Simply-typed Lambda Calculus

Todd Millstein

October 26, 2002

This document formally defines the call-by-value simply-typed lambda calculus (with booleans) and provides a proof of type soundness. It is meant only as a reference, and assumes familiarity with the basic notions involved.

1 Syntax

The metavariable x ranges over an infinite set of variable names. The metavariable e ranges over expressions (terms). The metavariable T ranges over types. The metavariable v ranges over values.

$$
\begin{array}{lcl}
e & ::= & x \mid \lambda x : T.e \mid e_1\ e_2 \\
 & & \text{true} \mid \text{false} \mid \text{if } e_1 \text{ then } e_2 \text{ else } e_3 \\
T & ::= & \text{Bool} \mid T_1 \rightarrow T_2 \\
v & ::= & \lambda x : T.e \mid \text{true} \mid \text{false}
\end{array}
$$

2 Operational Semantics

2.1 Substitution

The substitution function is defined below. We assume that renaming of bound variables is applied as necessary to make the side conditions of the third case hold.

$$
\begin{array}{lcll}
[x \mapsto e]x & = & e & \\
[x \mapsto e]x' & = & x' & \text{if } x \neq x' \\
[x \mapsto e](\lambda x' : T'.e') & = & \lambda x' : T'.[x \mapsto e]e' & \text{if } x \neq x' \text{ and } x' \text{ not free in } e \\
[x \mapsto e](e_1\ e_2) & = & [x \mapsto e]e_1\ [x \mapsto e]e_2 & \\
[x \mapsto e]\text{true} & = & \text{true} & \\
[x \mapsto e]\text{false} & = & \text{false} & \\
[x \mapsto e]\text{if } e_1 \text{ then } e_2 \text{ else } e_3 & = & \text{if } [x \mapsto e]e_1 \text{ then } [x \mapsto e]e_2 \text{ else } [x \mapsto e]e_3 &
\end{array}
$$

2.2 Inference Rules

The notation $e \longrightarrow e'$ means “expression e evaluates to e' in one step.”

$$\frac{}{(\lambda x : T.e)v \longrightarrow [x \mapsto v]e}\ \text{(E-AppRed)}$$

$$\frac{}{\text{if true then } e_2 \text{ else } e_3 \longrightarrow e_2}\ \text{(E-IfTrue)}$$

$$\frac{e_1 \longrightarrow e_1'}{e_1\ e_2 \longrightarrow e_1'\ e_2}\ \text{(E-App1)}$$

$$\frac{}{\text{if false then } e_2 \text{ else } e_3 \longrightarrow e_3}\ \text{(E-IfFalse)}$$

$$\frac{e \longrightarrow e'}{v\ e \longrightarrow v\ e'}\ \text{(E-App2)}$$

$$\frac{e_1 \longrightarrow e_1'}{\text{if } e_1 \text{ then } e_2 \text{ else } e_3 \longrightarrow \text{if } e_1' \text{ then } e_2 \text{ else } e_3}\ \text{(E-If)}$$

2.3 Stuck Expressions

An expression e is *stuck* if it is not a value but there is no e' such that $e \longrightarrow e'$. The stuck expressions can be thought of as the set of possible run-time "type" errors. The grammar of stuck expressions is as follows:

$$\begin{array}{lll} \text{stuck} & ::= & x \\ & & \text{stuck } e \mid \text{true } v \mid \text{false } v \\ & & v \text{ stuck} \\ & & \text{if stuck then } e_2 \text{ else } e_3 \\ & & \text{if } \lambda x : T.e \text{ then } e_2 \text{ else } e_3 \end{array}$$

3 Typechecking Rules

The metavariable Γ represents a *type environment*, which is a set of (variable name, type) pairs. Each pair with variable name x and type T is denoted $x : T$. We assume that a type environment has at most one pair for a given variable name; this can always be ensured via renaming of bound variables. If $\Gamma = \{x_1 : T_1, \ldots, x_n : T_n\}$, then we define dom($\Gamma$) = $\{x_1, \ldots, x_n\}$.

A judgement of the form $\Gamma \vdash e : T$ means "expression e has type T under the typing assumptions in Γ." If the Γ component is missing from a judgement, the type environment is assumed to be the empty set.

$$\frac{x : T \in \Gamma}{\Gamma \vdash x : T} \text{ (T-Var)} \qquad \frac{}{\Gamma \vdash \text{true} : \text{Bool}} \text{ (T-True)}$$

$$\frac{\Gamma \cup \{x : T_1\} \vdash e : T_2}{\Gamma \vdash (\lambda x : T_1.e) : T_1 \rightarrow T_2} \text{ (T-Abs)} \qquad \frac{}{\Gamma \vdash \text{false} : \text{Bool}} \text{ (T-False)}$$

$$\frac{\Gamma \vdash e_1 : T_2 \rightarrow T \quad \Gamma \vdash e_2 : T_2}{\Gamma \vdash e_1\ e_2 : T} \text{ (T-App)} \qquad \frac{\Gamma \vdash e_1 : \text{Bool} \quad \Gamma \vdash e_2 : T \quad \Gamma \vdash e_3 : T}{\Gamma \vdash \text{if } e_1 \text{ then } e_2 \text{ else } e_3 : T} \text{ (T-If)}$$

4 Type Soundness

Lemma (Canonical Forms):

a. If $\Gamma \vdash v : T_1 \rightarrow T_2$ then v has the form $\lambda x : T_1.e$.

b. If $\Gamma \vdash v :$ Bool then v is either true or false.

Proof: Immediate from rules T-Abs, T-True, and T-False, and the fact that no other typing rules apply to values.

Theorem (Progress): If $\vdash e : T$, then either e is a value or there exists e' such that $e \longrightarrow e'$ (equivalently, If $\vdash e : T$, then e is not stuck).
Proof: By (strong) induction on the depth of the derivation of $\vdash e : T$. Case analysis of the last rule in the derivation:

- Case T-Var: Then $e = x$ and $x : T \in \emptyset$, so we have a contradiction. Therefore, T-Var cannot be the last rule in the derivation.
- Case T-Abs: Then $e = \lambda x : T_1.e_1$, so e is a value.
- Case T-App: Then $e = e_1\ e_2$ and $\vdash e_1 : T_2 \rightarrow T$ and $\vdash e_2 : T_2$. By the inductive hypothesis, we have that either e_1 is a value or there exists e_1' such that $e_1 \longrightarrow e_1'$. Similarly, either e_2 is a value or there exists e_2' such that $e_2 \longrightarrow e_2'$. We perform a case analysis on these possibilities:

- Case there exists e_1' such that $e_1 \longrightarrow e_1'$: Then by E-App1 we have $e_1\ e_2 \longrightarrow e_1'\ e_2$.
- Case e_1 is a value v_1: There are two sub-cases.
  - Case there exists e_2' such that $e_2 \longrightarrow e_2'$: Then by E-App2 we have $v_1\ e_2 \longrightarrow v_1\ e_2'$.
  - Case e_2 is a value v_2: Since $\vdash e_1 : T_2 \to T$ and e_1 is a value v_1, by the Canonical Forms lemma we have that e_1 has the form $\lambda x : T'.e_3$. Therefore by E-AppRed we have $(\lambda x : T'.e_3)v_2 \longrightarrow [x \mapsto v_2]e_3$.

- Case T-True: Then $e =$ true, so e is a value.
- Case T-False: Then $e =$ false, so e is a value.
- Case T-If: Then $e =$ (if e_1 then e_2 else e_3) and $\vdash e_1 :$ Bool and $\vdash e_2 : T$ and $\vdash e_3 : T$. By the inductive hypothesis, we have that either e_1 is a value, or there exists e_1' such that $e_1 \longrightarrow e_1'$. In the latter case, by E-If we have that (if e_1 then e_2 else e_3) $\longrightarrow$ (if e_1' then e_2 else e_3). In the former case, by the Canonical Forms lemma we have that e_1 is either true or false. If e_1 is true, then by E-IfTrue we have that (if e_1 then e_2 else e_3) $\longrightarrow e_2$. If e_1 is false, then by E-IfFalse we have that (if e_1 then e_2 else e_3) $\longrightarrow e_3$.

Lemma (Weakening): If $\Gamma \vdash e : T$ and $x_0 \notin$ dom(Γ), then $\Gamma \cup \{x_0 : T_0\} \vdash e : T$.
Proof: By (strong) induction on the depth of the derivation of $\Gamma \vdash e : T$. Case analysis of the last rule in the derivation:

- Case T-Var: Then $e = x$ and $x : T \in \Gamma$. Since $x_0 \notin$ dom(Γ), we have that $x_0 \neq x$. Therefore $x : T \in \Gamma \cup \{x_0 : T_0\}$, so by T-Var we have $\Gamma \cup \{x_0 : T_0\} \vdash x : T$.
- Case T-Abs: Then $e = \lambda x_1 : T_1.e_2$ and $T = T_1 \to T_2$ and $\Gamma \cup \{x_1 : T_1\} \vdash e_2 : T_2$. We assume that $x_1 \neq x_0$, renaming x_1 if necessary. Since $x_0 \notin$ dom(Γ), also $x_0 \notin$ dom($\Gamma \cup \{x_1 : T_1\}$). Therefore by the inductive hypothesis we have $\Gamma \cup \{x_1 : T_1\} \cup \{x_0 : T_0\} \vdash e_2 : T_2$. So by T-Abs we have $\Gamma \cup \{x_0 : T_0\} \vdash (\lambda x_1 : T_1.e_2) : T_1 \to T_2$.
- Case T-App: Then $e = e_1\ e_2$ and $\Gamma \vdash e_1 : T_2 \to T$ and $\Gamma \vdash e_2 : T_2$. By the inductive hypothesis we have $\Gamma \cup \{x_0 : T_0\} \vdash e_1 : T_2 \to T$ and $\Gamma \cup \{x_0 : T_0\} \vdash e_2 : T_2$, so by T-App we have $\Gamma \cup \{x_0 : T_0\} \vdash e_1\ e_2 : T$.
- Case T-True: Then $e =$ true and $T =$ Bool. Therefore by T-True we have $\Gamma \cup \{x_0 : T_0\} \vdash$ true : Bool.
- Case T-False: Then $e =$ false and $T =$ Bool. Therefore by T-False we have $\Gamma \cup \{x_0 : T_0\} \vdash$ false : Bool.
- Case T-If: Then $e =$ (if e_1 then e_2 else e_3) and $\Gamma \vdash e_1 :$ Bool and $\Gamma \vdash e_2 : T$ and $\Gamma \vdash e_3 : T$. By the inductive hypothesis we have $\Gamma \cup \{x_0 : T_0\} \vdash e_1 :$ Bool and $\Gamma \cup \{x_0 : T_0\} \vdash e_2 : T$ and $\Gamma \cup \{x_0 : T_0\} \vdash e_3 : T$, so by T-If we have $\Gamma \cup \{x_0 : T_0\} \vdash$ (if e_1 then e_2 else e_3): T.

Lemma (Substitution): If $\Gamma \cup \{x : T\} \vdash e' : T'$ and $\Gamma \vdash v : T$, then $\Gamma \vdash [x \mapsto v]e' : T'$.
Proof: By (strong) induction on the depth of the derivation of $\Gamma \cup \{x : T\} \vdash e' : T'$. Case analysis of the last rule in the derivation:

- Case T-Var: Then $e' = x'$ and $x' : T' \in \Gamma \cup \{x : T\}$. There are two subcases:
  - Case $x' = x$: Then $[x \mapsto v]e' = [x \mapsto v]x = v$. Since we assume that $\Gamma \cup \{x : T\}$ has at most one element for each variable name, we have that $T' = T$. Finally, since $\Gamma \vdash v : T$, this case is proven.
  - Case $x' \neq x$: Then $[x \mapsto v]e' = x'$. Since $x' : T' \in \Gamma \cup \{x : T\}$ and $x' \neq x$, we have $x' : T' \in \Gamma$. Therefore by T-Var we have $\Gamma \vdash x' : T'$.

- Case T-Abs: Then $e' = \lambda x_0 : T_0.e_1$ and $T' = T_0 \to T_1$ and $\Gamma \cup \{x : T\} \cup \{x_0 : T_0\} \vdash e_1 : T_1$. Since $\Gamma \vdash v : T$, by Weakening (renaming x_0 if necessary) we have $\Gamma \cup \{x_0 : T_0\} \vdash v : T$, so by the inductive hypothesis we have $\Gamma \cup \{x_0 : T_0\} \vdash [x \mapsto v]e_1 : T_1$. Therefore by T-Abs we have $\Gamma \vdash \lambda x_0 : T_0.[x \mapsto v]e_1 : T_0 \to T_1$. Since we can assume that $x \neq x_0$ and x_0 not free in v, performing renaming as necessary, we have $[x \mapsto v]e' = \lambda x_0 : T_0.[x \mapsto v]e_1$, so the result follows.

- Case T-App: Then $e' = e_1\ e_2$ and $\Gamma \cup \{x : T\} \vdash e_1 : T_2 \to T'$ and $\Gamma \cup \{x : T\} \vdash e_2 : T_2$. Then by the inductive hypothesis we have $\Gamma \vdash [x \mapsto v]e_1 : T_2 \to T'$ and $\Gamma \vdash [x \mapsto v]e_2 : T_2$, so by T-App we have $\Gamma \vdash [x \mapsto v]e_1\ [x \mapsto v]e_2 : T'$. Since $[x \mapsto v](e_1\ e_2) = [x \mapsto v]e_1\ [x \mapsto v]e_2$, the result follows.

- Case T-True: Then $e' =$ true and $T' =$ Bool. Then by T-True we have $\Gamma \vdash$ true : Bool. Since $[x \mapsto v]$true = true, the result follows.

- Case T-False: Then $e' =$ false and $T' =$ Bool. Then by T-False we have $\Gamma \vdash$ false : Bool. Since $[x \mapsto v]$false = false, the result follows.

- Case T-If: Then $e' =$ (if e_1 then e_2 else e_3) and $\Gamma \cup \{x : T\} \vdash e_1$: Bool and $\Gamma \cup \{x : T\} \vdash e_2 : T'$ and $\Gamma \cup \{x : T\} \vdash e_3 : T'$. By the inductive hypothesis we have $\Gamma \vdash [x \mapsto v]e_1$: Bool and $\Gamma \vdash [x \mapsto v]e_2 : T'$ and $\Gamma \vdash [x \mapsto v]e_3 : T'$, so by T-If we have $\Gamma \vdash$ (if $[x \mapsto v]e_1$ then $[x \mapsto v]e_2$ else $[x \mapsto v]e_3$): T'. Since $[x \mapsto v]$(if e_1 then e_2 else e_3) = (if $[x \mapsto v]e_1$ then $[x \mapsto v]e_2$ else $[x \mapsto v]e_3$), the result follows.

Theorem (Type Preservation): If $\Gamma \vdash e : T$ and $e \longrightarrow e'$, then $\Gamma \vdash e' : T$.
Proof: By (strong) induction on the depth of the derivation of $\Gamma \vdash e : T$. Case analysis of the last rule in the derivation:

- Case T-Var: Then $e = x$. By inspection of the operational semantics, there is no e' such that $x \longrightarrow e'$, so this case is satisfied trivially.

- Case T-Abs: Similar to the previous case.

- Case T-App: Then $e = e_1\ e_2$ and $\Gamma \vdash e_1 : T_2 \to T$ and $\Gamma \vdash e_2 : T_2$. We're given that $e \longrightarrow e'$. Case analysis of the last rule used in the derivation of this reduction step:

  - Case E-App1: Then $e' = e_1'\ e_2$ and $e_1 \longrightarrow e_1'$. By the inductive hypothesis we have that $\Gamma \vdash e_1' : T_2 \to T$. Therefore, by T-App we have $\Gamma \vdash e_1'\ e_2 : T$.
  - Case E-App2: Then $e' = e_1\ e_2'$ and $e_2 \longrightarrow e_2'$. By the inductive hypothesis we have that $\Gamma \vdash e_2' : T_2$. Therefore, by T-App we have $\Gamma \vdash e_1\ e_2' : T$.
  - Case E-AppRed: Then $e_1 = \lambda x : T_1.e_3$ and $e_2 = v$ and $e' = [x \mapsto v]e_3$. Since $\Gamma \vdash e_1 : T_2 \to T$ and $e_1 = \lambda x : T_1.e_3$, by inspection of the typing rules we have that $T_1 = T_2$, so we have $\Gamma \vdash \lambda x : T_2.e_3 : T_2 \to T$. By inspection, this derivation must end with rule T-Abs. Therefore we have that $\Gamma \cup \{x : T_2\} \vdash e_3 : T$. Since $\Gamma \vdash e_2 : T_2$ and $e_2 = v$ we have $\Gamma \vdash v : T_2$. Therefore by the Substitution lemma we have $\Gamma \vdash [x \mapsto v]e_3 : T$.

- Case T-True: Then $e =$ true. By inspection, there is no e' such that true $\longrightarrow e'$, so this case is satisfied trivially.

- Case T-False: Similar to the previous case.

- Case T-If: Then $e =$ (if e_1 then e_2 else e_3) and $\Gamma \vdash e_1$: Bool and $\Gamma \vdash e_2 : T$ and $\Gamma \vdash e_3 : T$. We're given that $e \longrightarrow e'$. Case analysis of the last rule used in the derivation of this reduction step:

  - Case E-IfTrue: Then $e' = e_2$, so we have $\Gamma \vdash e' : T$.
  - Case E-IfFalse: Then $e' = e_3$, so we have $\Gamma \vdash e' : T$.

– Case E-If: Then (if e_1 then e_2 else e_3) $\longrightarrow$ (if e_1' then e_2 else e_3), where $e_1 \longrightarrow e_1'$. By the inductive hypothesis we have $\Gamma \vdash e_1'$: Bool. Therefore by T-If we have $\Gamma \vdash$ (if e_1' then e_2 else e_3): T.

Theorem (Type Soundness #1): If $\vdash e : T$ then either e is a value or there exists e' such that $e \longrightarrow e'$ and $\vdash e' : T$.
Proof: Since $\vdash e : T$, by Progress either e is a value or there exists e' such that $e \longrightarrow e'$. In the latter case, by Type Preservation we have $\vdash e' : T$.

Let $\xrightarrow{*}$ denote the reflexive, transitive closure of the $\longrightarrow$ relation.
Corollary (Type Soundness #2): If $\vdash e : T$ and the evaluation of e terminates, then there exists v such that $e \xrightarrow{*} v$ and $\vdash v : T$.
Proof: Since $\vdash e : T$, by Type Soundness #1 we have that either e is a value or there exists e' such that $e \longrightarrow e'$ and $\vdash e' : T$. Since the evaluation of e terminates, some evaluation of e has finite length (number of reduction steps). We prove this corollary by induction on the length of this evaluation of e.

- Case length = 0: Then there does not exist e' such that $e \longrightarrow e'$, so e must be a value. Therefore, this case is proven by taking $v = e$.

- Case length = n, where $n > 0$: Then there is at least one reduction step in the evaluation, so e is not a value. Therefore there exists e' such that $e \longrightarrow e'$ and $\vdash e' : T$. Since the evaluation of e terminates, so does the evaluation of e'. Further, the evaluation of e' has length $n - 1$. Therefore, by the inductive hypothesis we have that there exists v such that $e' \xrightarrow{*} v$ and $\vdash v : T$. Since $e \longrightarrow e'$ and $e' \xrightarrow{*} v$, we have $e \xrightarrow{*} v$.