CSEP 521
Applied Algorithms

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Lecture 8

Network Flow

Announcements

- Reading for this week
  - 6.8, 7.1, 7.2 [7.3-7.4 will not be covered]
  - Next week: 7.5-7.12 Application of Network Flow.
- Final exam, March 18, 6:30 pm. At UW.
  - 2 hours
  - In class (CSE 303 / CSE 305)
  - Comprehensive
    - 67% post midterm, 33% pre midterm

Bellman-Ford Shortest Paths Algorithm

- Computes shortest paths from a starting vertex
- Allows negative cost edges
  - Negative cost cycles identified
- Runtime O(nm) Dijkstra O(m log n)
- Easy to code

Bellman Ford Algorithm, Version 2

foreach w

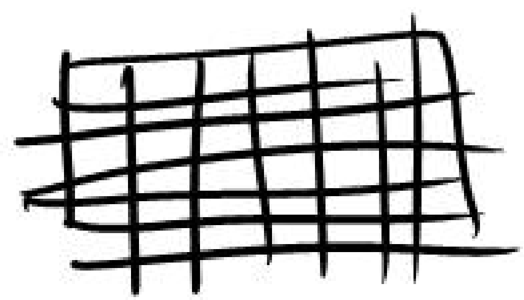
M[0, w] = infinity;

M[0, v] = 0;

for i = 1 to n-1

foreach w

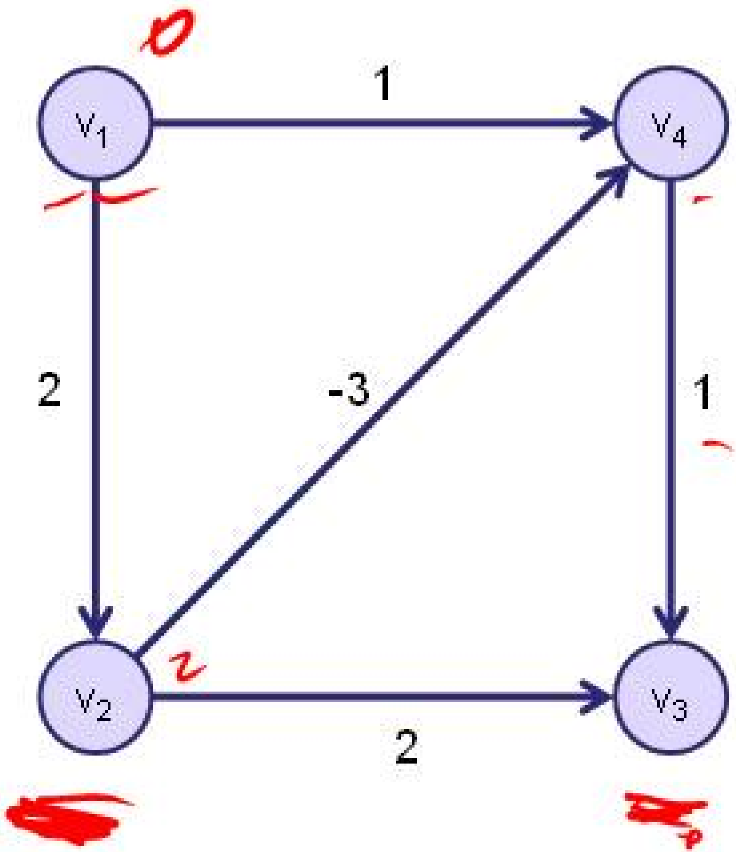
$M[i, w] = \min(M[i-1, w], \min_x(M[i-1,x] + cost[x,w]))$



Bellman Ford Algorithm, Version 3

```
foreach w
        M[w] = infinity;
M[v] = 0;
for i = 1 to n-1
        foreach w
                M[w] = min(M[w], min_x(M[x] + cost[x,w]))
```

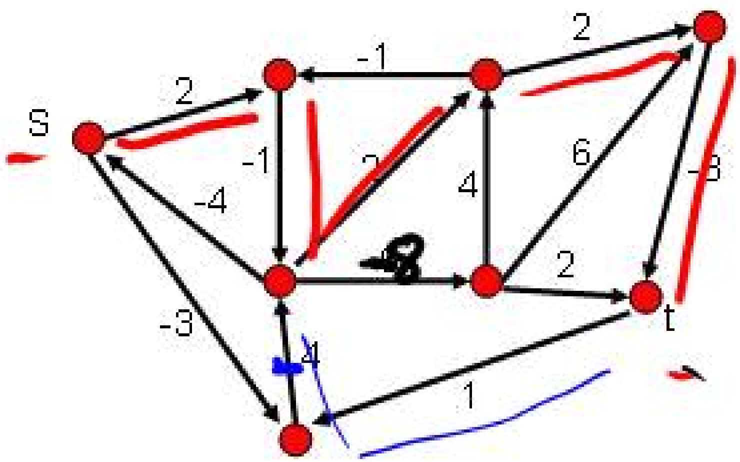
Bellman Ford Example



Algorithm 2				
i	v_1	v_2	v_3	v_4
0	0	∞	∞	∞
1	0	2	∞	1
2	0	2	4	-1
3	0	2	0	-1

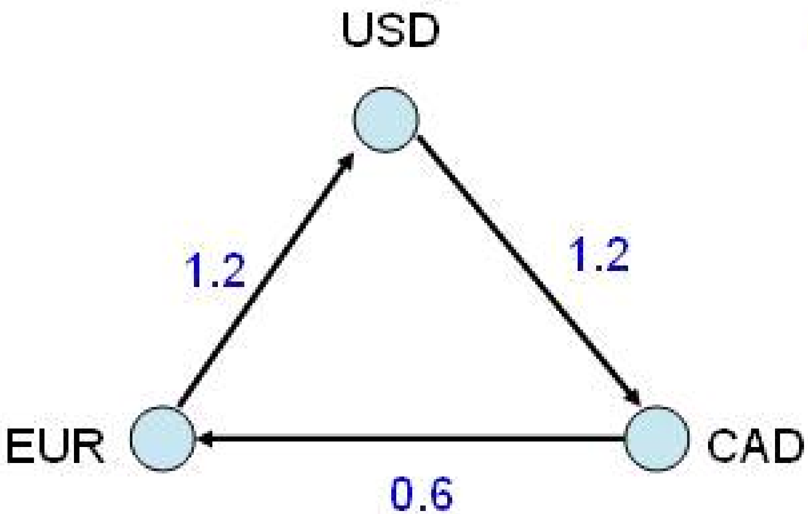
Algorithm 3				
i	v_1	v_2	v_3	v_4
0	0	∞	∞	∞
1	0	2	4	-1
2	0	2	0	-1
3				

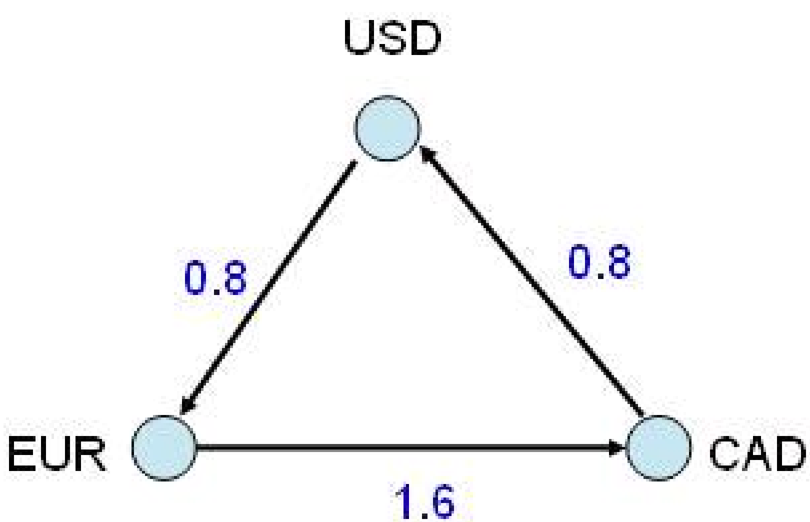
Finding the longest path in a graph



Longest – Negate Weights
– Short Paths.
– Negate Answer.

Foreign Exchange Arbitrage

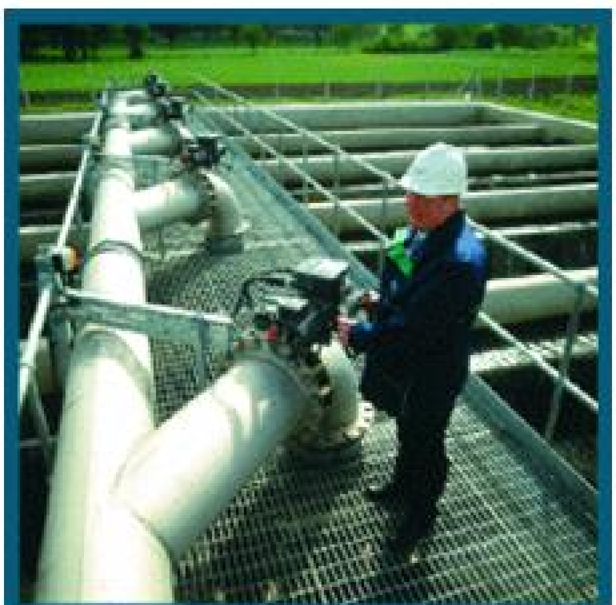




	USD	EUR	CAD
USD	------	0.8	1.2
EUR	1.2	------	1.6
CAD	0.8	0.6	-----

Cycle product >1

Network Flow



Outline

- Network flow definitions
- Flow examples
- Augmenting Paths
- Residual Graph
- Ford Fulkerson Algorithm
- Cuts
- Maxflow-MinCut Theorem

$c(e) \geq 0$

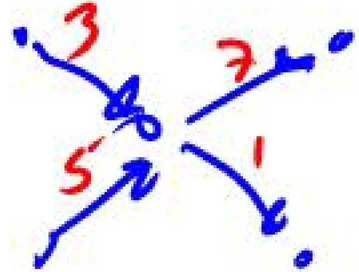
Network Flow Definitions

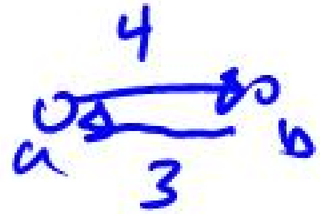
Assign flow

- Capacity
- Source, Sink s, t

$c(e) \geq f(e) \geq 0$

- Capacity Condition



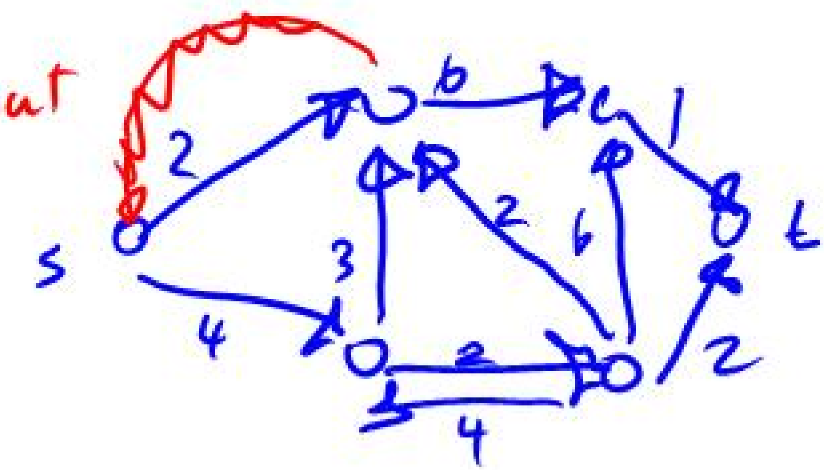


- Conservation Condition

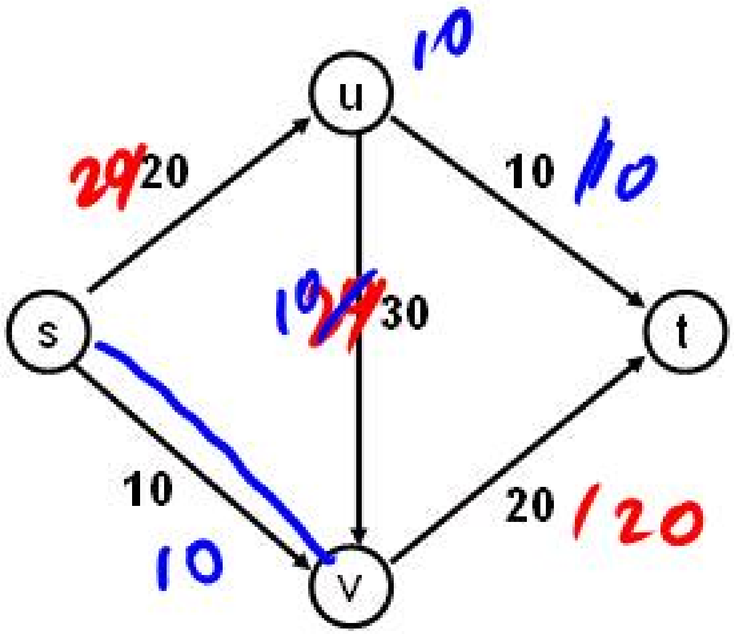
Flow in = flow out

- Value of a flow

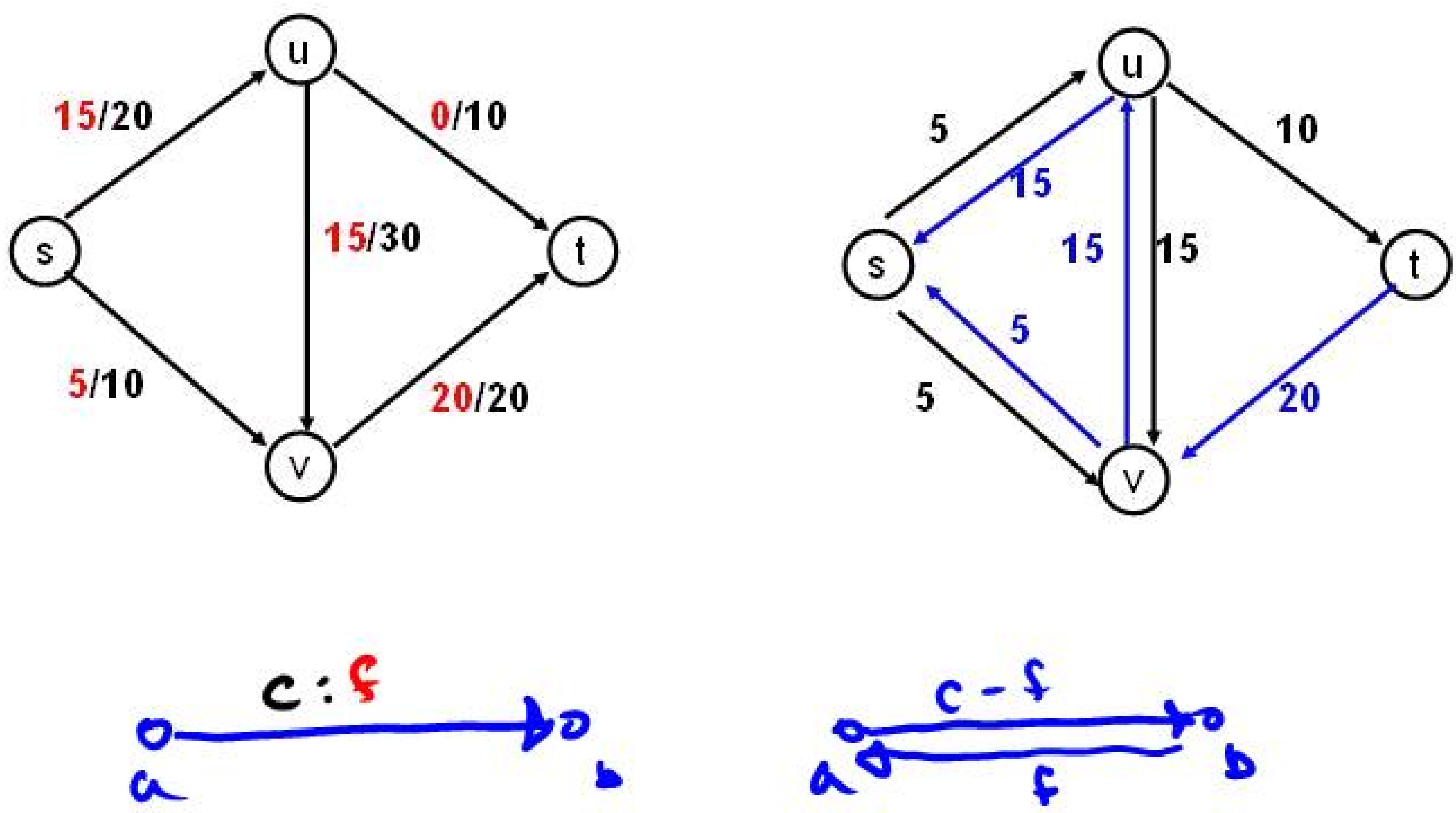
Σ flows out of s



Flow Example



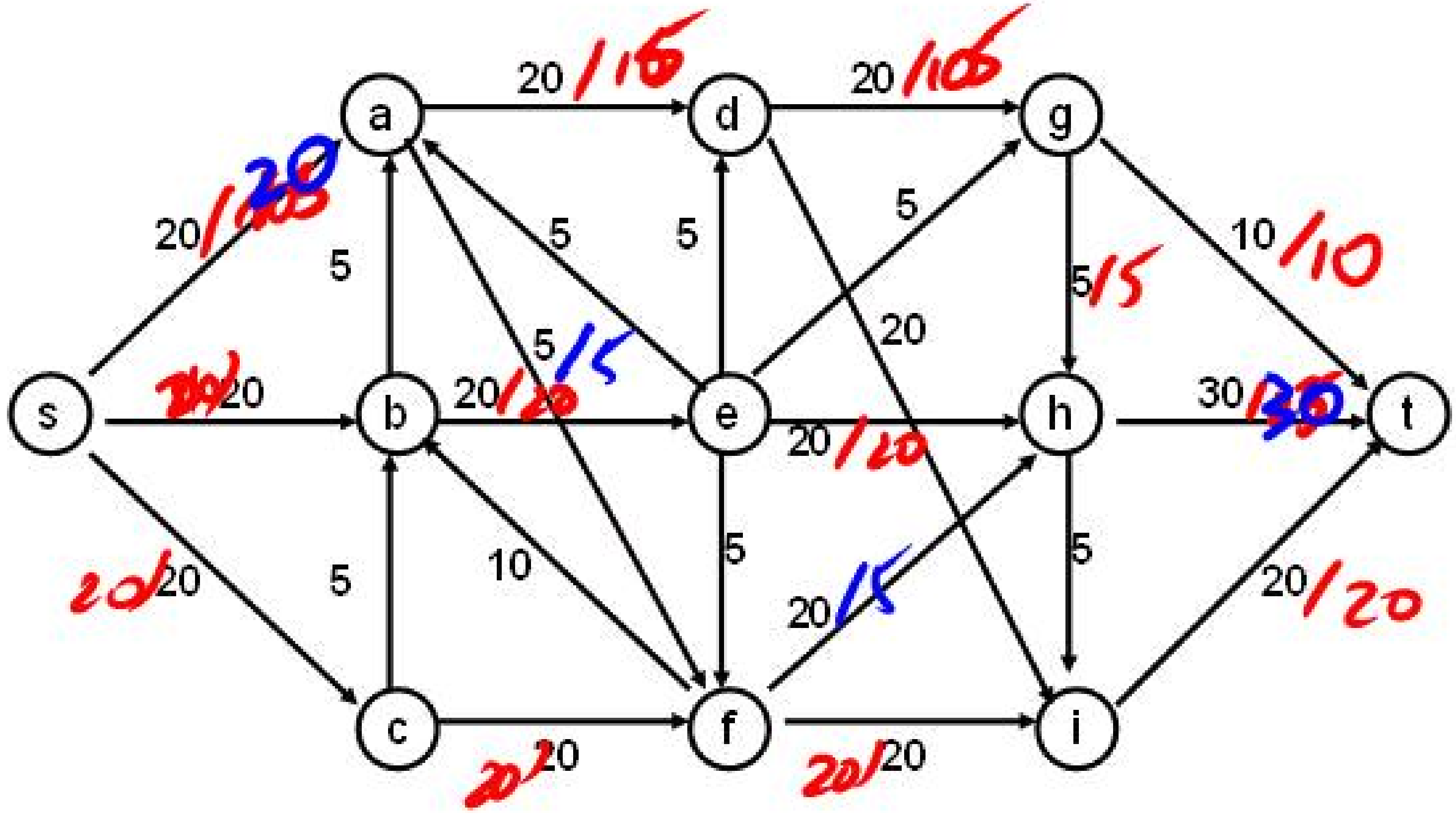
Flow assignment and the residual graph



Network Flow Definitions

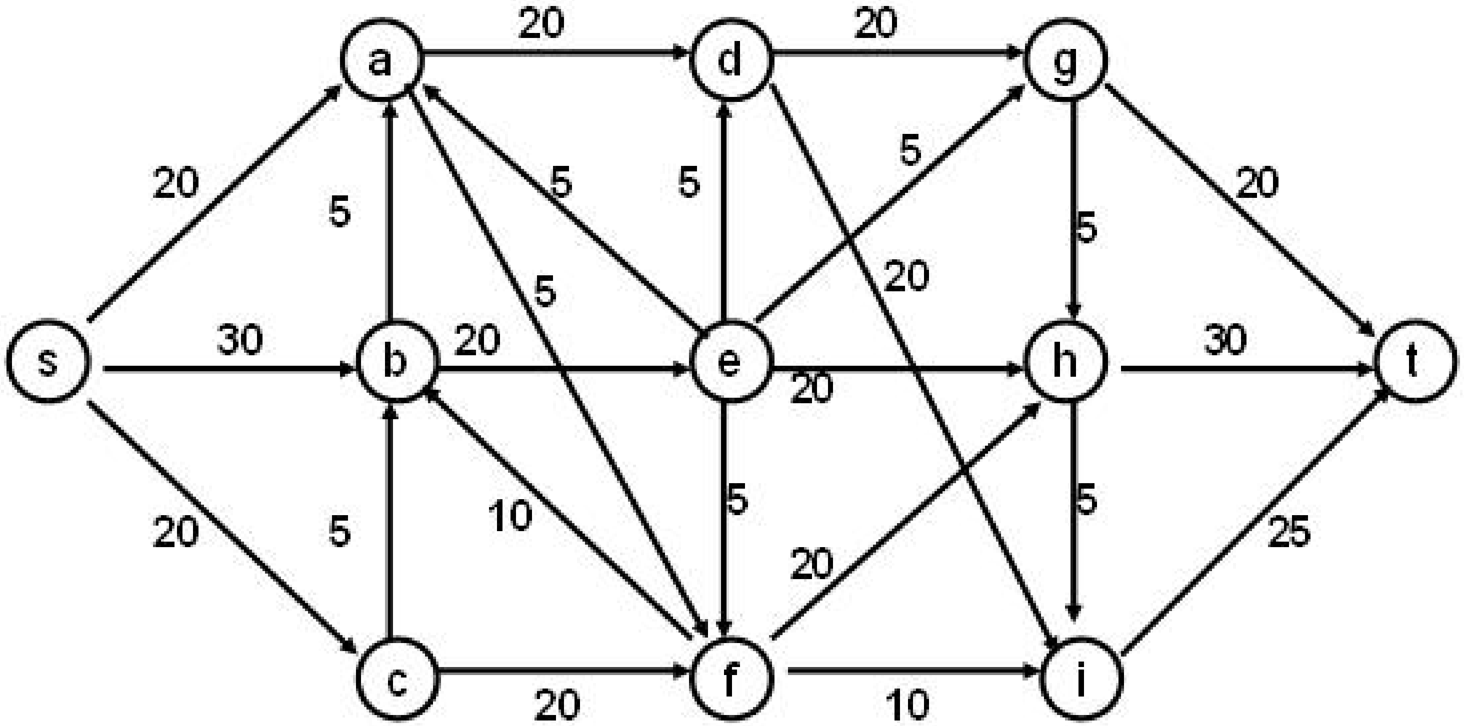
- Flowgraph: Directed graph with distinguished vertices s (source) and t (sink)
- Capacities on the edges, c(e) >= 0
- Problem, assign flows f(e) to the edges such that:
  - 0 <= f(e) <= c(e)
  - Flow is conserved at vertices other than s and t
    - Flow conservation: flow going into a vertex equals the flow going out
  - The flow leaving the source is a large as possible

Flow Example



Find a maximum flow

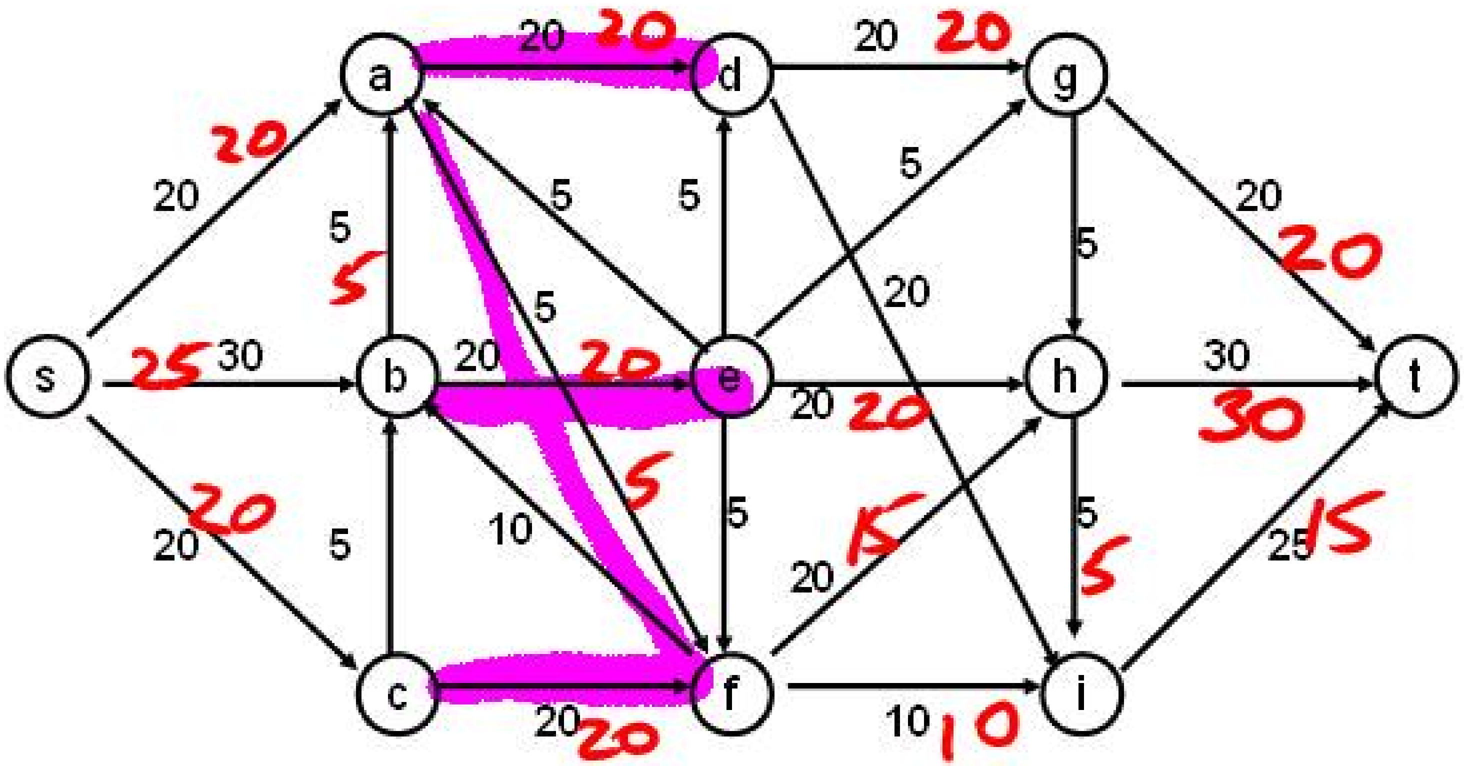
Value of flow:



Construct a maximum flow and indicate the flow value

65

Find a maximum flow

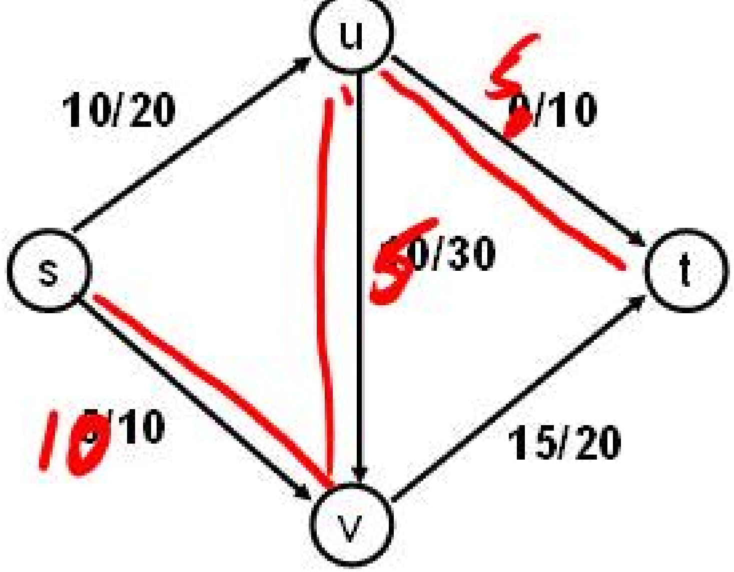


Augmenting Path Algorithm

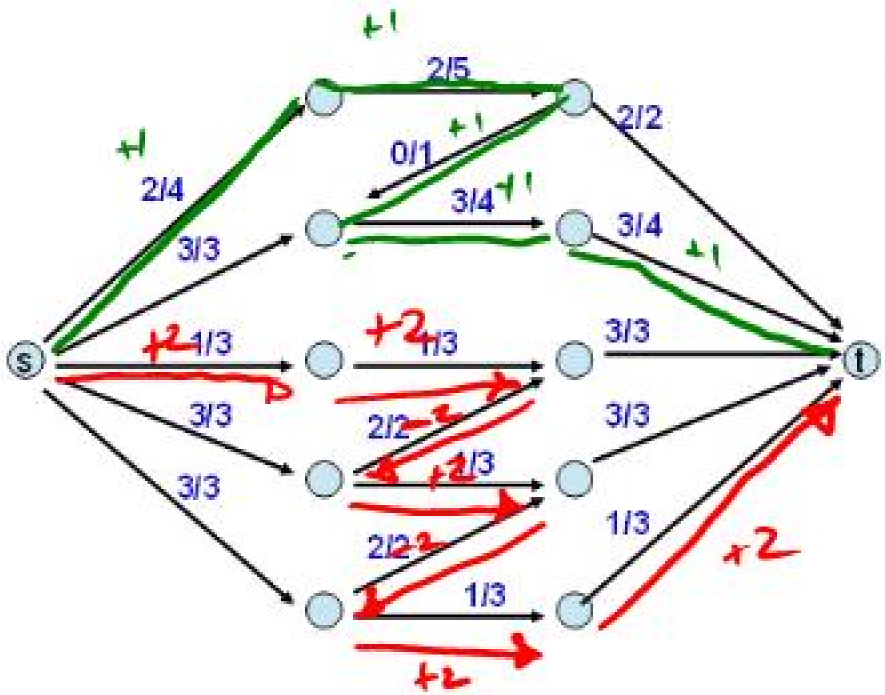
- Augmenting path
  - Vertices $v_1, v_2, \ldots, v_k$
    - $v_1 = s$, $v_k = t$
    - Possible to add b units of flow between v_j and v_{j+1} for j = 1 … k-1

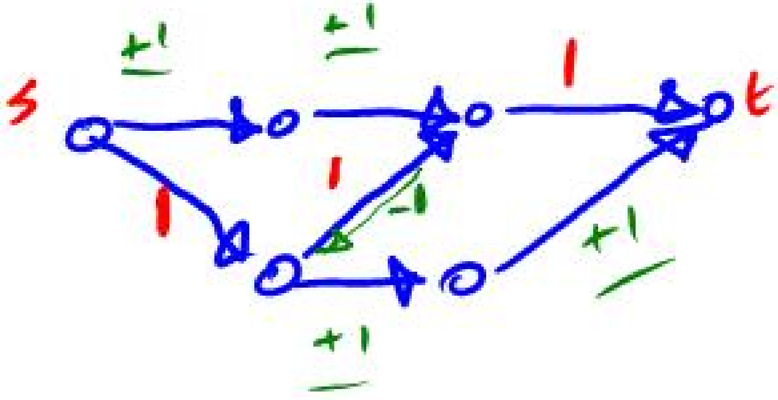
s, u, v, t (5)

s, v, u, t (5)



Find two augmenting paths

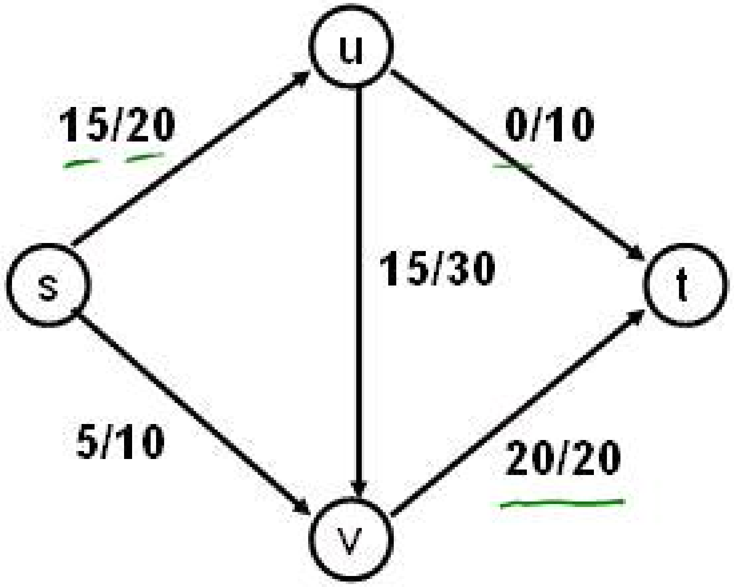


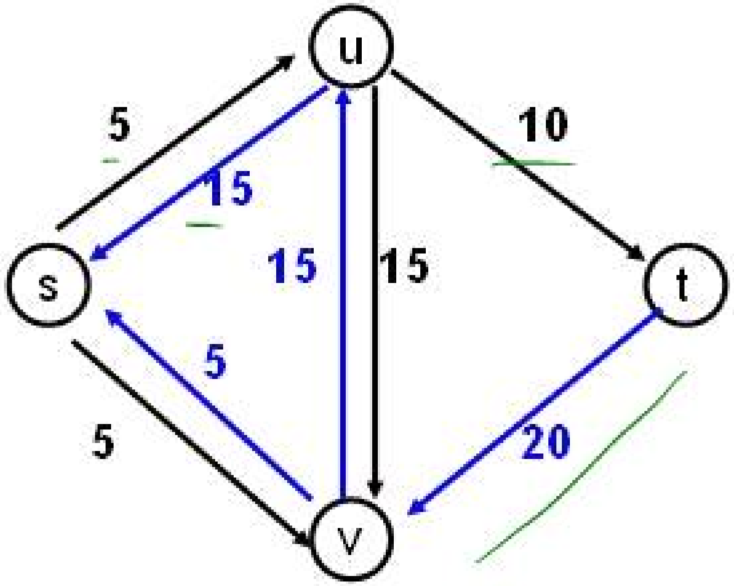


Residual Graph

- Flow graph showing the remaining capacity
- Flow graph G, Residual Graph G_R
  - G: edge e from u to v with capacity c and flow f
  - G_R: edge e' from u to v with capacity c – f
  - G_R: edge e'' from v to u with capacity f

Residual Graph

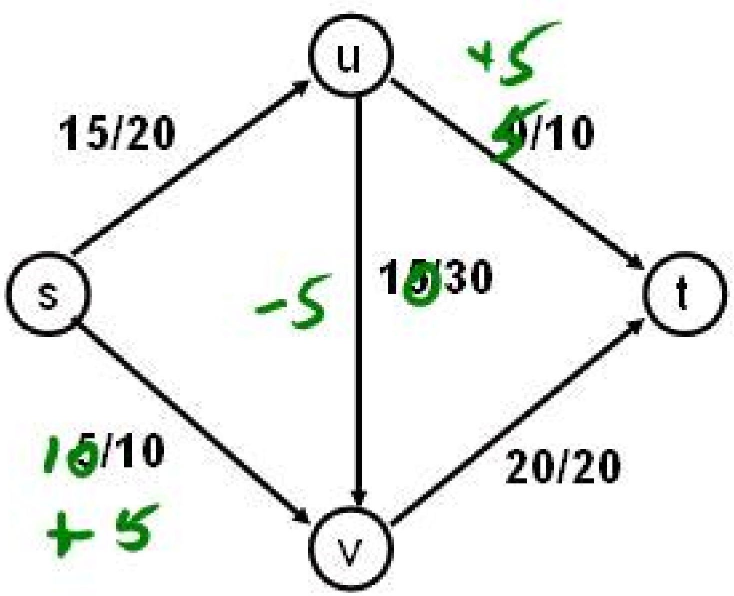


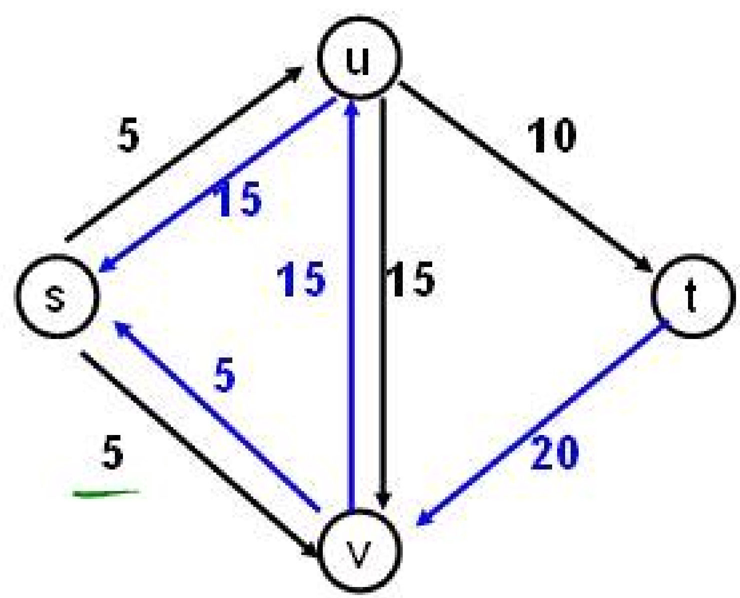


s, v, u, t (5)

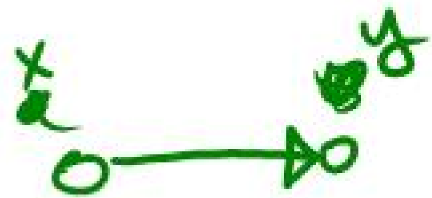
Augmenting Path Lemma

- Let P = $v_1, v_2, \ldots, v_k$ be a path from s to t with minimum capacity b in the residual graph.
- b units of flow can be added along the path P in the flow graph.





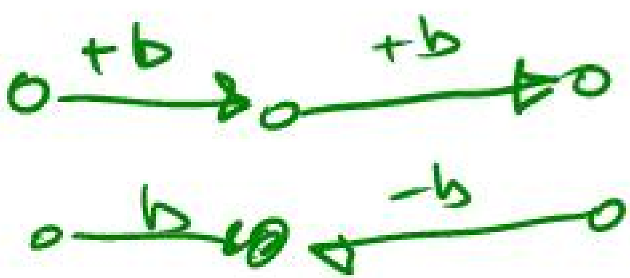
Proof



- Add b units of flow along the path P
- What do we need to verify to show we have a valid flow after we do this?
  - Capacity constraints
    $c(e) \geq f(e) \geq 0$
  - Conservation of flow.

case 1
forward edge
$c - f \geq b$

case 2
backward edge
$f \geq b$



Ford-Fulkerson Algorithm (1956)

while not done

Construct residual graph G_R

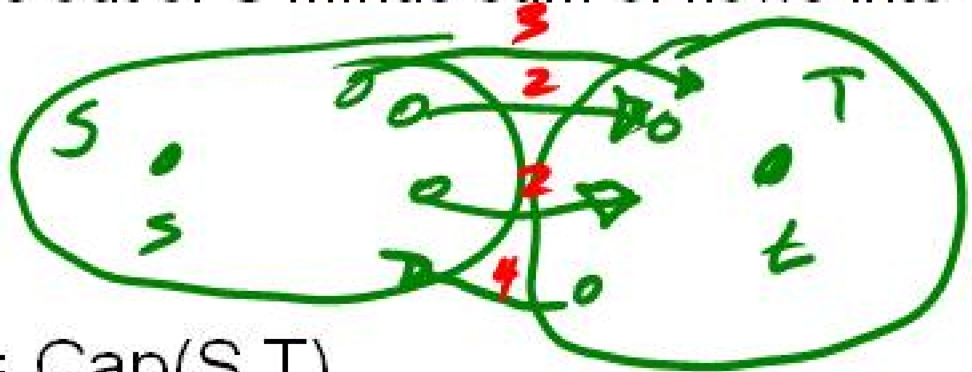
Find an s-t path P in G_R with capacity $b > 0$

Add b units along in G

If the sum of the capacities of edges leaving S is at most C, then the algorithm takes at most C iterations

Cuts in a graph

- Cut: Partition of V into disjoint sets S, T with s in S and t in T.
- Cap(S,T): sum of the capacities of edges from S to T
- Flow(S,T): net flow out of S
  - Sum of flows out of S minus sum of flows into S

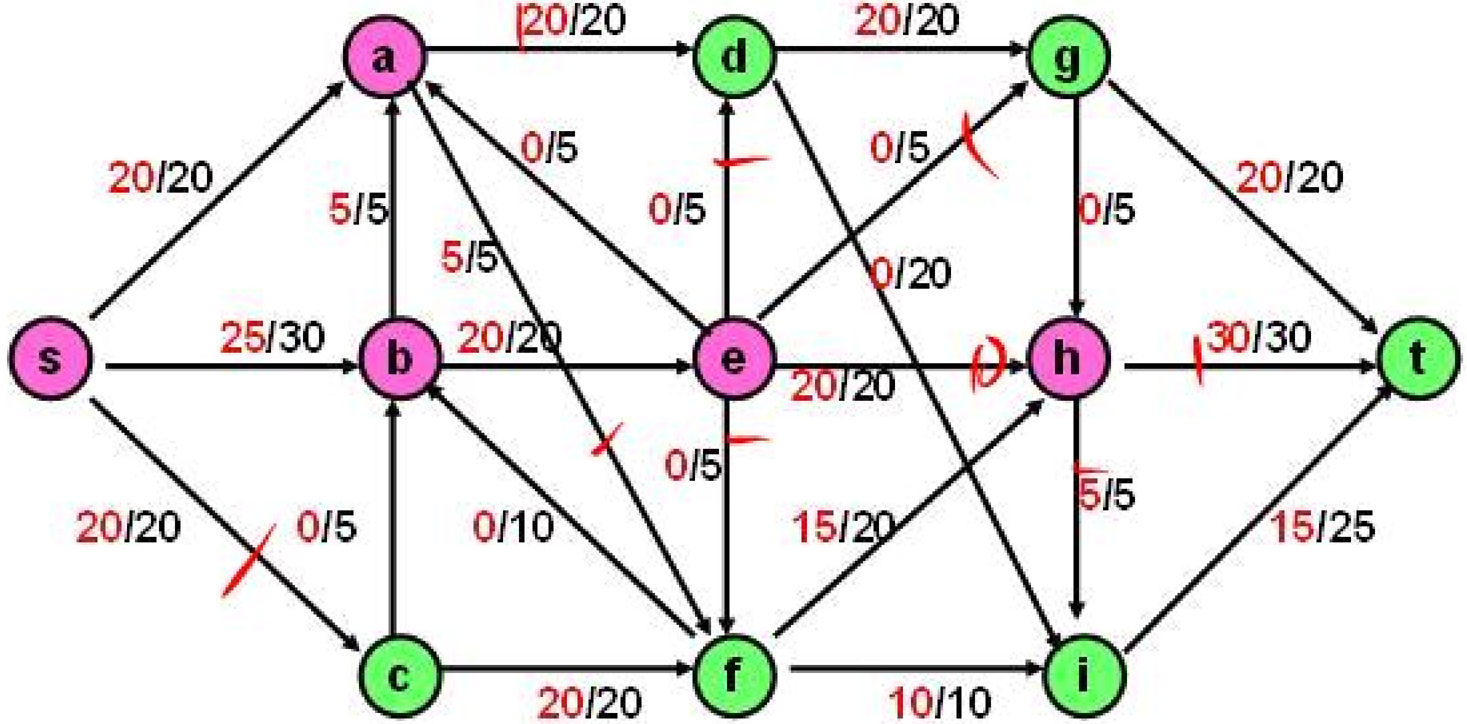


- Flow(S,T) <= Cap(S,T)

3 + 2 + 2 − 4

What is Cap(S,T) and Flow(S,T)

S={s, a, b, e, h}, T = {c, f, i, d, g, t}



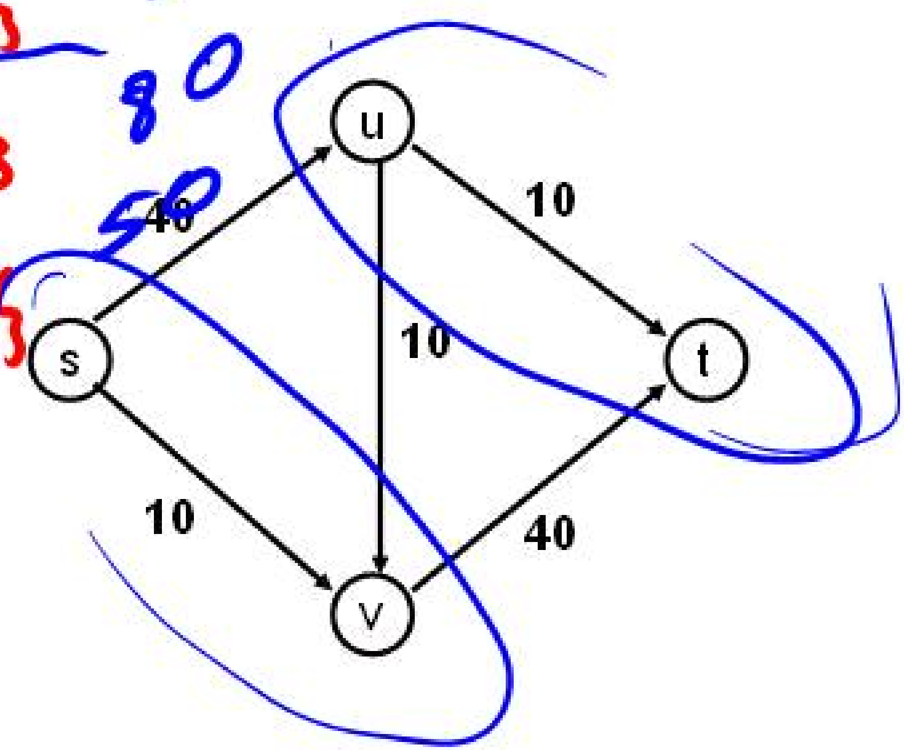
Minimum value cut

{s} {u, v, t} 50

{s, u} {v, t} 30

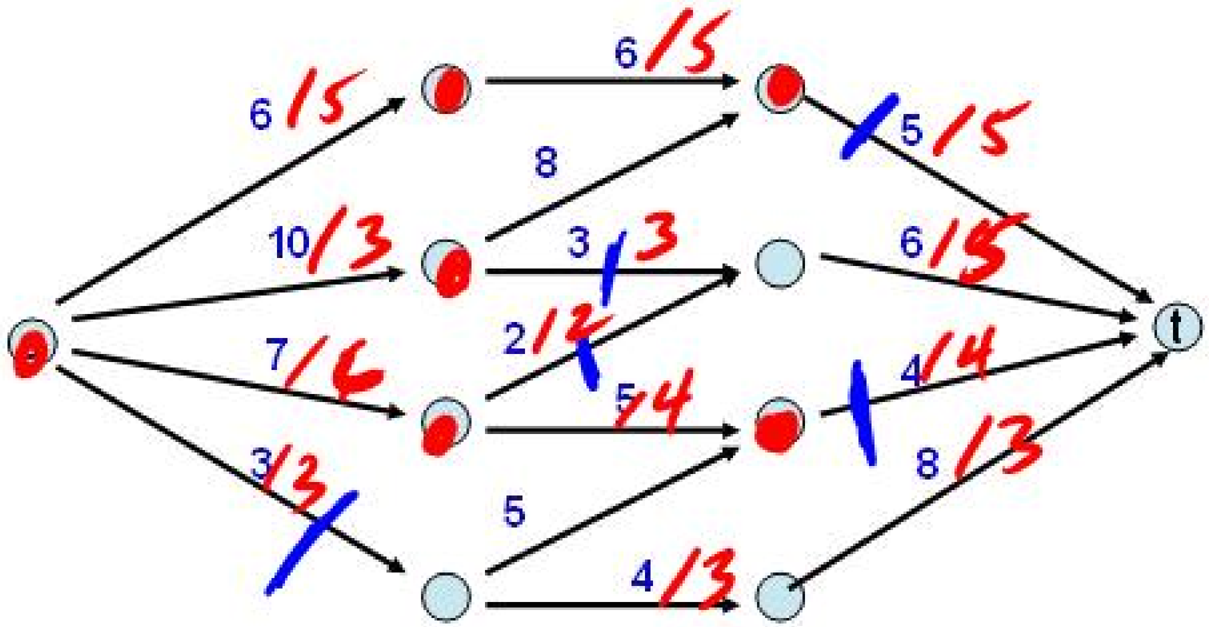
{s, v} {u, t} 80

{s, u, v} {t} 50



Find a minimum value cut

cap(S,T) ≥ flow(S,T)
cap(S,T) = flow(S,T)

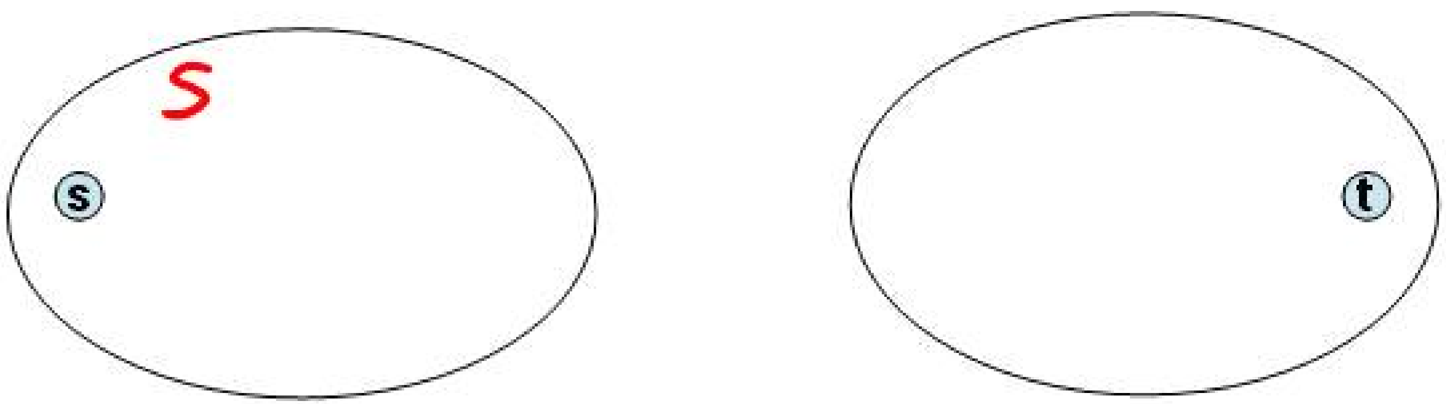


MaxFlow – MinCut Theorem

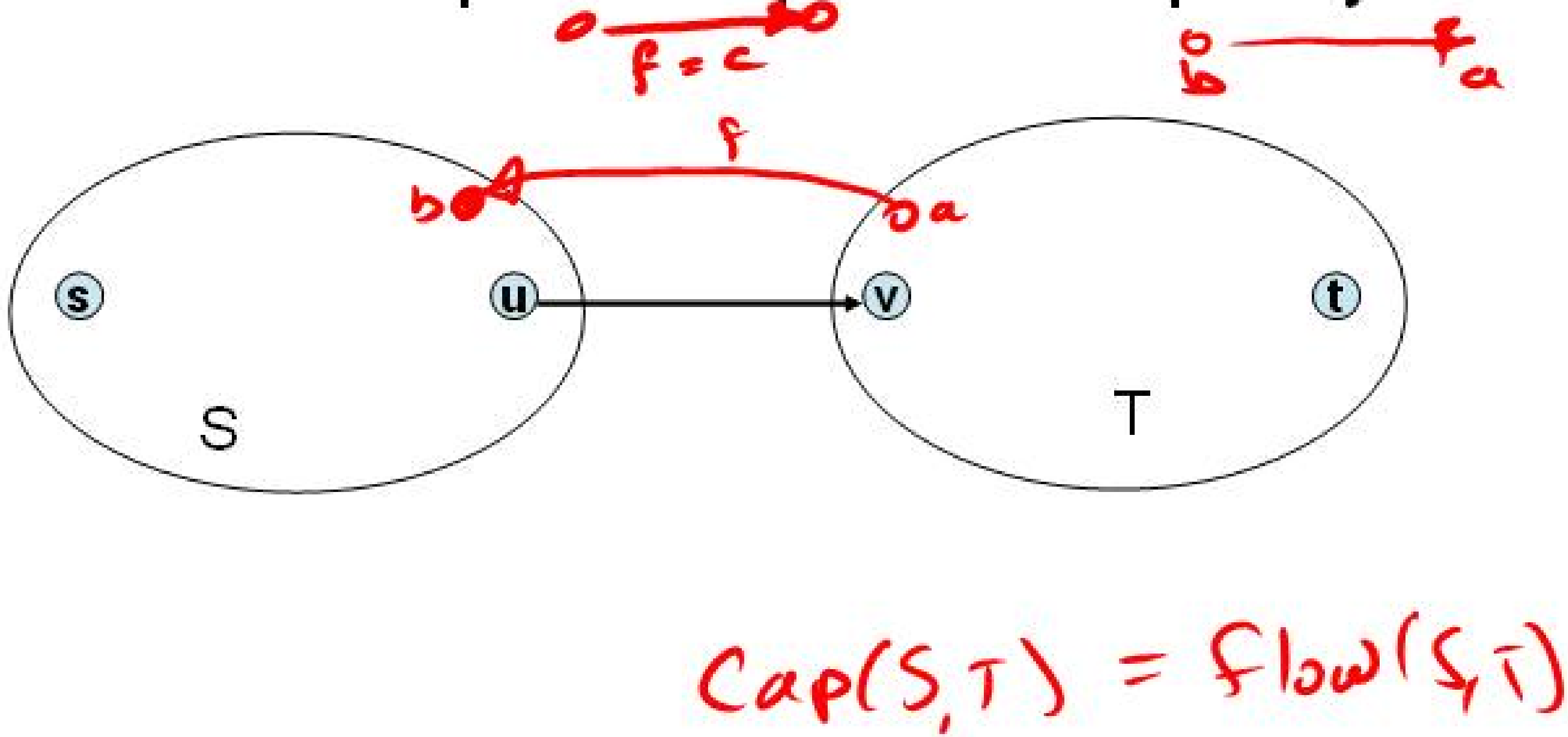
- Let S, T be a cut, and F a flow
  - Cap(S,T) >= Flow(S,T)
- If Cap(S,T) = Flow(S,T)
  - S, T must be a minimum cut
  - F must be a maximum flow
- The amazing Ford-Fulkerson theorem shows that there is always a cut that matches a flow, and also shows how their algorithm finds the flow

MaxFlow – MinCut Theorem

- There exists a flow which has the same value of the minimum cut
- Proof: Consider a flow where the residual graph has no s-t path with positive capacity
- Let S be the set of vertices in G_R reachable from s with paths of positive capacity



Let S be the set of vertices in G_R reachable from s with paths of positive capacity



What can we say about the flows and capacity between u and v?

Max Flow - Min Cut Theorem

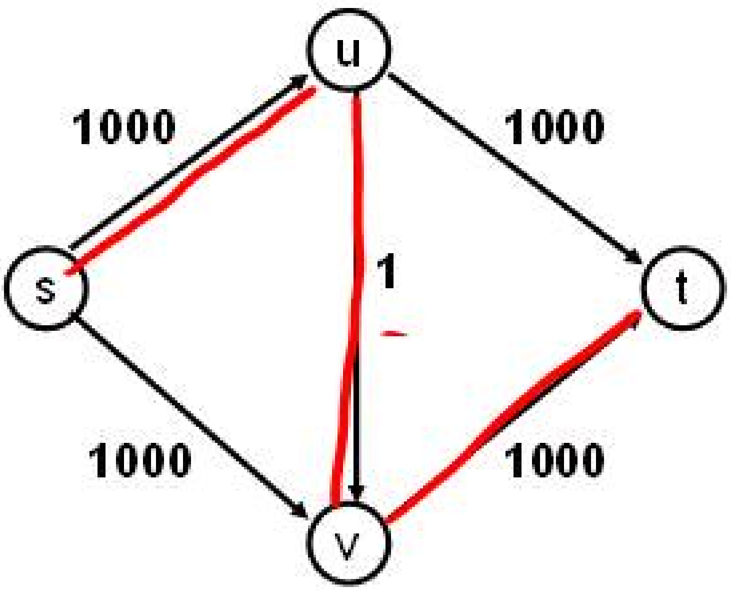
- Ford-Fulkerson algorithm finds a flow where the residual graph is disconnected, hence FF finds a maximum flow.

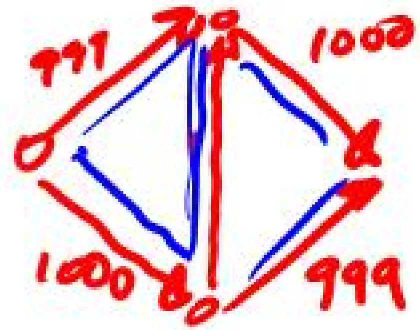
- If we want to find a minimum cut, we begin by looking for a maximum flow.

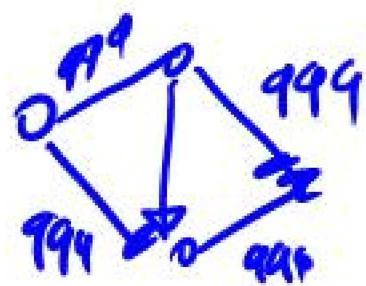
Performance

2000 Augmentations

- The worst case performance of the Ford-Fulkerson algorithm is horrible







Better methods of finding augmenting paths

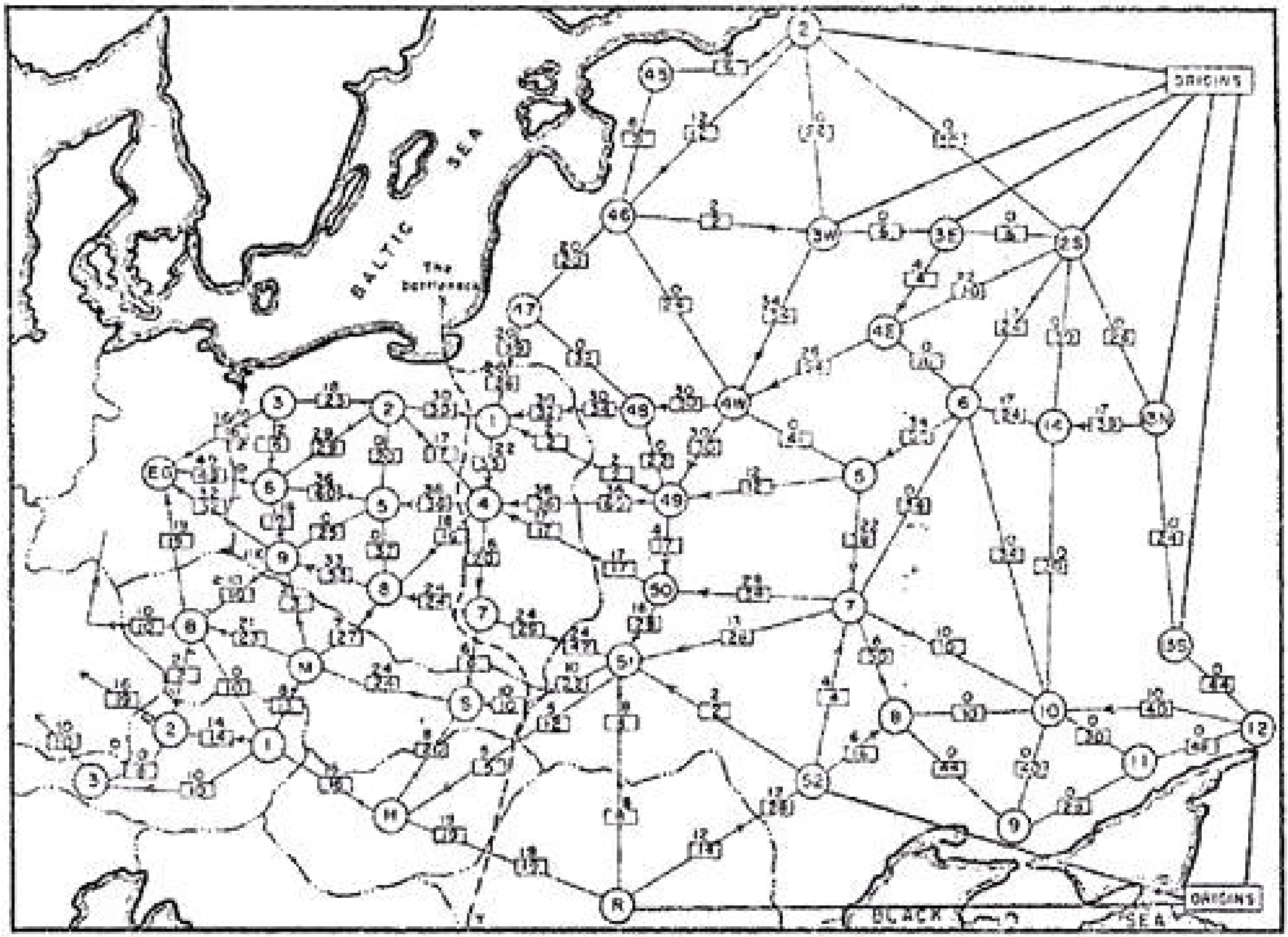
Runtime: Max flow have size C

$O(mC)$

- Find the maximum capacity augmenting path
  - $O(m^2 \log(C))$ time algorithm for network flow
- Find the shortest augmenting path
  - $O(m^2 n)$ time algorithm for network flow
- Find a blocking flow in the residual graph
  - $O(mn \log n)$ time algorithm for network flow

1956

History



Reference: *On the history of the transportation and maximum flow problems.* Alexander Schrijver in Math Programming, 91: 3, 2002.

Problem Reduction

- Reduce Problem A to Problem B
  - Convert an instance of Problem A to an instance of Problem B
  - Use a solution of Problem B to get a solution to Problem A
- Practical
  - Use a program for Problem B to solve Problem A
- Theoretical
  - Show that Problem B is at least as hard as Problem A

Problem Reduction Examples

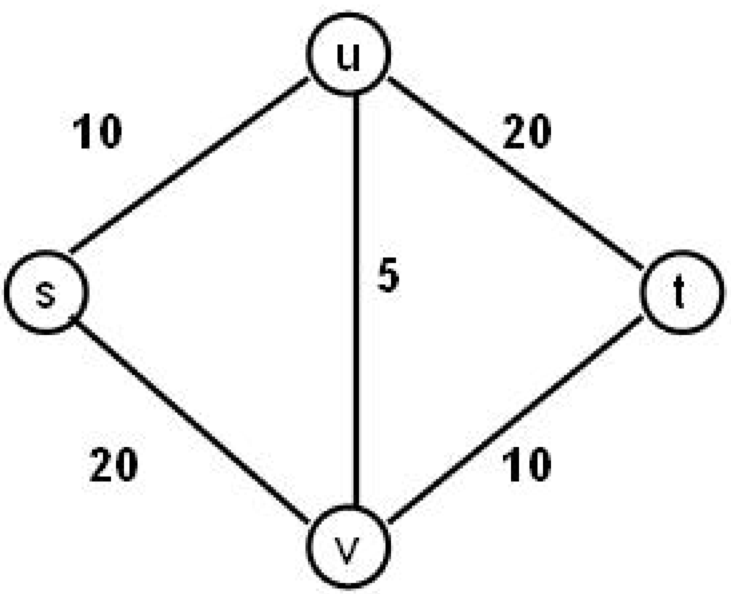
- Reduce the problem of finding the Maximum of a set of integers to finding the Minimum of a set of integers

Find the maximum of: 8, -3, 2, 12, 1, -6

Construct an equivalent minimization problem

Undirected Network Flow

- Undirected graph with edge capacities
- Flow may go either direction along the edges (subject to the capacity constraints)



Construct an equivalent flow problem

Bipartite Matching

- A graph G=(V,E) is bipartite if the vertices can be partitioned into disjoints sets X,Y
- A matching M is a subset of the edges that does not share any vertices
- Find a matching as large as possible