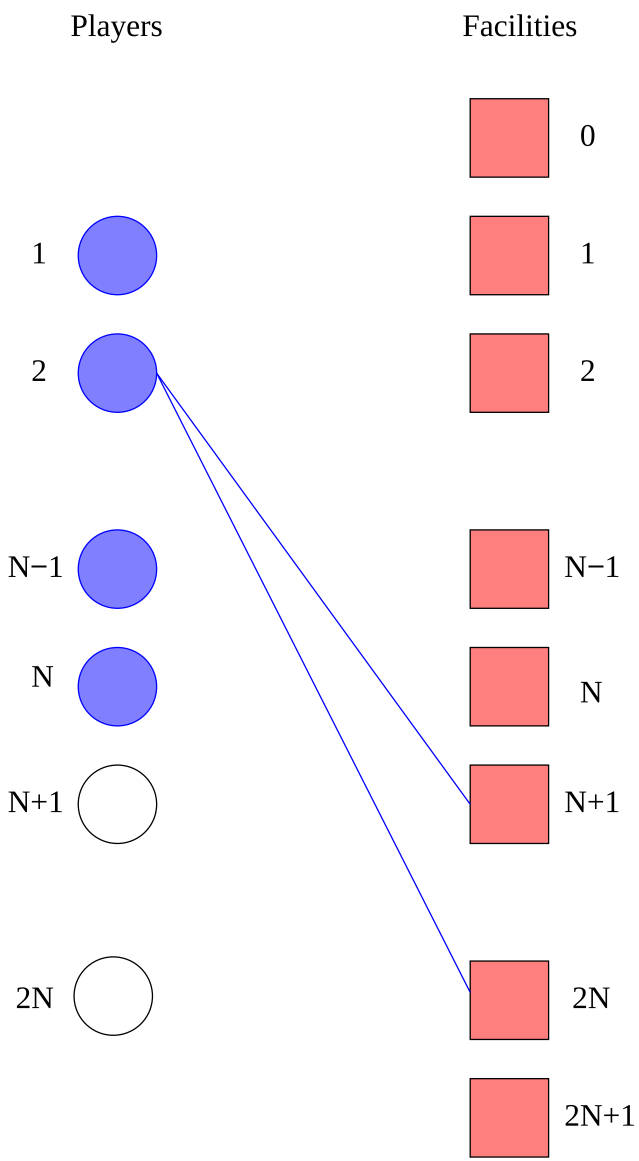

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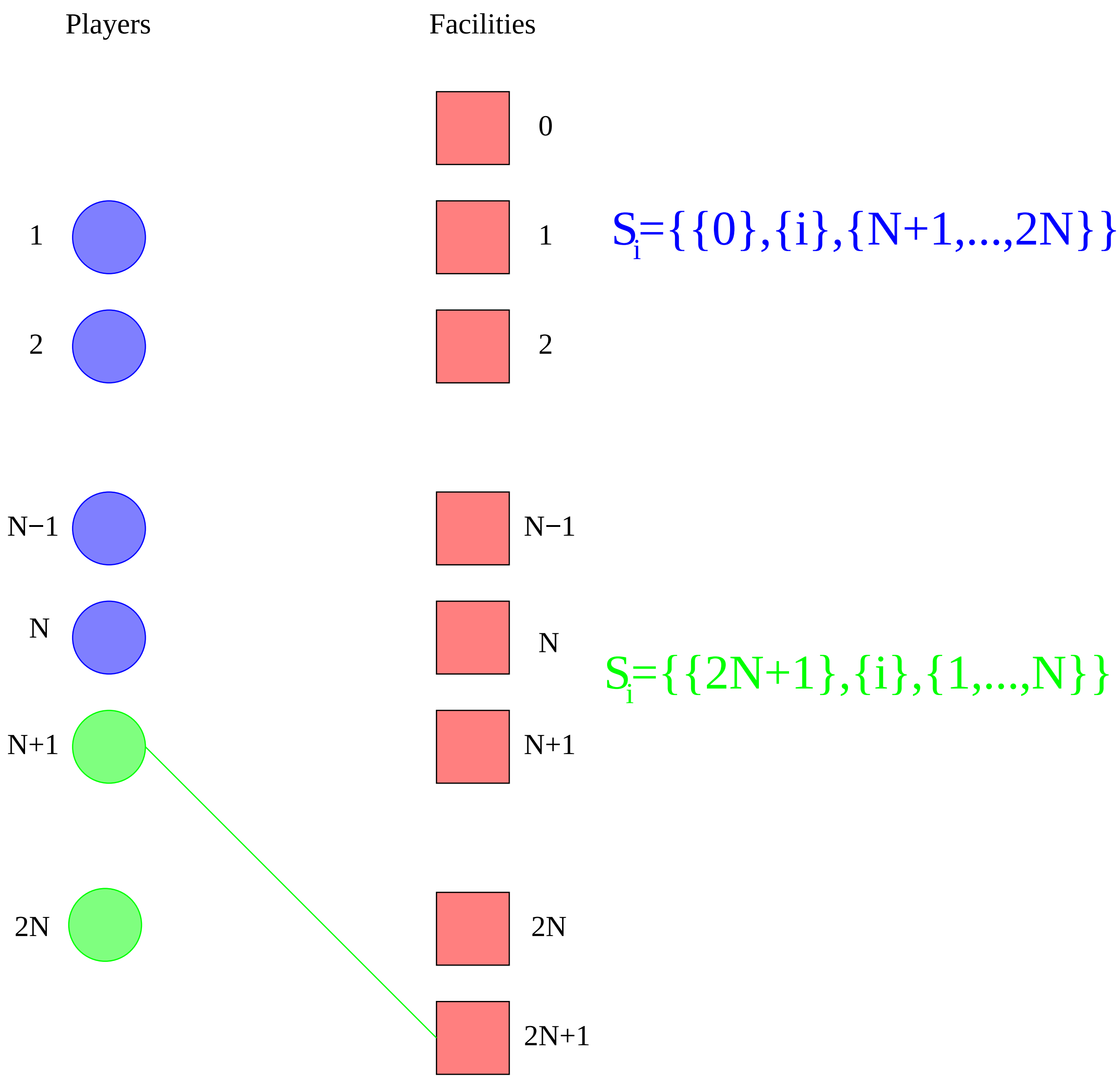

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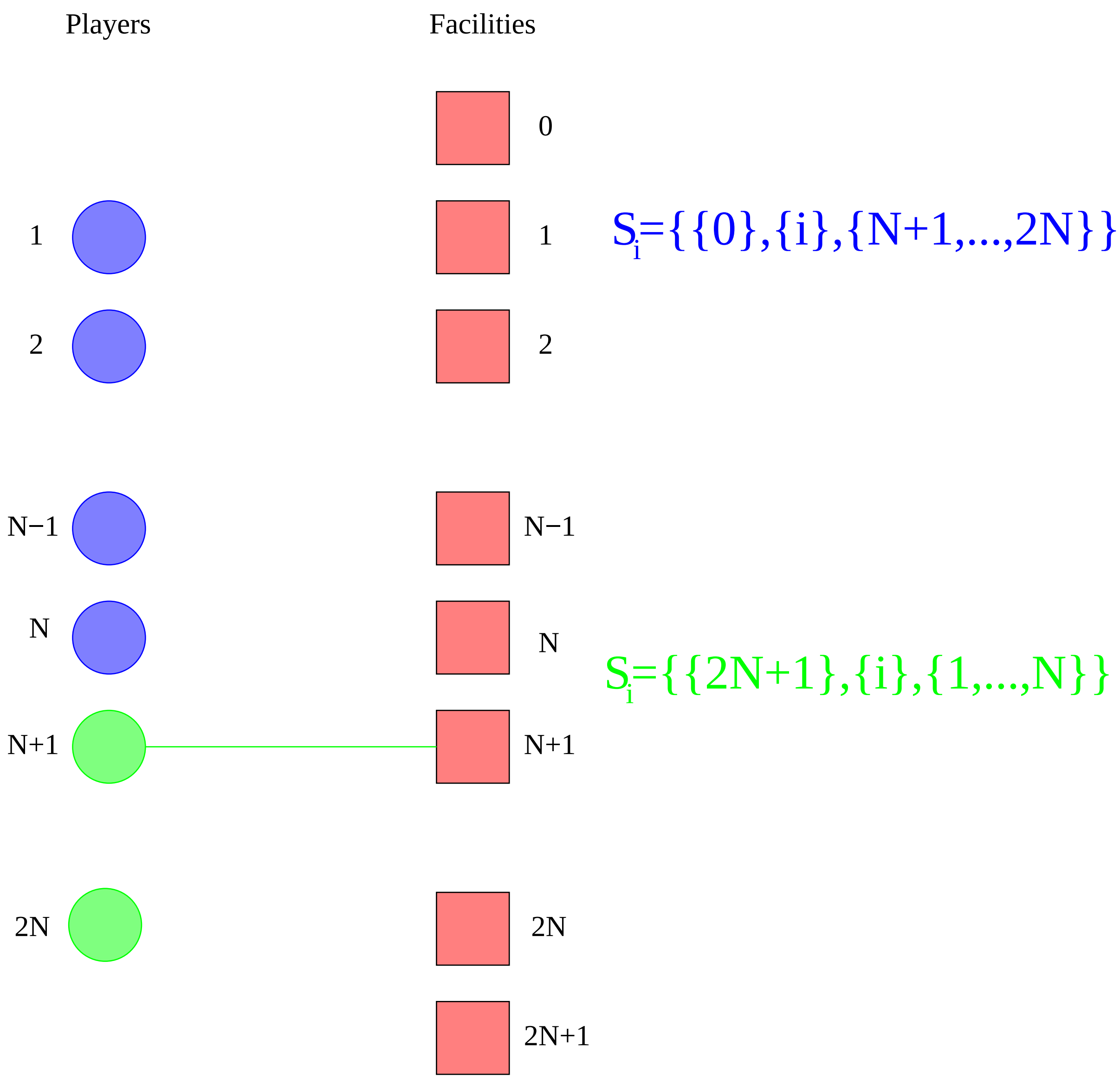

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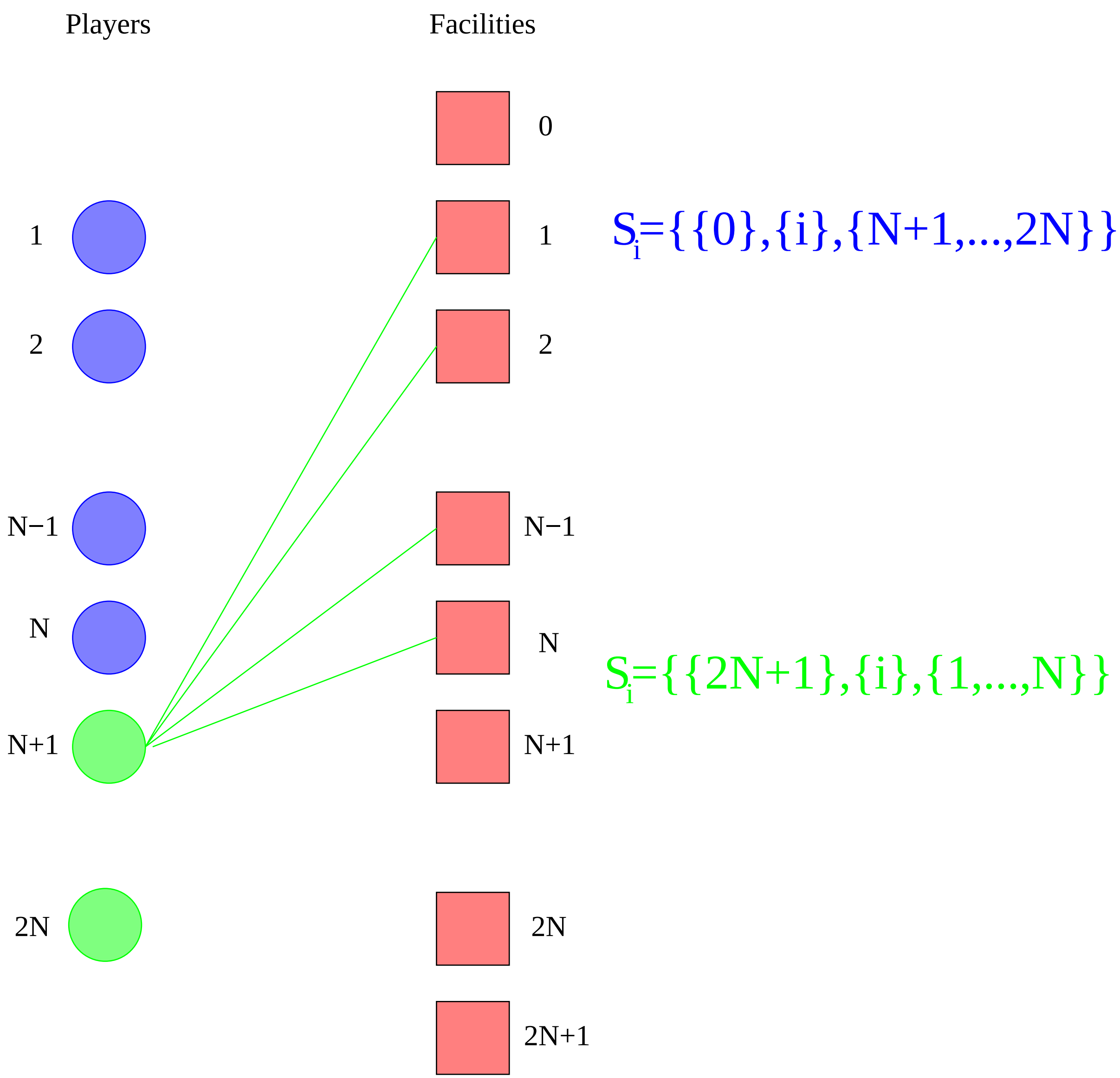

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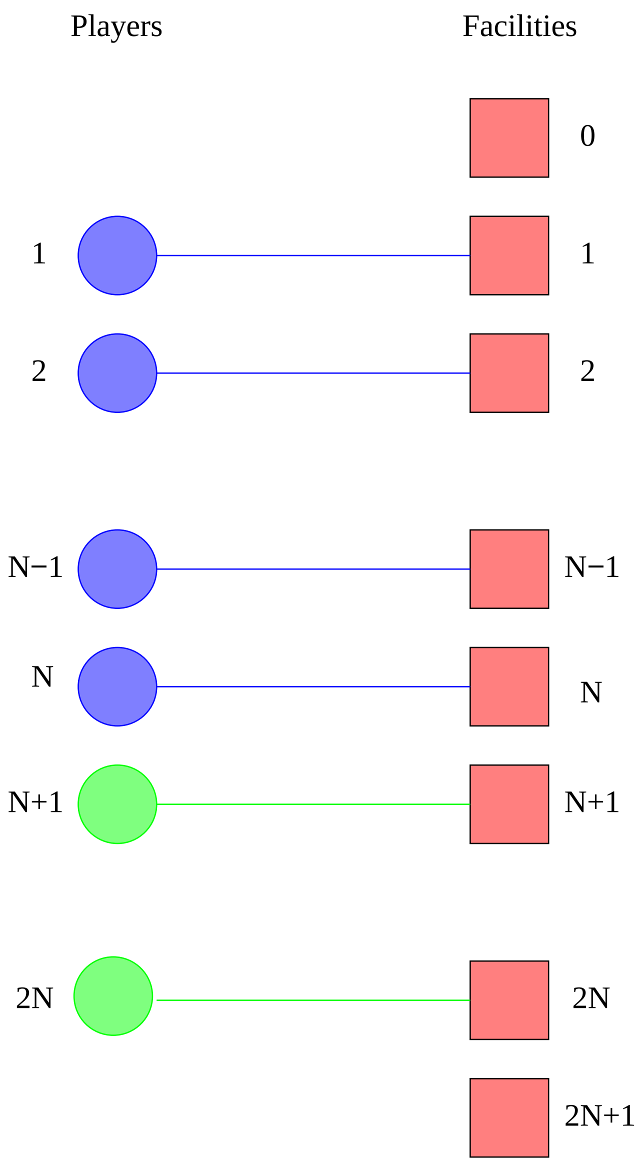
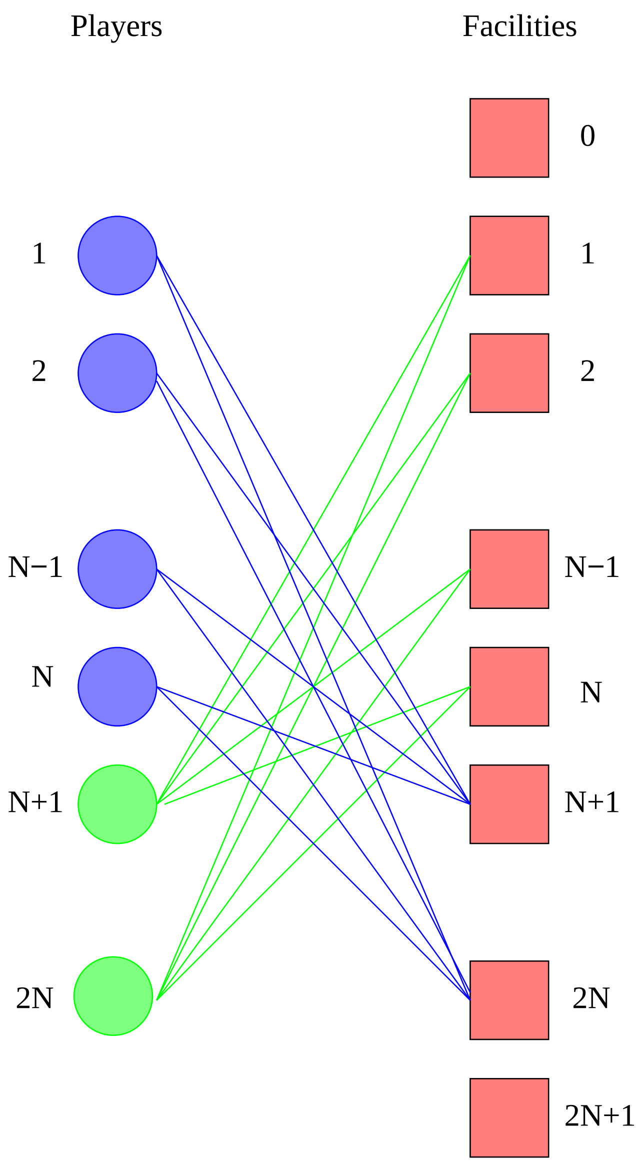


opt=2N

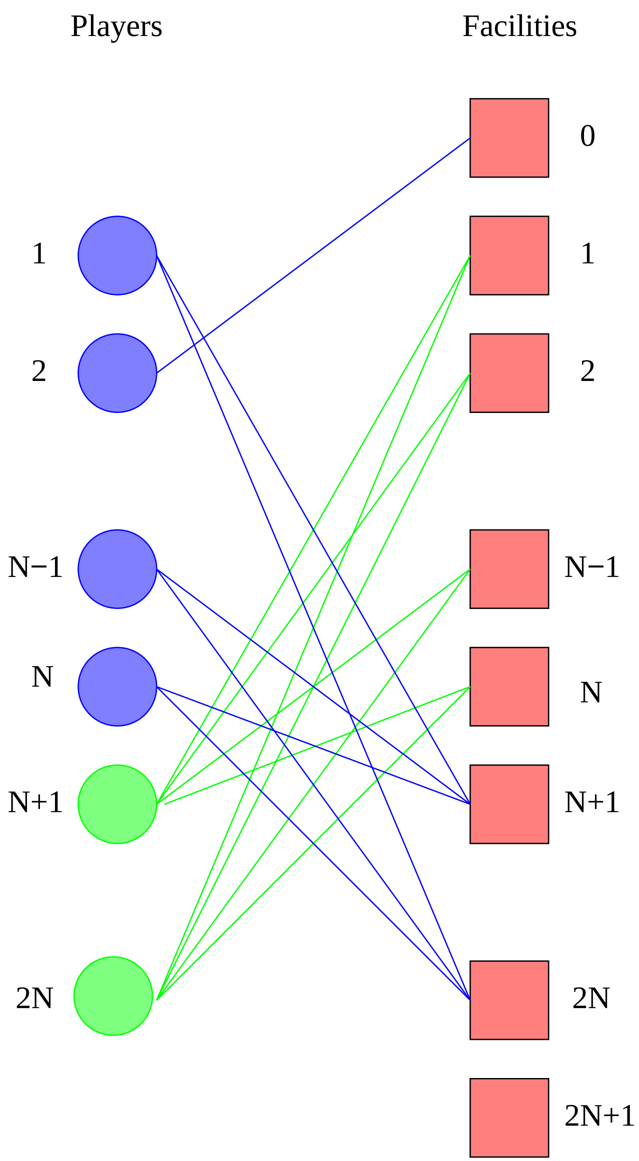
Linear Congestion Games - LBs



$S_i = \{\{0\},\{i\},\{N+1,\ldots,2N\}\}$

$S_i = \{\{2N+1\},\{i\},\{1,\ldots,N\}\}$

Linear Congestion Games - LBs



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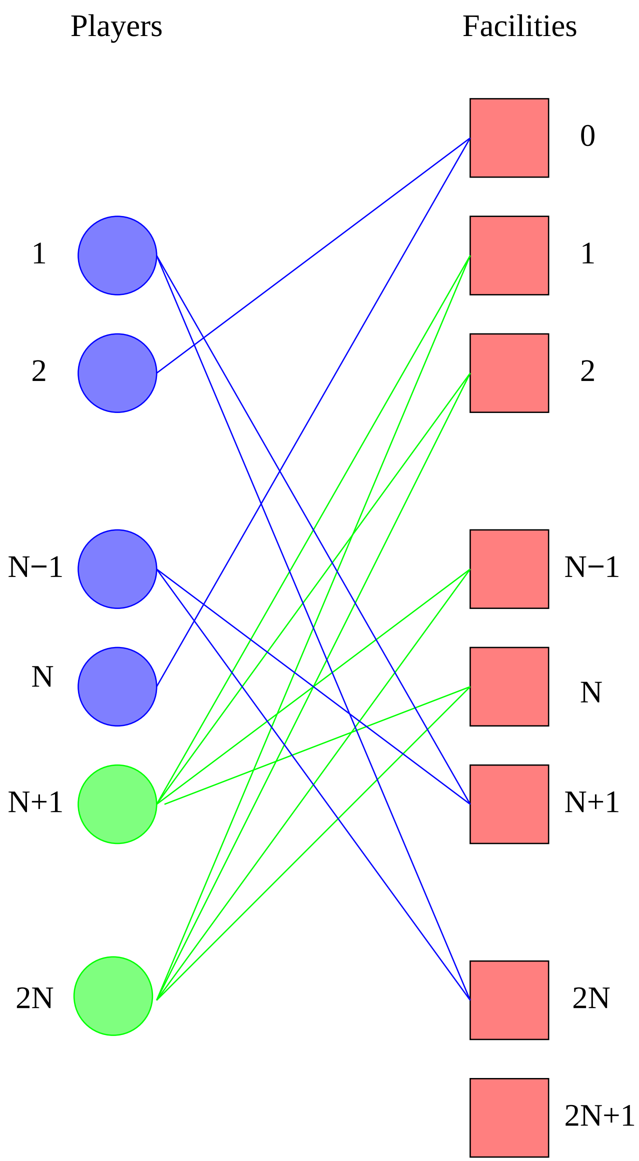

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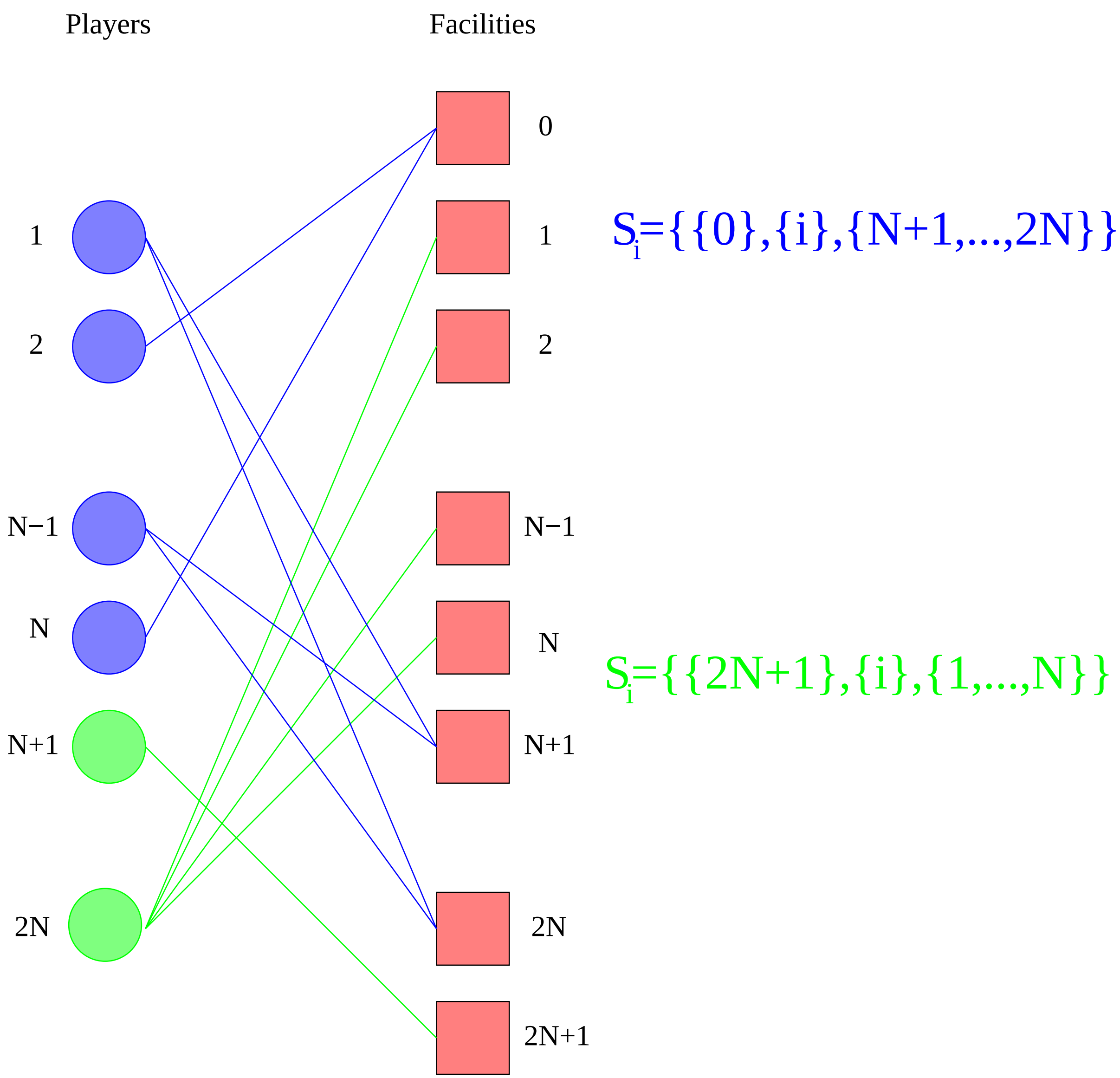

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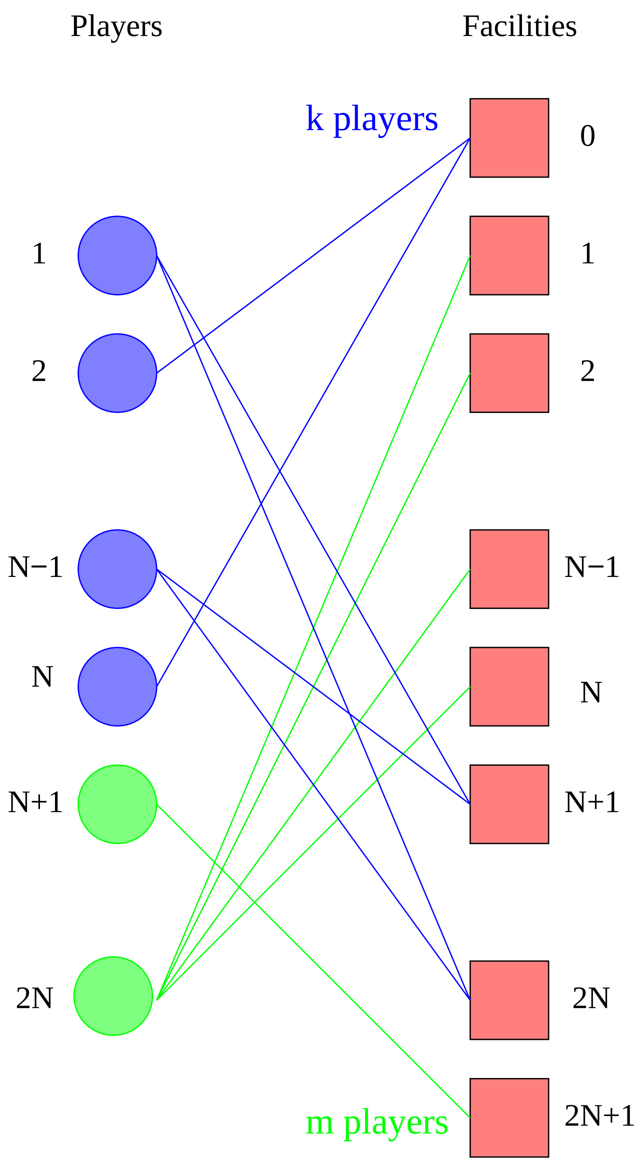

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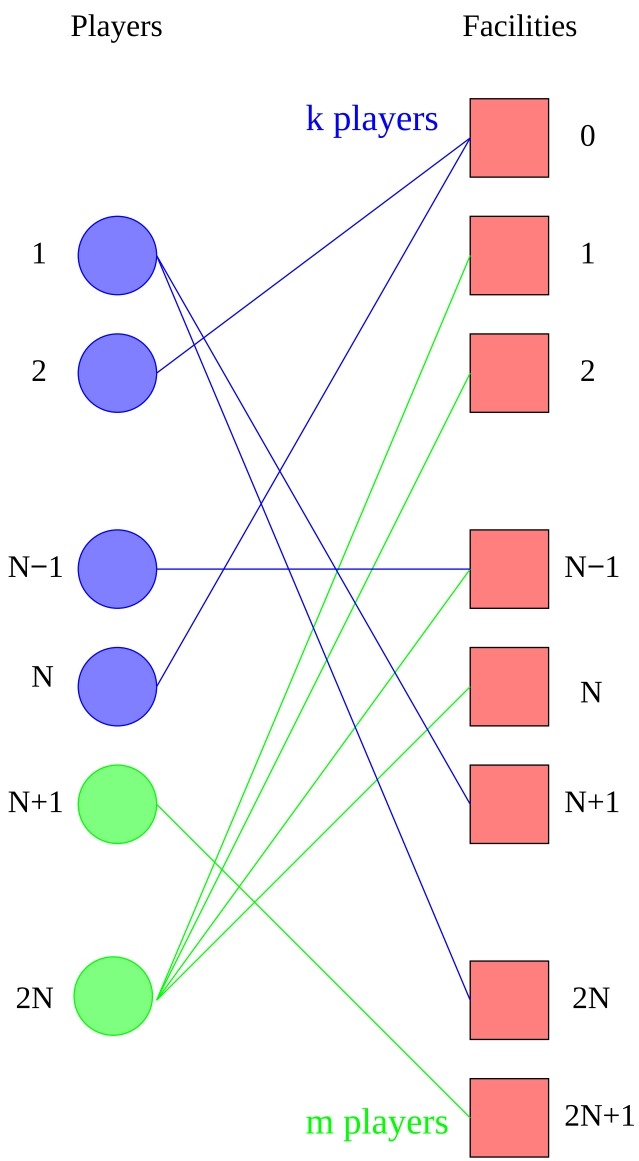

$k > N - m$

$cost > k^2 + m^2 + N = \Omega(N^2)$

Questions

1. What do players converge to?
   - Find Potential Functions? Characterize Sink Equilibria?
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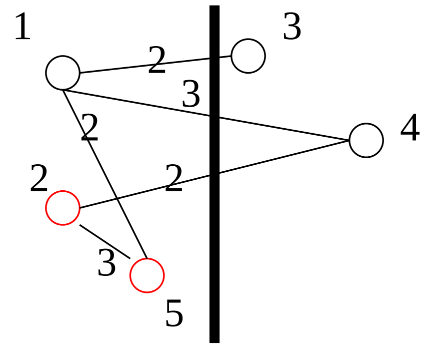
   - Price of Sinking, Price of Anarchy?
3. Speed of Convergence to Sink Equilibria?
   - PLS-Complete?
4. Convergence to Approximate Solutions?
   - Deterministic and Random Walks?

Rest of the Talk

- Cut Games: Convergence on Random and Determinist Best-response Paths.
- Valid-utility games

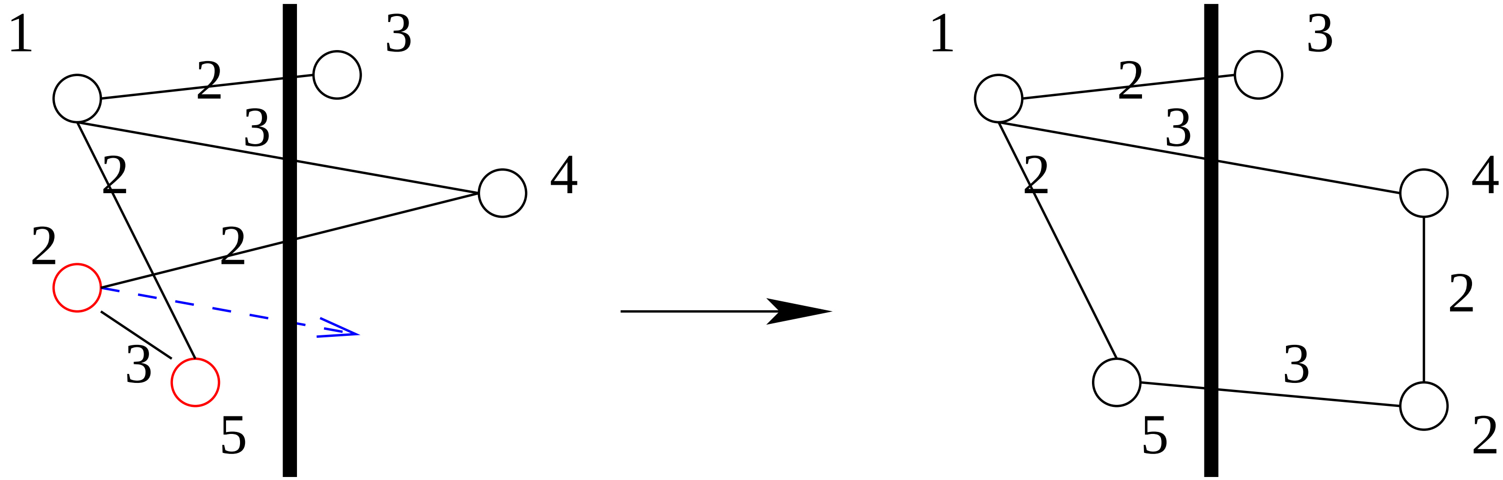
A Cut game: The Party Affiliation Game

- Cut game:
  - Players: Nodes of the graph.
  - Player's strategy $\in \{1, -1\}$ (Republican or Democrat)
  - An action profile corresponds to a cut.
  - Payoff: Total Contribution in the cut.
  - Change Party if you gain.



Cut Value: 7
2 and 5 are unhappy.

The Cut Game: Nash equilibria



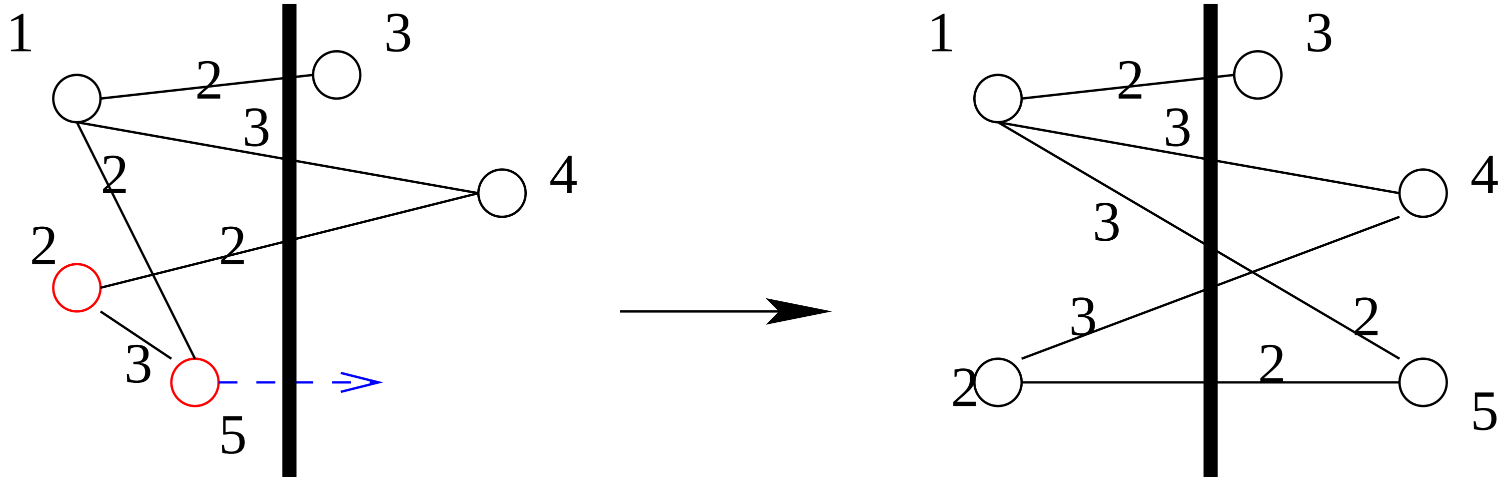
Cut Value: 7

2 and 5 are unhappy.

Cut Value: 8

Pure Nash Equilibrium.

The Cut Game: Nash equilibria



Cut Value: 7
2 and 5 are unhappy.

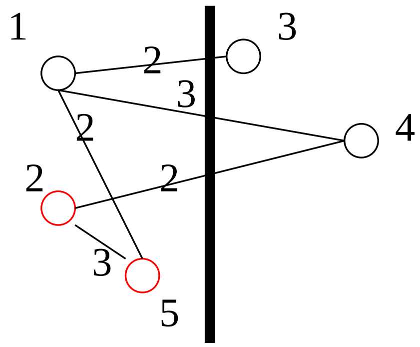
Cut Value: 12
The Optimum.

- Social Function:
  - The cut value.

Price of Anarchy for this instance: $\frac{12}{8} = 1.5$.

The Cut game

- Cut game:

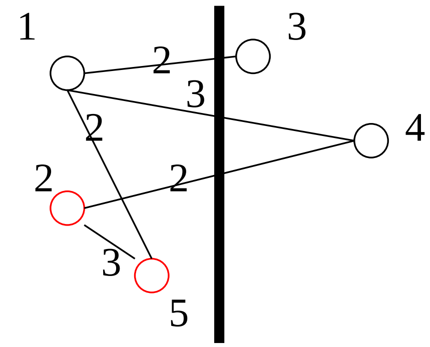


Cut Value: 7
2 and 5 are unhappy.

- Social Function:
  - The Cut Value
  - Total Happiness
- Price of anarchy: at most 2.
- Local search algorithm for Max-Cut!

The Cut game

- Cut game:



Cut Value: 7
2 and 5 are unhappy.

- Social Function:
  - The Cut Value
- Convergence:
  - Finding local optimum for Max-Cut is PLS-complete (Schaffer, Yannakakis [1991]).

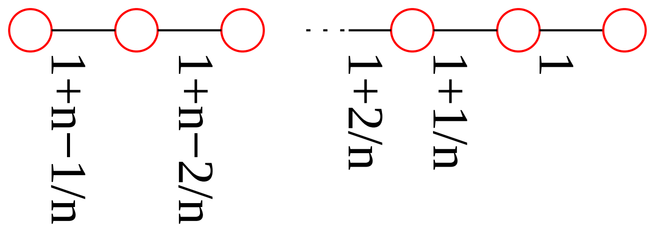
Cut Game: Walks to Nash equilibria

- Unweighted graphs After $O(n^2)$ steps, we converge to a Nash equilibrium.
- Weighted graphs: It is PLS-complete.
  - PLS-Complete problems and tight PLS reduction (Johnson, Papadimitriou, Yannakakis [1988]).
  - Tight PLS reduction from Max-Cut (Schaffer, Yannakakis [1991])
  - There are some states that are exponentially far from any Nash equilibrium.

Question: Are there long poor covering walks?

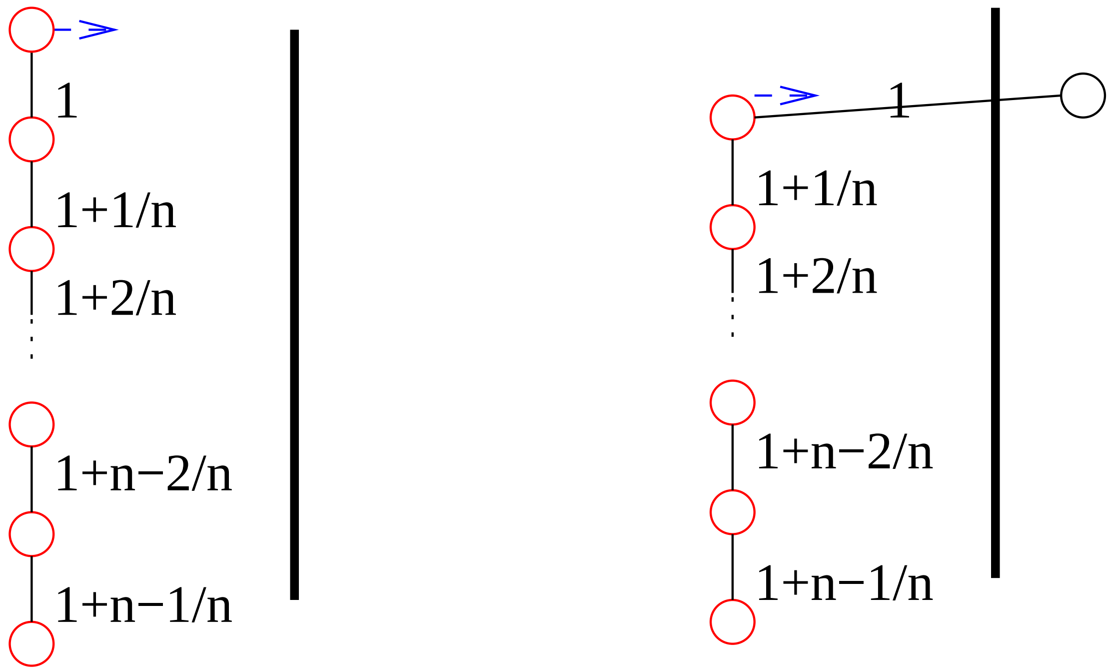
Cut Game: A Bad Example

- Consider graph G, a line of n vertices. The weight of edges are $1, 1+\frac{1}{n}, 1+\frac{2}{n}, \ldots, 1+\frac{n-1}{n}$. Vertices are labelled $1, \ldots, n$ throughout the line. Consider the round of best responses:



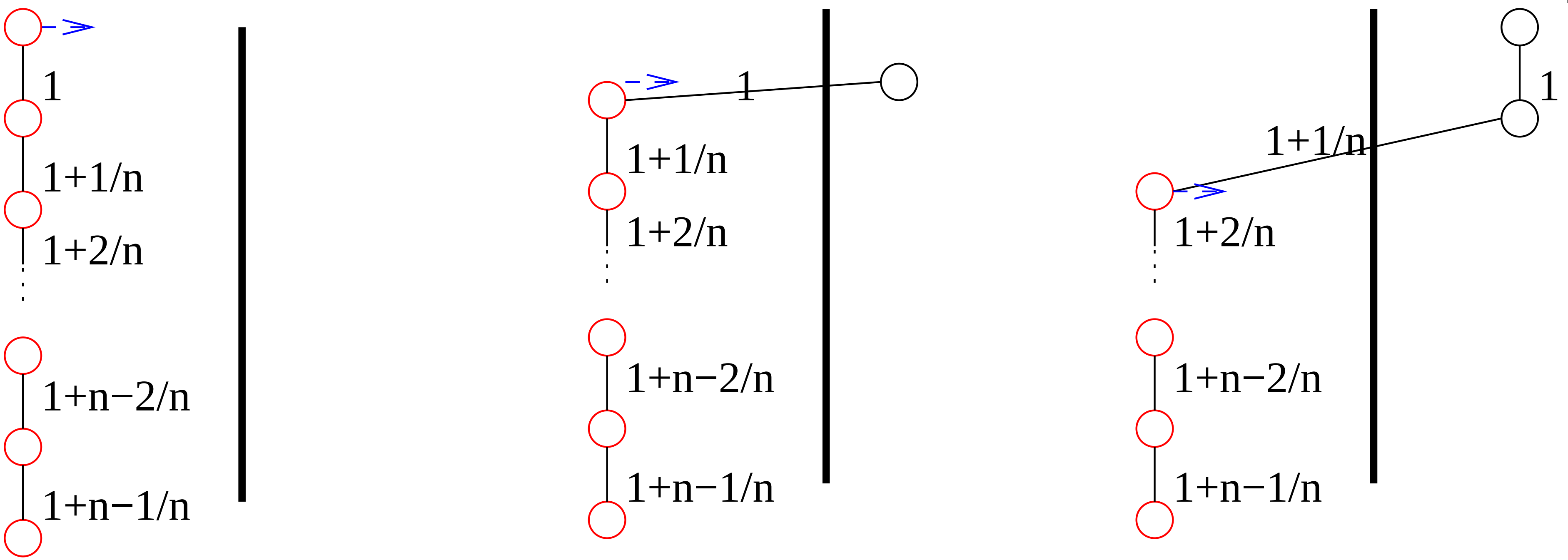
- Theorem: In the above example, the cut value after k rounds is $O(\frac{k}{n})$ of the optimum.

A Bad Example: Illustration



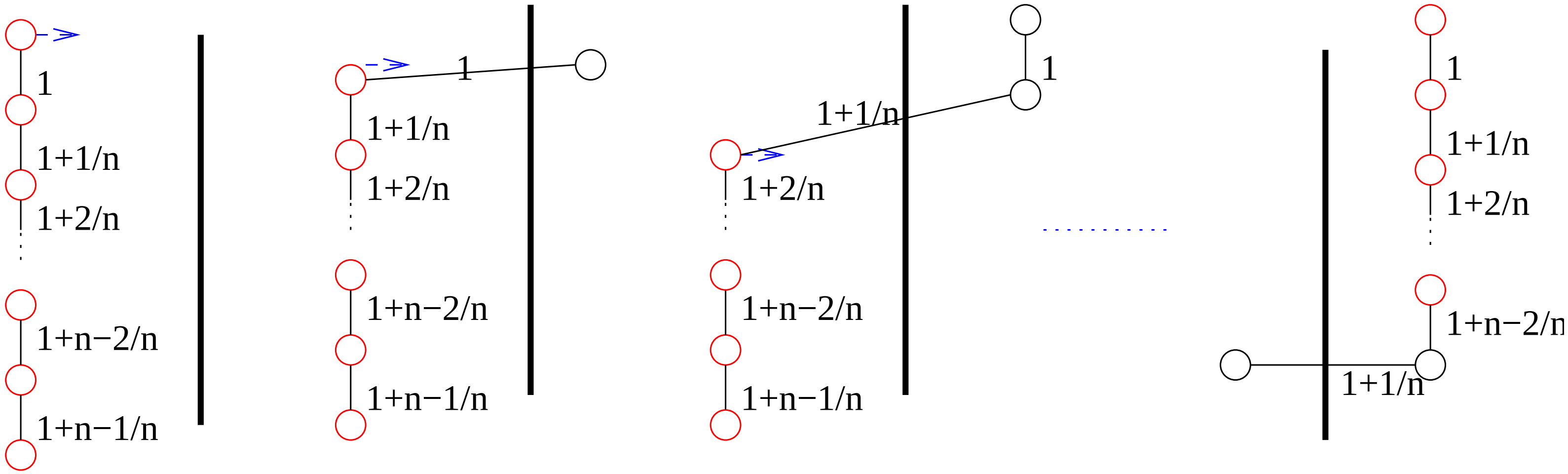
After one move.

A Bad Example: Illustration



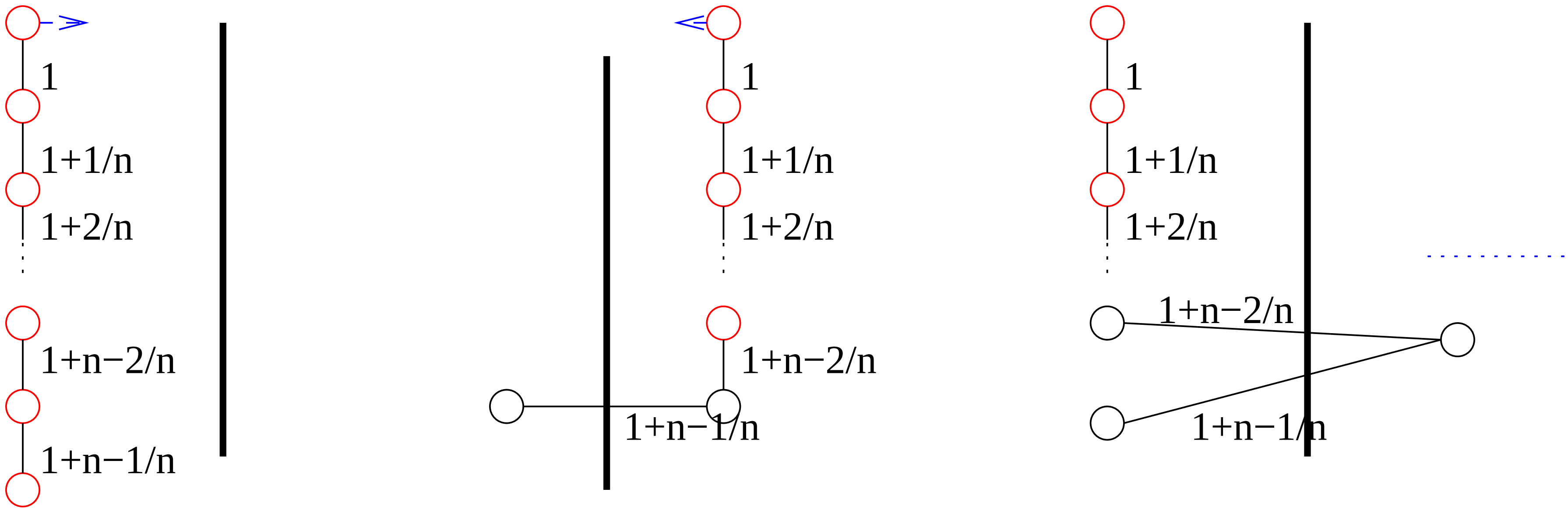
After two moves.

A Bad Example: Illustration



After n moves (one round)

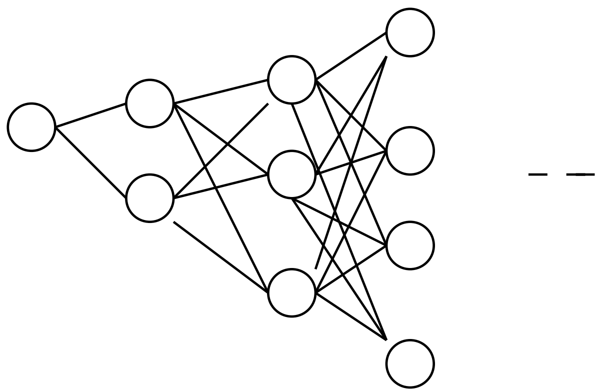
A Bad Example: Illustration



After two rounds.

Unweighted Cut Game: A Bad Example

- Let graph G be the following bipartite graph with $V(G) = \bigcup_{i=1}^{t} \bigcup_{j=1}^{i} \{\{v_{i,j}\}\}$, and $E(G) = \bigcup_{i=1}^{t-1} \bigcup_{j=1}^{i} \bigcup_{l=1}^{i+1} \{\{v_{i,j}, v_{i+1,l}\}\}$.



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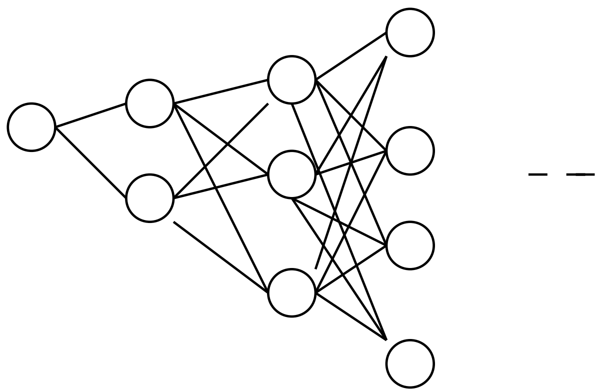

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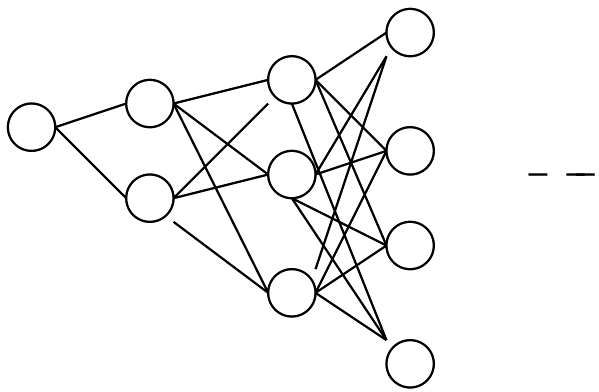

- Theorem: In the above example, the cut value after k rounds is $O(\frac{k}{\sqrt{n}})$ of the optimum.
  In unweighted graphs, the value of the cut after an $O(n)$-covering walk is a constant-factor of the optimum cut.

Random One-round walks

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- Proof Sketch: The sum of payoffs of nodes after their moves is $\frac{1}{2}$-approximation. In a random ordering, with a constant probability a node occurs after $\frac{3}{4}$ of its neighbors. The expected contribution of a node in the cut is a constant-factor of its total weight.

Exponentially Long Poor Walks

- Theorem: (M., Sidiropoulos[2004]) There exists a weighted graph $G = (V(G), E(G))$, with $|V(G)| = \Theta(n)$, and a k-covering walk $\mathcal{P}$ in the state graph, for some k exponentially large in n, such that the value of the cut at the end of $\mathcal{P}$, is at most $O(1/n)$ of the optimum cut.

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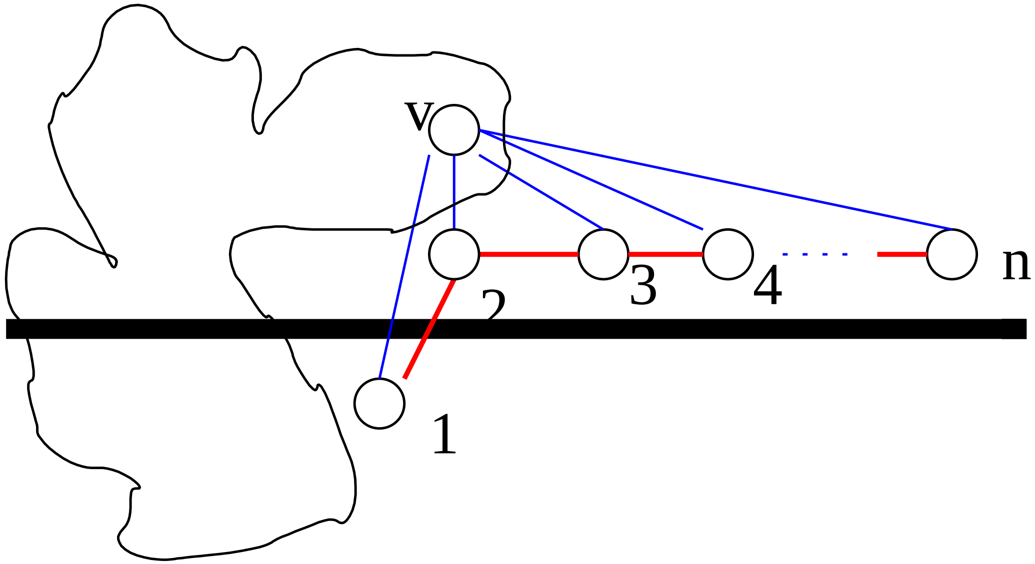
- Proof Sketch:
  Use the example for the exponentially long paths to the Nash equilibrium in the cut game. Find a player, v, that moves exponentially many times. Add a line of n vertices to this graph and connect all the vertices to player v.

Poor Long Walk: Illustration



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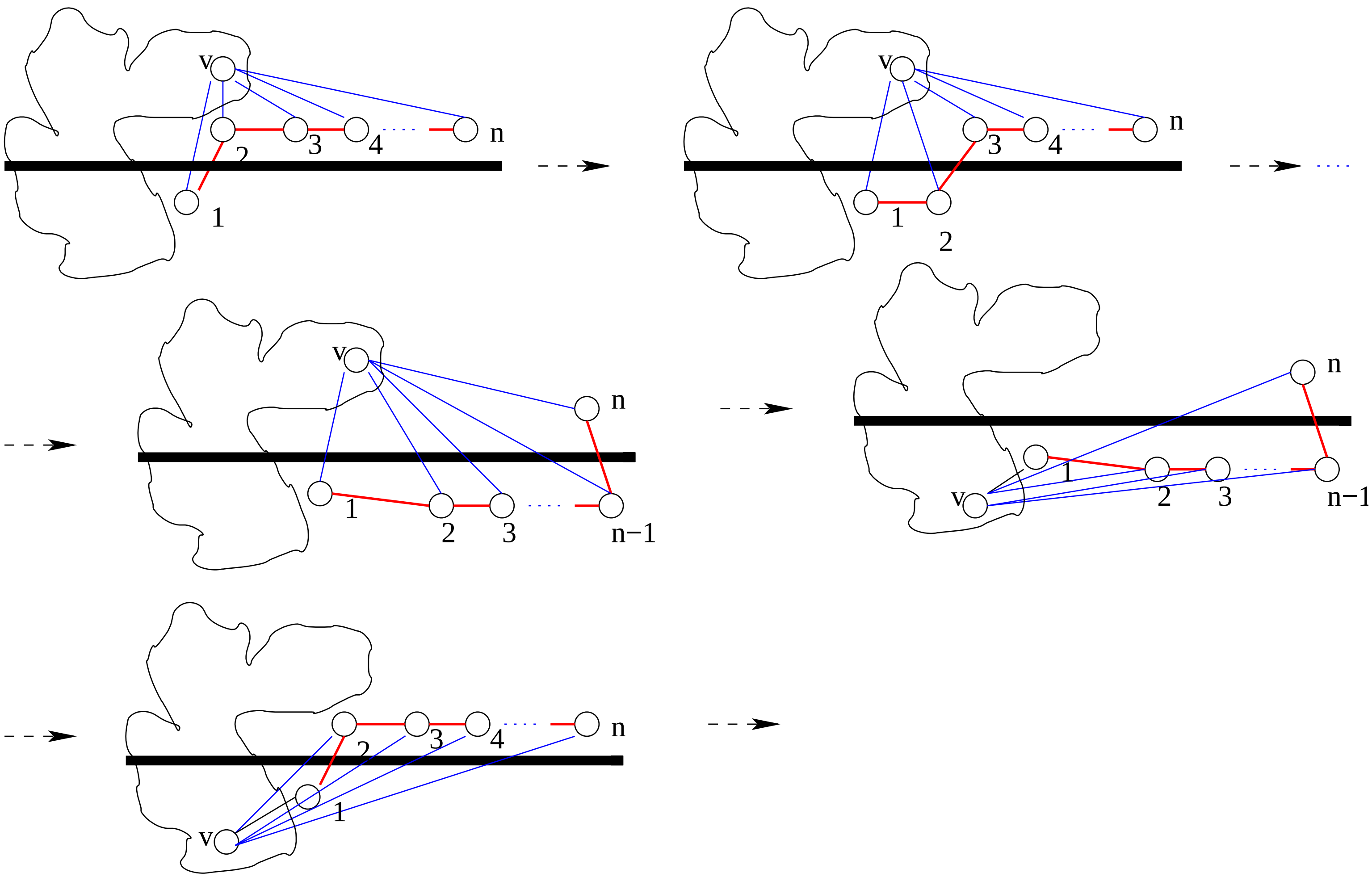

Mildly Greedy Players

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- Message: Mildly Greedy Players converge faster.

Questions

1. What do players converge to?
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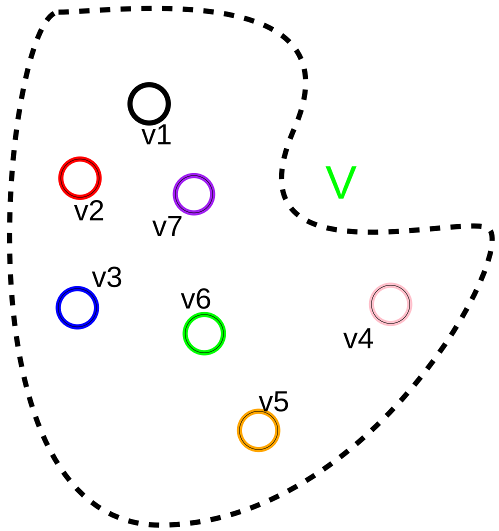
Rest of the Talk

- Valid-utility games: Price of Sinking.
- Valid-utility games: PLS-Completeness.

Valid-Utility Games

- Strategy of each player is a subset of a groundset.



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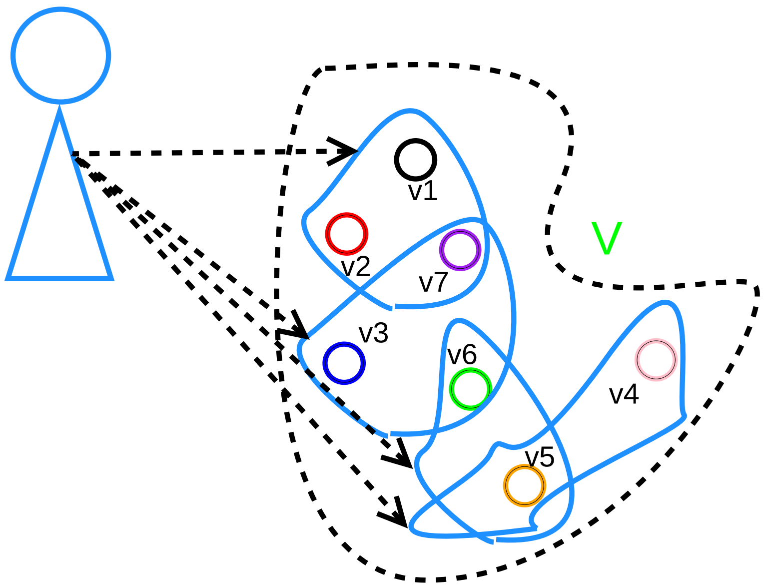

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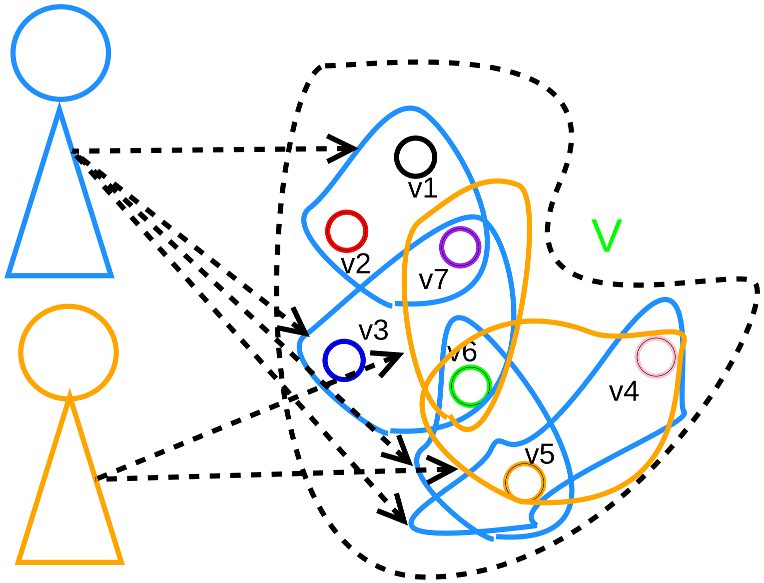

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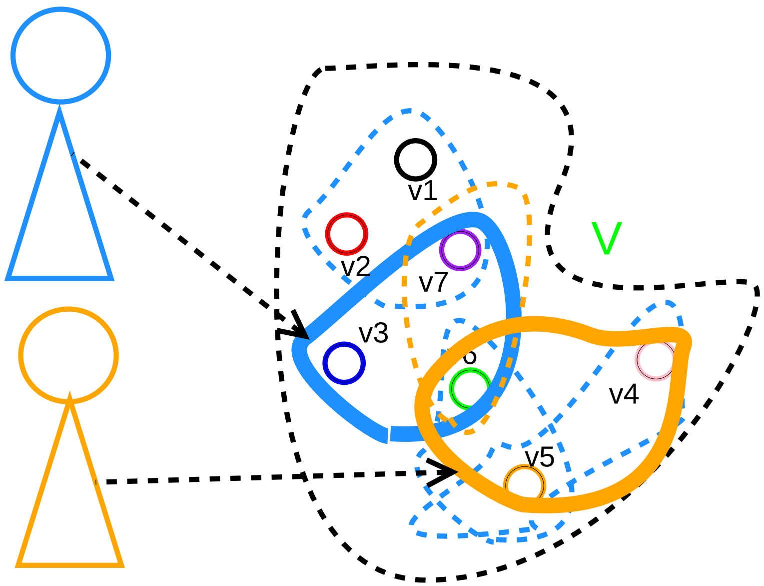

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- The payoff of any player is at least the change that he makes in the social function by playing.
- The sum of payoffs is at most the social function.
- In basic-utility games, the payoff is equal to the change that a player makes...

Examples

- Several examples, including the facility location game, market sharing games (Goemans, Li, M. Thottan), and a distributed caching game (Fleischer, Goemans, M. Sviridenko)
- Market Sharing Game (Goemans, Li, M., Thottan [2004])
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  - The value of Market is divided equally between players.

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Valid-utility Games: Price of Anarchy

- Theorem:(Vetta[2002]) The price of anarchy (of a mixed Nash equilibrium) in valid-utility games is at most 2.
- Theorem:(Vetta[2002]) Pure Nash equilibria exists for basic-utility games and Nash dynamics converges to a Nash equilibrium.

Valid-utility Games: Price of Sinking

- Theorem: The price of sinking in valid-utility games is between n and $n + 1$.

Basic-Utility Games: Convergence

- Theorem:(M.,Vetta[2004]) In basic-utility games, after one round of selfish behavior of players, they converge to a $\frac{1}{3}$-optimal solution.
- Theorem: In basic-utility games, after a random walk of length $O(n \log n)$ of best responses of players, they converge to a $\frac{1}{2} - \epsilon$-optimal solution.

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Exponential Convergence to Sink Eq.

- Theorem: Finding a state in the sink equilibrium of a valid-utility game is PLS-Complete.
- Theorem: There are states that are exponentially far from any state in a sink equilibrium.

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5. TAKE YOUR FAVORITE GAME and ANSWER THESE QUESTIONS.