CSEP 521
Applied Algorithms

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Lecture 7

Dynamic Programming

Announcements

- Reading for this week
  - 6.1-6.8

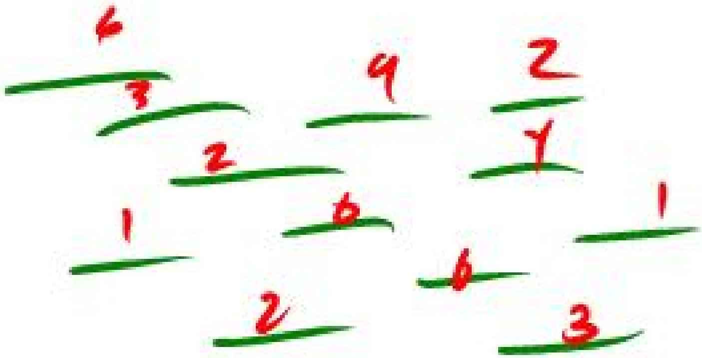
Midterm Return – Mean ~ 37

Review from last week

$I_1, \dots I_n$

Weighted Interval Scheduling

v_j



$Opt[j]$ – max value solution from $I_1 \dots I_j$

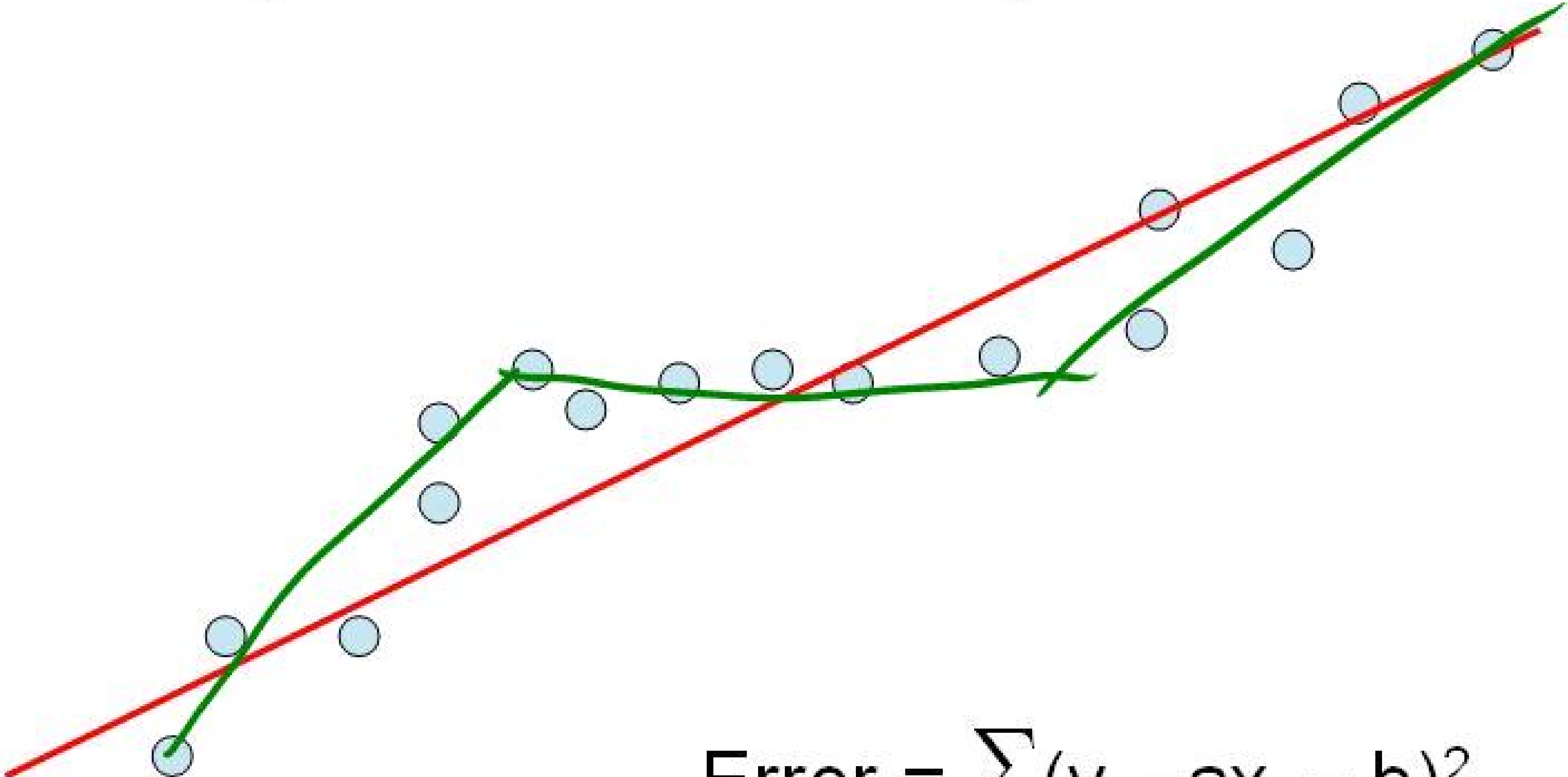
$Opt[j] = \max(Opt[j-1], v_j + Opt[j-p[j]])$

Case analysis

I_j is used

I_j is not used.

Optimal linear interpolation



$$\text{Error} = \sum (y_i - ax_i - b)^2$$

Subset Sum Problem

- Let $w_1, \ldots, w_n$ = {6, 8, 9, 11, 13, 16, 18, 24}
- Find a subset that has as large a sum as possible, without exceeding 50

$S[j, K]$ - subset of $w_1 \cdots w_j$ sums to exactly K

$$S[j, K] = S[j-1, K] \text{ OR } S[j-1, K - w_j]$$

Counting electoral votes

$$c[j,k] = c[j-1,k] + c[j-1, k - v_j]$$

$$c[51, 269]$$

$$c[0, x] = 0$$

$$c[0, 0] = 1$$

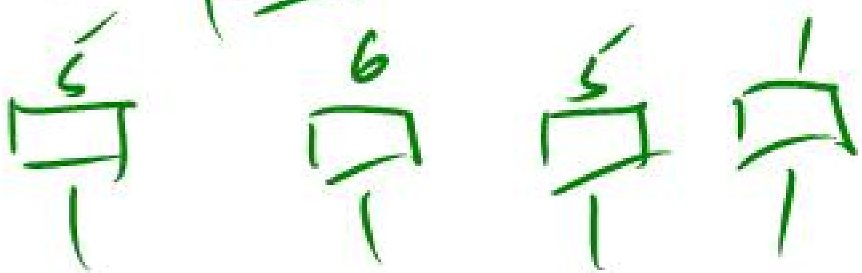
$$c[x, 0] = 1$$

Dynamic Programming Examples

- Examples
  - Optimal Billboard Placement
    - Text, Solved Exercise, Pg 307
  - Linebreaking with hyphenation
    - Compare with HW problem 6, Pg 317
  - String approximation
    - Text, Solved Exercise, Page 309

Billboard Placement

- Maximize income in placing billboards
  - $b_i = (p_i, v_i)$, v_i: value of placing billboard at position p_i
- Constraint:
  - At most one billboard every five miles
- Example
  - {(6,5), (8,6), (12, 5), (14, 1)}



Design a Dynamic Programming Algorithm for Billboard Placement

- Compute Opt[1], Opt[2], . . ., Opt[n]
- What is Opt[k]?

Input $b_1, \ldots, b_n$, where $b_i = (p_i, v_i)$, position and value of billboard i

Optimal line breaking and hyphen-ation

- Problem: break lines and insert hyphens to make lines as balanced as possible
- Typographical considerations:
  - Avoid excessive white space
  - Limit number of hyphens
  - Avoid widows and orphans
  - Etc.

Penalty Function

- Pen(i, j) – penalty of starting a line a position i, and ending at position j

Opt-i-mal line break-ing and hyph-en-a-tion is com-put-ed with dy-nam-ic pro-gram-ming

- Key technical idea
  - Number the breaks between words/syllables

Longest Common Subsequence

Longest Common Subsequence

- $C=c_1 \ldots c_g$ is a subsequence of $A=a_1 \ldots a_m$ if C can be obtained by removing elements from A (but retaining order)
- LCS(A, B): A maximum length sequence that is a subsequence of both A and B

```
ocurranec                     attacggct

occurrence                    tacgacca
```

ocurrnc

Determine the LCS of the following strings

BARTHOLEMEWSIMPSON

KRUSTYTHECLOWN

RTHOWN

String Alignment Problem

- Align sequences with gaps

```
CAT TGA  AT

CAGAT AGGA
```

- Charge δ_x if character x is unmatched
- Charge γ_{xy} if character x is matched to character y

Note: the problem is often expressed as a minimization problem, with $\gamma_{xx} = 0$ and $\delta_x > 0$

LCS Optimization

- $A = a_1a_2\ldots a_m$
- $B = b_1b_2\ldots b_n$

- Opt[j, k] is the length of $LCS(a_1a_2\ldots a_j, b_1b_2\ldots b_k)$

Express Opt[j, k] in terms
of Opt[j-1, k], Opt[j-1, k-1], Opt[j, k-1]

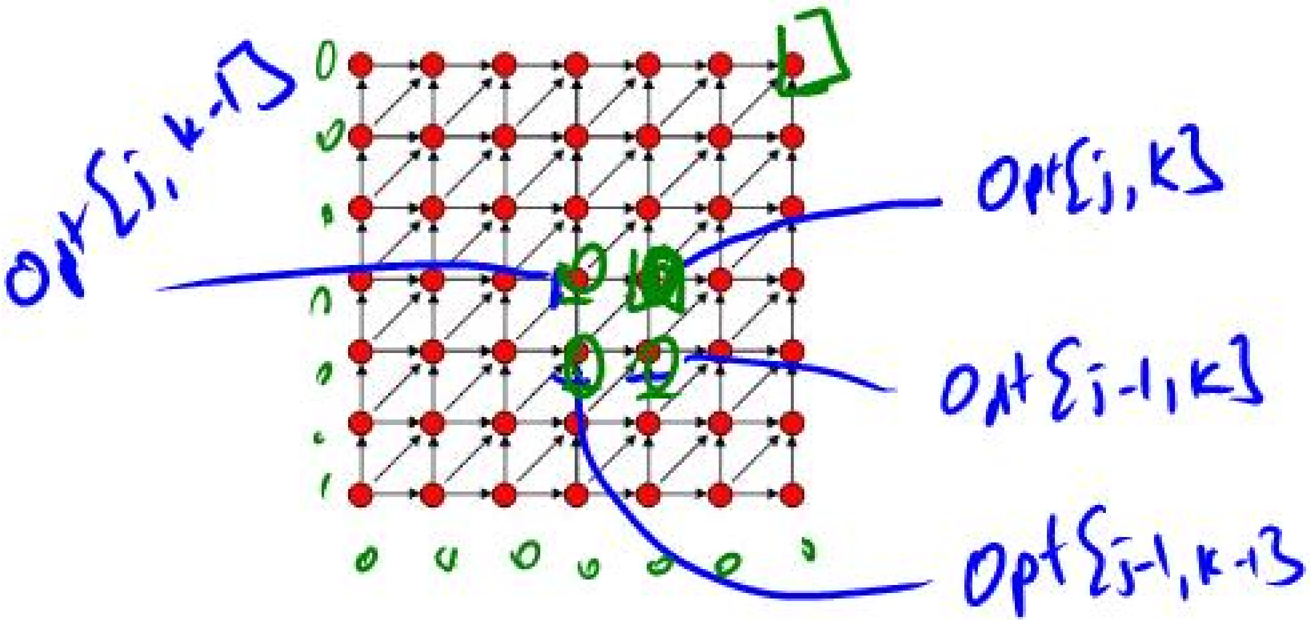
Optimization recurrence

If $a_j = b_k$, $Opt[j,k] = 1 + Opt[j-1, k-1]$

If $a_j \neq b_k$, $Opt[j,k] = \max(Opt[j-1,k], Opt[j,k-1])$

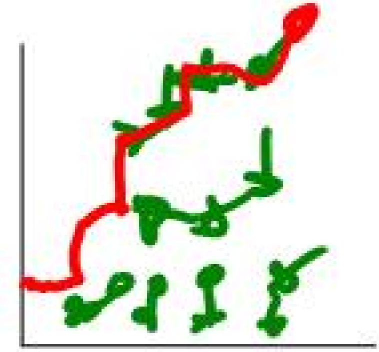
$Opt[n,m]$

Dynamic Programming Computation



for i = 1 to n
    for j = 1 to m
        Opt[i, j] = ...

Storing the path information



$b_1 \ldots b_n$

$a_1 \ldots a_m$

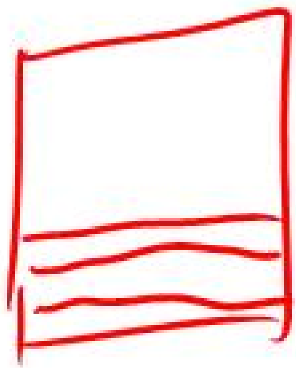
```
A[1..m],  B[1..n]
for i := 1 to m     Opt[i, 0] := 0;
for j := 1 to n     Opt[0,j] := 0;
Opt[0,0] := 0;
for i := 1 to m
        for j := 1 to n
                if A[i] = B[j]  {  Opt[i,j] := 1 + Opt[i-1,j-1];  Best[i,j] := Diag; }
                else if Opt[i-1, j] >= Opt[i, j-1]
                        {  Opt[i, j] := Opt[i-1, j], Best[i,j] := Left; }
                else    {  Opt[i, j] := Opt[i, j-1], Best[i,j] := Down; }
```

How good is this algorithm?

- Is it feasible to compute the LCS of two strings of length 100,000 on a standard desktop PC? Why or why not.

100,000 × 100,000

10,000,000,000



Observations about the Algorithm

- The computation can be done in O(m+n) space if we only need one column of the Opt values or Best Values

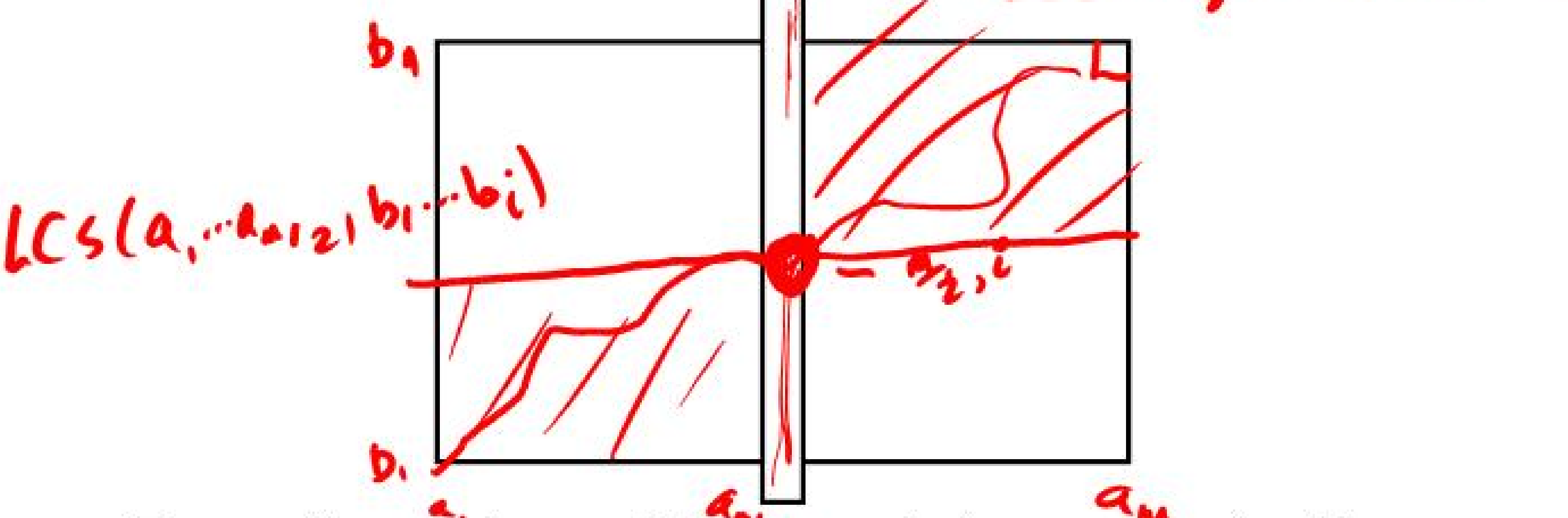
- The algorithm can be run from either end of the strings

Computing LCS in O(nm) time and O(n+m) space

- Divide and conquer algorithm
- Recomputing values used to save space

Divide and Conquer Algorithm

- Where does the best path cross the middle column?



- For a fixed i, and for each j, compute the LCS that has a_i matched with b_j

Constrained LCS

- $LCS_{i,j}(A,B)$: The LCS such that
  - $a_1,\ldots,a_i$ paired with elements of $b_1,\ldots,b_j$
  - $a_{i+1},\ldots a_m$ paired with elements of $b_{j+1},\ldots,b_n$

- $LCS_{4,3}$(abbacbb, cbbaa)

abba cbb

cbb aa

A = RRSSRTTRTS

B=RTSRRSTST

1

Compute $\underbrace{LCS_{5,0}(A,B)}_{\cancel{3}\ 4}$, $\underbrace{LCS_{5,1}(A,B)}_{1+4}$, …, $LCS_{5,9}(A,B)$

A = RRSSRTTRTS

B=RTSRRSTST

Compute $LCS_{5,0}(A,B)$, $LCS_{5,1}(A,B)$, …, $LCS_{5,9}(A,B)$

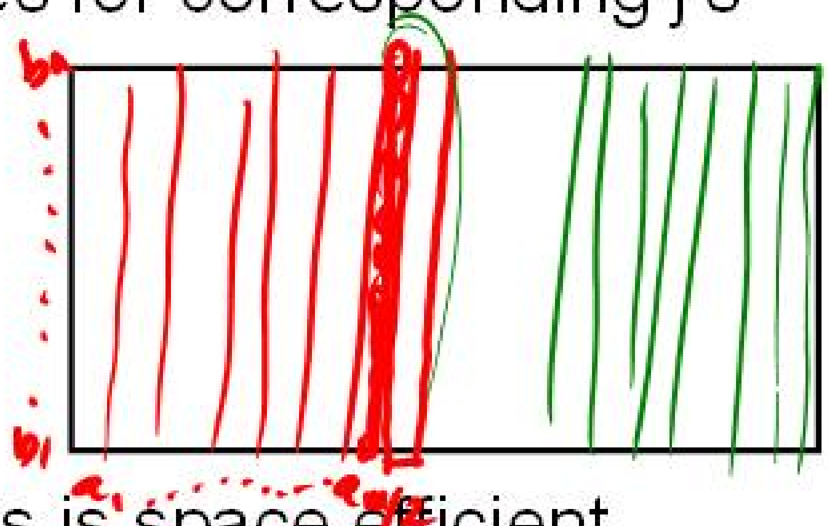
j	left	right	
0	0	4	4
1	1	4	5
2	1	3	4
3	2	3	5
4	3	3	4
5	3	2	5
6	3	2	5
7	3	1	4
8	4	1	5
9	4	0	4

RRSSR TTRTS

RTSR RSTST

Computing the middle column

- From the left, compute LCS($a_1 \ldots a_{m/2}, b_1 \ldots b_j$)
- From the right, compute LCS($a_{m/2+1} \ldots a_m, b_{j+1} \ldots b_n$)
- Add values for corresponding j's



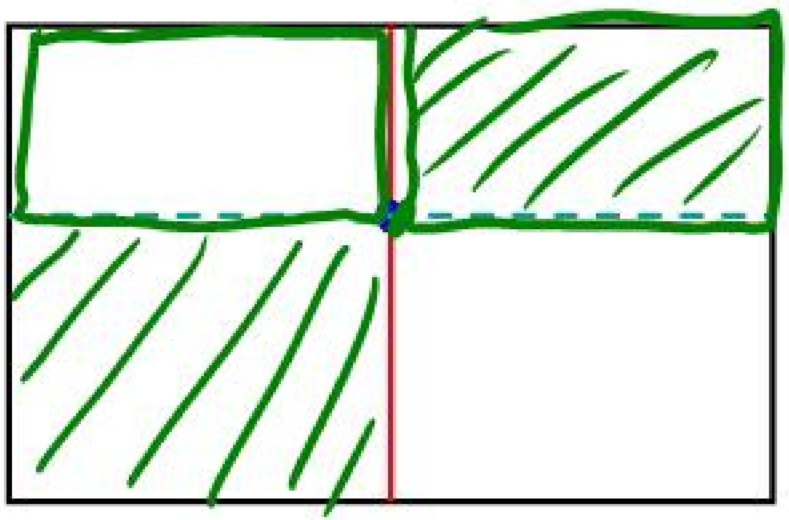
- Note – this is space efficient

Divide and Conquer

- $A = a_1,\ldots,a_m$ $\qquad B = b_1,\ldots,b_n$
- Find j such that
  - LCS($a_1 \ldots a_{m/2}$, $b_1 \ldots b_j$) and
  - LCS($a_{m/2+1} \ldots a_m$, $b_{j+1} \ldots b_n$) yield optimal solution
- Recurse

Algorithm Analysis

- $T(m,n) = T(m/2, j) + T(m/2, n-j) + cnm$



Prove by induction that $T(m,n) <= 2cmn$

Induction - on m

Base case - $m = 1$

Assume $T(k,n) \le 2ckn$ for $k < m$

$$T(m,n) = T\left(\frac{m}{2}, j\right) + T\left(\frac{m}{2}, n-j\right) + cmn$$

$$\le 2c\frac{m}{2}j + 2c\frac{m}{2}(n-j) + cmn$$

$$= 2c\frac{m}{2}(j + (n-j)) + cmn$$

$$= cmn + cmn = 2cmn$$

Memory Efficient LCS Summary

- We can afford O(nm) time, but we can't afford O(nm) space
- If we only want to compute the length of the LCS, we can easily reduce space to O(n+m)
- Avoid storing the value by recomputing values
  - Divide and conquer used to reduce problem sizes

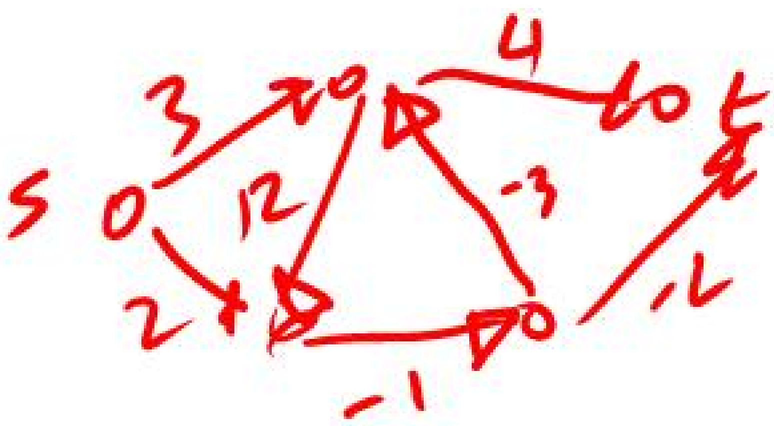
Shortest Paths with Dynamic Programming

Shortest Path Problem

- Dijkstra's Single Source Shortest Paths Algorithm
  - O(mlog n) time, positive cost edges
- General case – handling negative edges
- If there exists a negative cost cycle, the shortest path is not defined
- Bellman-Ford Algorithm
  - O(mn) time for graphs with negative cost edges

Lemma

- If a graph has no negative cost cycles, then the **shortest** paths are **simple** paths
- Shortest paths have at most n-1 edges



Shortest paths with a fixed number of edges

- Find the shortest path from v to w with exactly k edges

Express as a recurrence

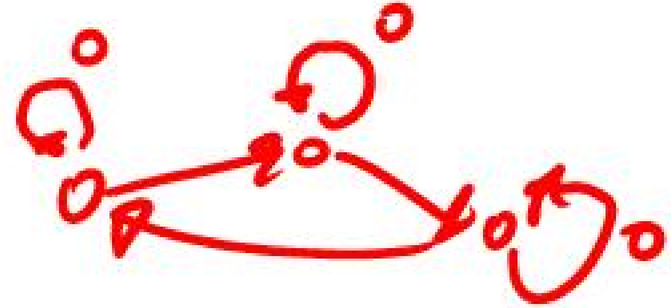
- $Opt_k(w) = \min_x [Opt_{k-1}(x) + c_{xw}]$
- $Opt_0(w) = 0$ if $v=w$ and infinity otherwise

Find shortest path distance
from v to w.

Algorithm, Version 1

```
foreach w
        M[0, w] = infinity;
M[0, v] = 0;
for i = 1 to n-1
        foreach w
                M[i, w] = min_x(M[i-1,x] + cost[x,w]);
```

Algorithm, Version 2



```
foreach w
        M[0, w] = infinity;
M[0, v] = 0;
for i = 1 to n-1
        foreach w
                M[i, w] = min(M[i-1, w], min_x(M[i-1,x] + cost[x,w]))
```

Bell-Ford

Algorithm, Version 3

```
foreach w
        M[w] = infinity;
M[v] = 0;
for i = 1 to n-1
        foreach w
                M[w] = min(M[w], min_x(M[x] + cost[x,w]))
```

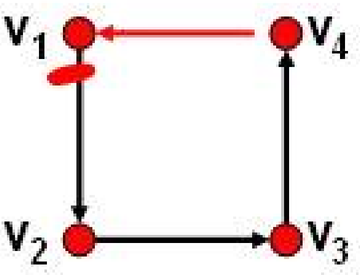
Correctness Proof for Algorithm 3

- Key lemma – at the end of iteration i, for all w, M[w] <= M[i, w];

- Reconstructing the path:
  - Set P[w] = x, whenever M[w] is updated from vertex x

If the pointer graph has a cycle, then the graph has a negative cost cycle

- If P[w] = x then M[w] >= M[x] + cost(x,w)
  - Equal when w is updated
  - M[x] could be reduced after update
- Let $v_1, v_2, \ldots v_k$ be a cycle in the pointer graph with (v_k, v_1) the last edge added
  - Just before the update
    - $M[v_j] >= M[v_{j+1}] + cost(v_{j+1}, v_j)$ for $j < k$
    - $M[v_k] > M[v_1] + cost(v_1, v_k)$
  - Adding everything up
    - $0 > cost(v_1, v_2) + cost(v_2, v_3) + \ldots + cost(v_k, v_1)$

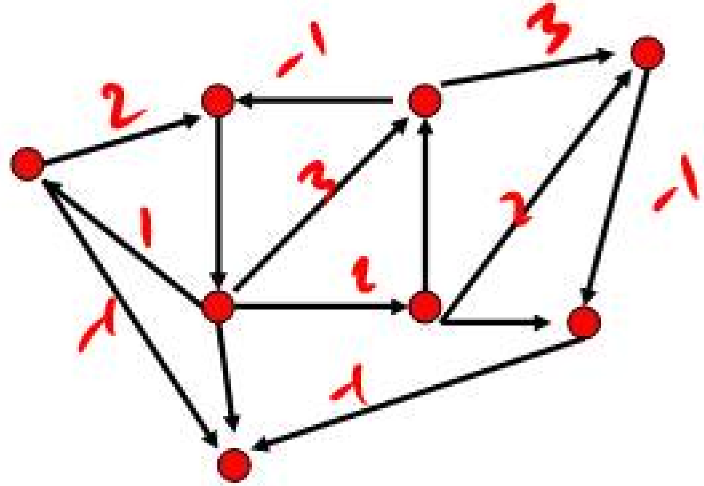


Negative Cycles

- If the pointer graph has a cycle, then the graph has a negative cycle
- Therefore: if the graph has no negative cycles, then the pointer graph has no negative cycles

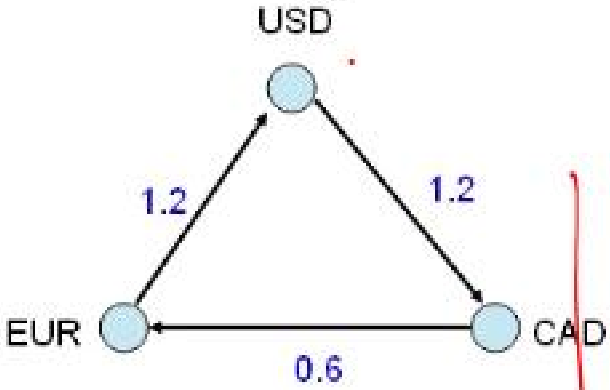
Finding negative cost cycles

- What if you want to find negative cost cycles?

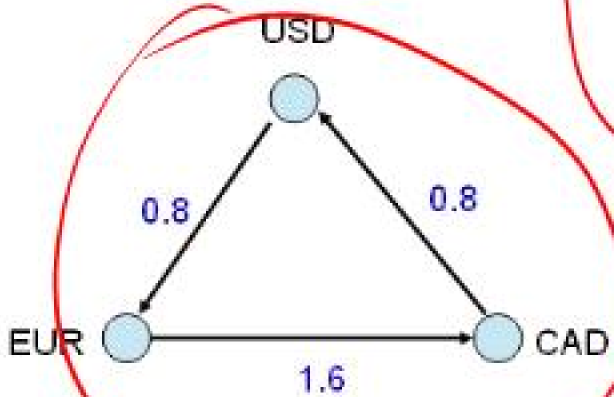


Foreign Exchange Arbitrage

$1.2 \times .6 \times 1.2$
$= .864$



	USD	EUR	CAD
USD	------	0.8	1.2
EUR	1.2	------	1.6
CAD	0.8	0.6	-----



$.8 \times 1.6 \times .8 = 1.024$