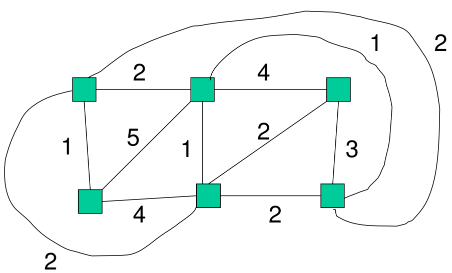
CSEP 521
Applied Algorithms
Autumn 2009

Traveling Salesman Problem

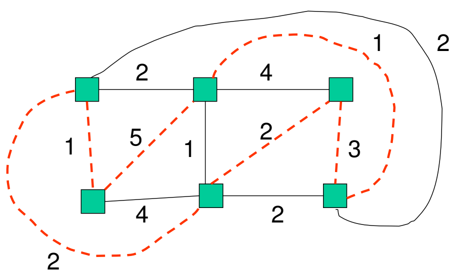
Traveling Salesman Problem

- Input: Undirected Graph G = (V,E) and a cost function C from E to the reals. C(e) is the cost of edge e.
- Output: A cycle that visits each vertex exactly once and is minimum total cost.

Example



Example



Cost = 1 + 5 + 1 + 3 + 2 + 2 = 14

Variations

- Hamiltonian Cycle
  - Is there a cycle that visits each vertex exactly once
  - Ignores costs
- Triangle inequality constraint
  - $C(u,v) \leq C(u,x) + C(x,v)$
- Euclidean Traveling Salesman
  - Vertices are points on the plane and the cost is the Euclidian distance between them
  - Implies triangle inequality

Applications

- Telescope planning
- Route planning
  - coin pickup
  - mail delivery
  - book order pickup in the Amazon warehouse
- Circuit board drilling

Why Traveling Salesman?

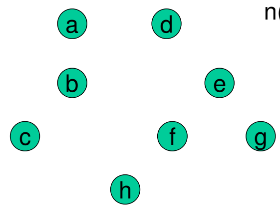
- Old well-studied problem
- Example of an NP-hard problem
  - These problems are very hard to solve exactly
  - No polynomial time algorithms known to exist
- Interesting and effective approximation algorithms
  - Good practical algorithms
  - Simple algorithms with provable approximation bounds

Approximation Alg. vs. Heuristic

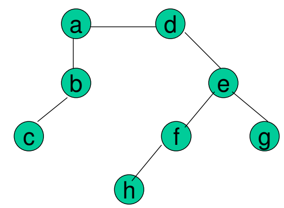
- Approximation Algorithm
  - There is a provable guarantee of how close the algorithm's result is to the optimal solution.
- Heuristic
  - The algorithm finds a solutions but there is no guarantee how good the solution is.
  - Heuristics often outperform provable approximation algorithms.

A Simple Approximation Algorithm

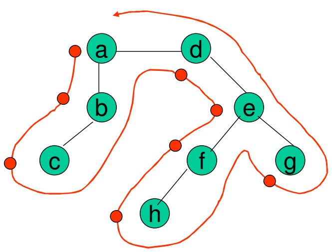
Euclidean distance
n(n-1)/2 edges



1. Find a Minimum Spanning Tree

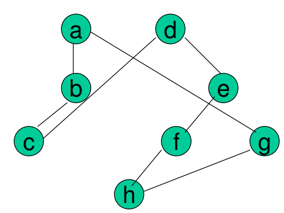


2. Depth-First Search of Tree



Marking Order = a, b, c, d, e, f, h, g

3. Connect Vertices in Marking Order



Marking Order = a, b, c, d, e, f, h, g

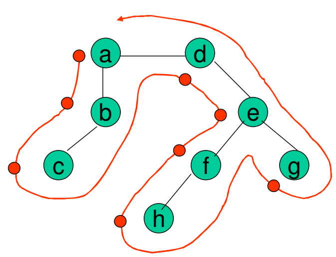
Evaluation

- Time and Storage
  - Time $O(n^2 \log n)$ with Kruskal's Algorithm
  - Storage $O(n^2)$
- Quality of Solution H
  - $C(H) \leq 2\,C(H^*)$ where H* is an optimal tour
  - This is a "2-approximation algorithm"
- Same approximation bound applies to case of triangle inequality

Proof of Approximation Bound

- Setup
  - T minimum spanning tree
  - W the depth-first walk of T
  - H the tour computed by the algorithms
  - H* an optimal tour

Depth-First Walk



$C(W) = 2\,C(T)$
$C(H) \leq C(W)$
triangle inequality

```
Depth-first walk = a,b,c,b,a,d,e,f,h,f,e,g,e,d,a
   Marking order = a,b,c,    d,e,f,h,     g
```


Proof of Approximation Bound

1. $C(W) = 2\,C(T)$
2. $C(H) \leq C(W)$, triangle inquality
3. $C(H) \leq 2\,C(T)$, last two lines
4. $C(T) \leq C(H^*)$, minus an edge H* is a spanning tree
5. $C(H) \leq 2\,C(H^*)$, last two lines

Solving TSP Exactly

- Branch-and-Bound
  - $n < 25$?
- Linear Programming
  - $n < 100$
- Cutting Plane Methods for Euclidian case
  - $n < 15{,}000$ with "concord"
  - see http://www.math.princeton.edu/tsp/

Solving TSP Approximately

- 3/2 – approximation algorithm of Christofedes
- Polynomial approximation scheme for Euclidian TSP by Aurora (1998), Mitchell (1999)
  - To get within $(1+\varepsilon)$ of optimal can be done in time polynomial in $1/\varepsilon$ and n.
  - These are not practical

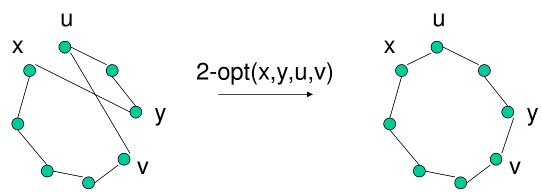
Solving TSP Approximately, Practically

- Local Search
  - Lin-Kernighan method
- Simulated Annealing
- Genetic Algorithms
- Neural Networks

Local Search Algorithms

- Start with an initial solution that is usually easy to find, but is not necessarily good.
- Repeatedly modify the current solution to a better nearby one. Until no nearby one is better.

2-Opt Neighborhood



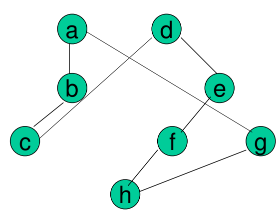
2-opt Algorithm

Lin-Kernighan (1973)

Find an initial tour T
1. For every pair of distinct edges (x,y), (u,v) in T
   if C(x,u) + C(y,v) < C(x,y) + C(u,v) then
      T := T – {(x,y),(u,v)} union {(x,u),(y,v)}
      exit for loop and go to 1
Return T

Example of LK

Euclidian case



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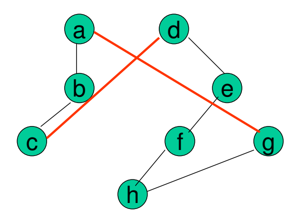
Euclidian case



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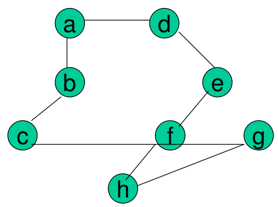

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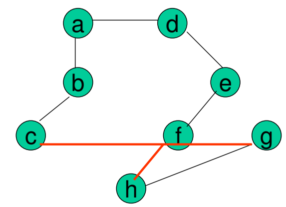

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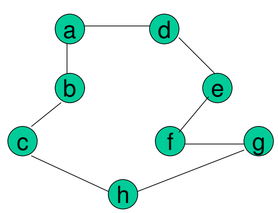

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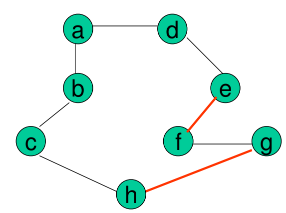

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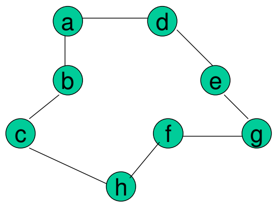

Lin-Kernighan

- Empirical $O(n^{2.2})$ time
- Finds optimal in most examples < 100 points
- Excellent Implementations
  - Can easily handle hundreds of thousands of points