# CSEP 521
# Applied Algorithms

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Lecture 9

Network Flow Applications

# Announcements

- Reading for this week
  - 7.5-7.12. Network flow applications
  - Next week: Chapter 8. NP-Completeness
- Final exam, March 18, 6:30 pm. At UW.
  - 2 hours
  - In class (CSE 303 / CSE 305)
  - Comprehensive
    - 67% post midterm, 33% pre midterm

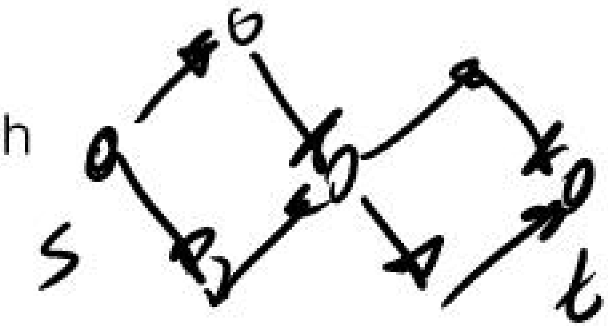
# Network Flow


# Review

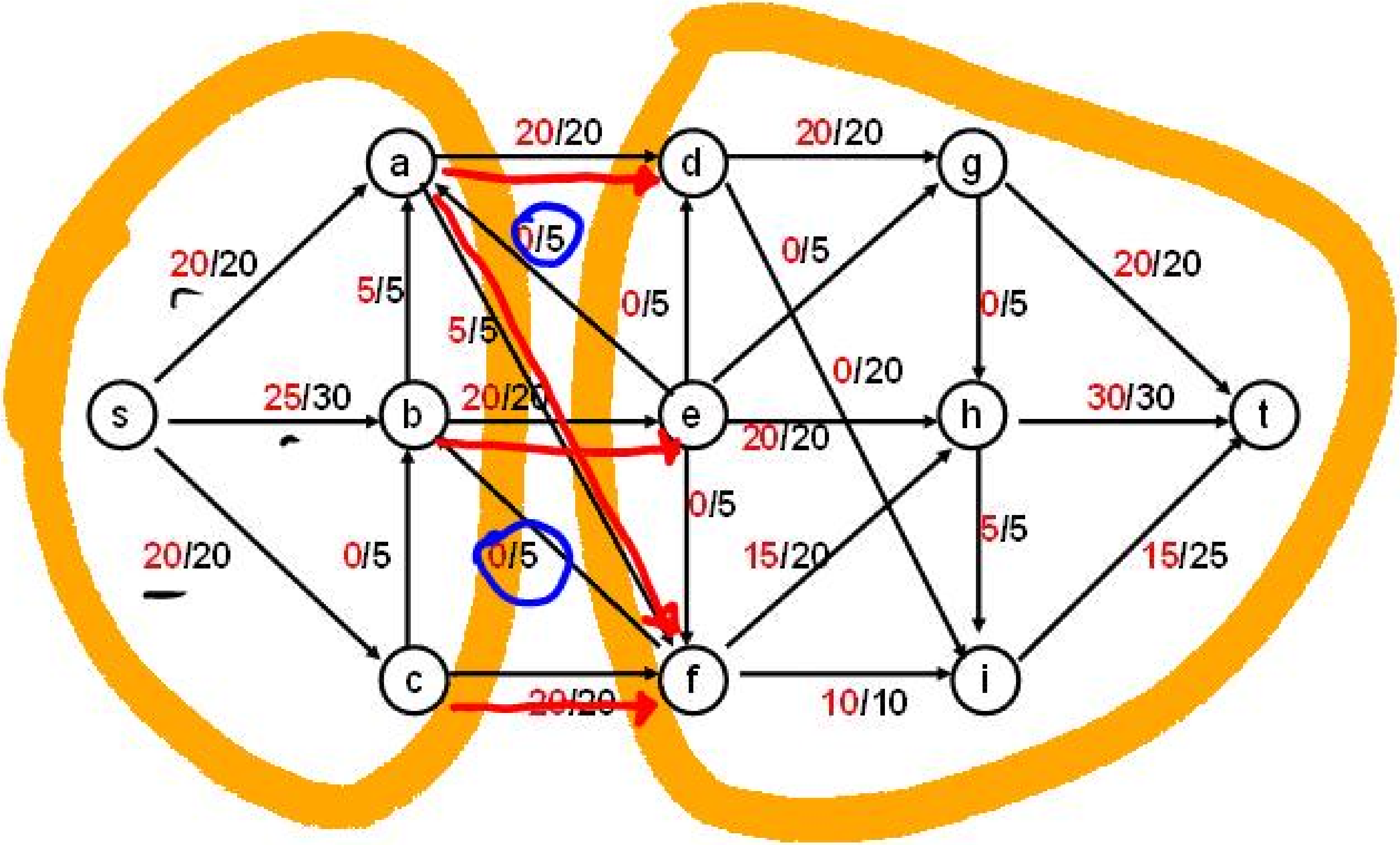
- Network flow definitions
- Flow examples
- Augmenting Paths
- Residual Graph
- Ford Fulkerson Algorithm
- Cuts
- Maxflow-MinCut Theorem

# Network Flow Definitions

- Flowgraph: Directed graph with distinguished vertices s (source) and t (sink)
- Capacities on the edges, c(e) >= 0
- Problem, assign flows f(e) to the edges such that:
  - 0 <= f(e) <= c(e)
  - Flow is conserved at vertices other than s and t
    - Flow conservation: flow going into a vertex equals the flow going out
  - The flow leaving the source is a large as possible

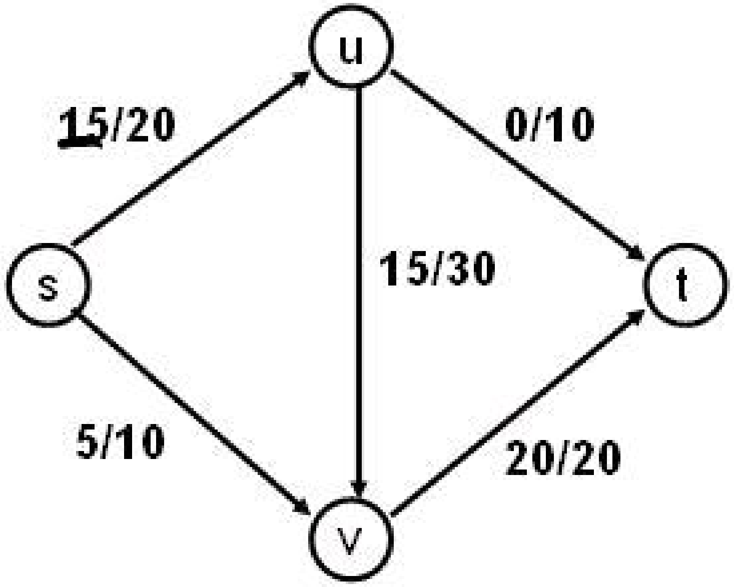
# Find a maximum flow

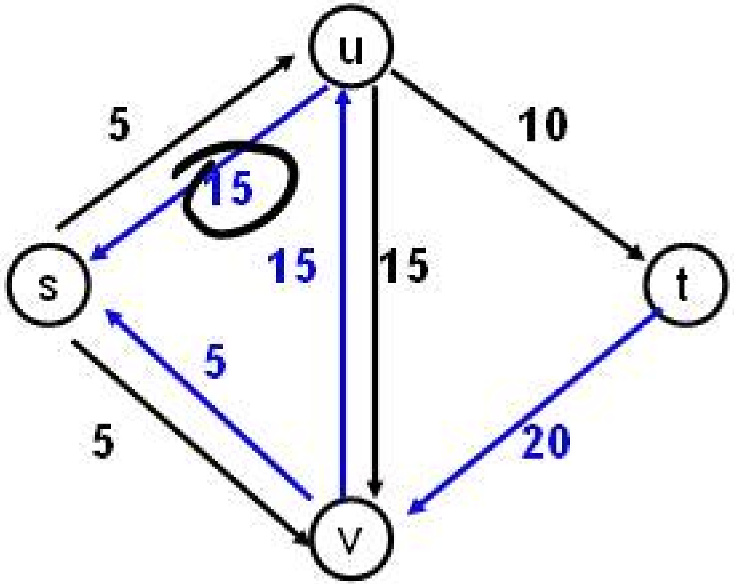


# Residual Graph

- Flow graph showing the remaining capacity
- Flow graph G, Residual Graph $G_R$
  - G: edge e from u to v with capacity c and flow f
  - $G_R$: edge e' from u to v with capacity c – f
  - $G_R$: edge e'' from v to u with capacity f

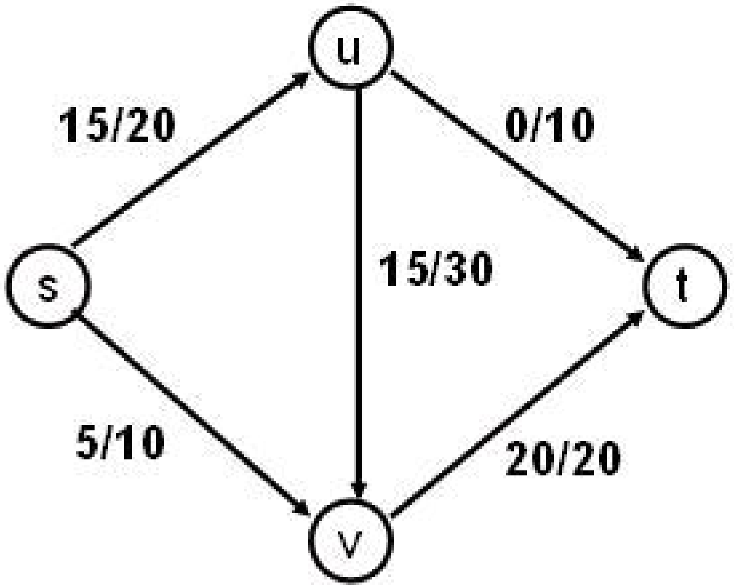
# Residual Graph

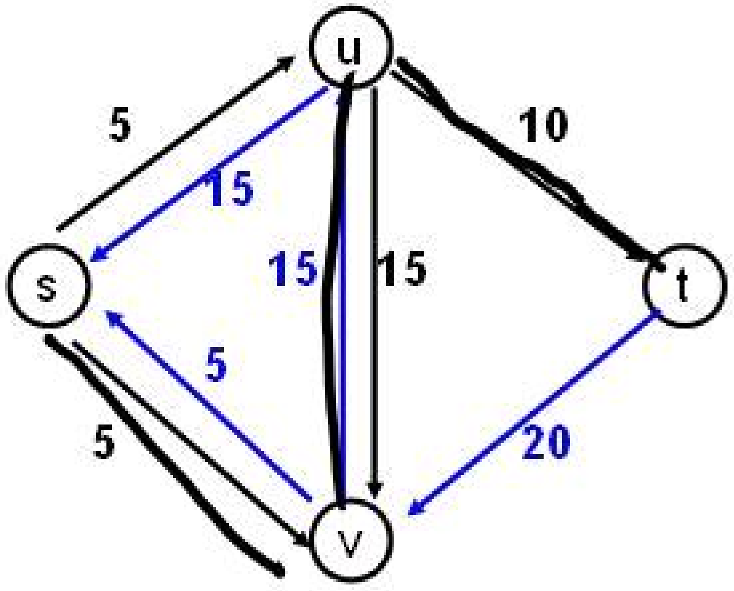


# Augmenting Path Lemma

- Let $P = v_1, v_2, \ldots, v_k$ be a path from s to t with minimum capacity b in the residual graph.
- b units of flow can be added along the path P in the flow graph.



# Ford-Fulkerson Algorithm (1956)

while not done

Construct residual graph $G_R$

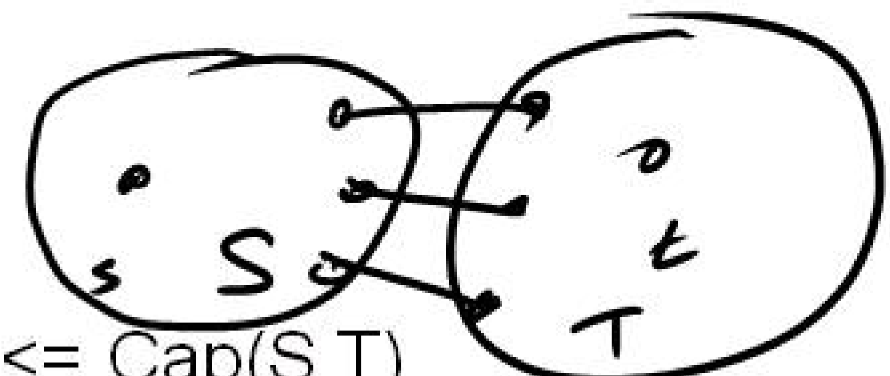
Find an s-t path P in $G_R$ with capacity $b > 0$

Add b units along in G

If the sum of the capacities of edges leaving S is at most C, then the algorithm takes at most C iterations

# Cuts in a graph

- Cut: Partition of V into disjoint sets S, T with s in S and t in T.
- Cap(S,T): sum of the capacities of edges from S to T
- Flow(S,T): net flow out of S
  - Sum of flows out of S minus sum of flows into S




- Flow(S,T) <= Cap(S,T)

# Ford Fulkerson MaxFlow – MinCut Theorem

- There exists a flow which has the same value of the minimum cut
  - Shows that a cut is the dual of the flow
  - Proves that the augmenting paths algorithm finds a maximum flow
  - Gives an algorithms for finding the minimum cut

# Better methods of for constructing a network flow

- Improved methods for finding augmenting paths or blocking flows
- Goldberg's Preflow-Push algorithm
  - Text, section 7.4

$O(nm)$

Efficient Network flow

Algorithms

# Applications of Network Flow

# Problem Reduction

- Reduce Problem A to Problem B
  - Convert an instance of Problem A to an instance of Problem B
  - Use a solution of Problem B to get a solution to Problem A
- Practical
  - Use a program for Problem B to solve Problem A
- Theoretical
  - Show that Problem B is at least as hard as Problem A

# Problem Reduction Examples

- Reduce the problem of finding the longest path in a directed graph to the problem of finding a shortest path in a directed graph

① Negate Weights graph.

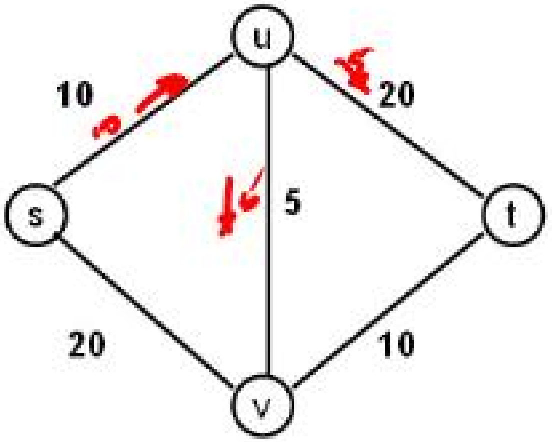
Call S. P. A.

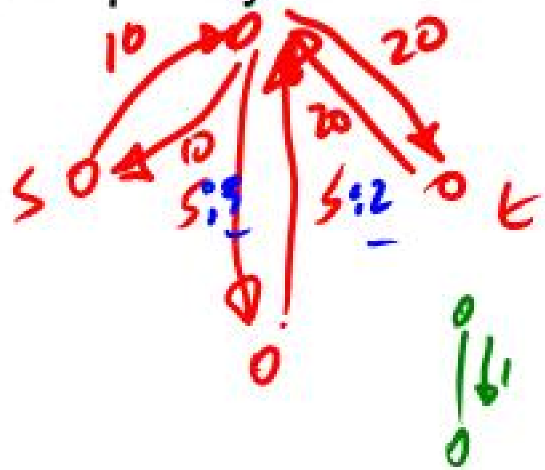
② Reverse sign. Return PATH

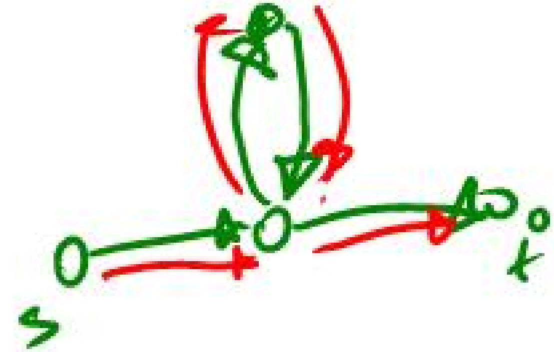
Construct an equivalent minimization problem

# Undirected Network Flow

- Undirected graph with edge capacities
- Flow may go either direction along the edges (subject to the capacity constraints)



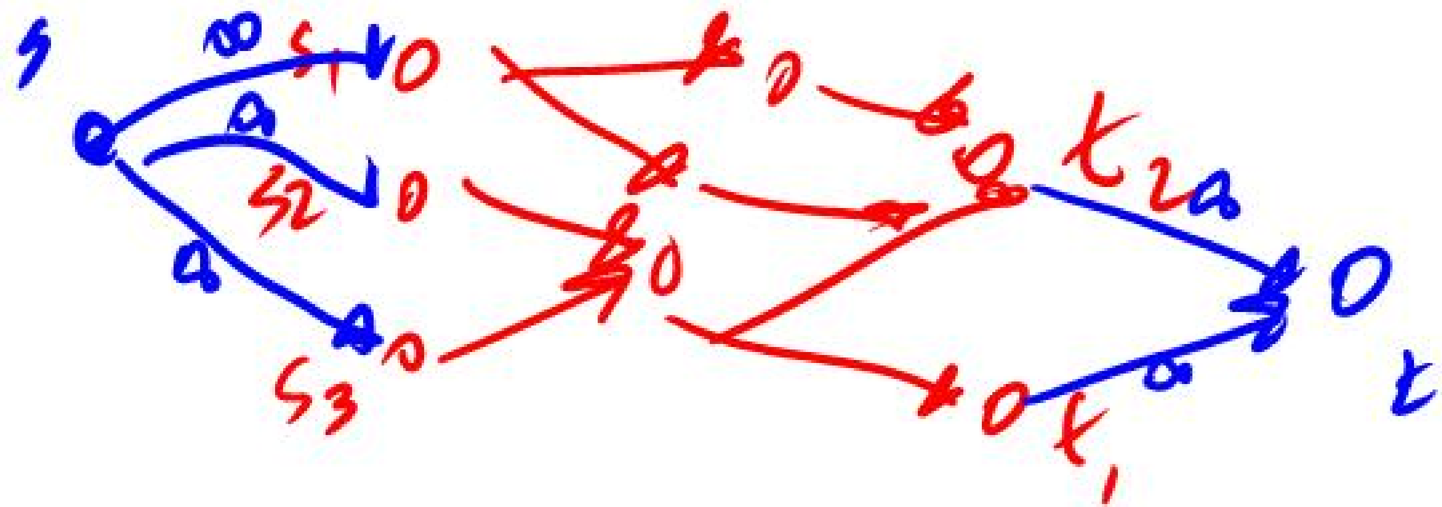




Construct an equivalent flow problem

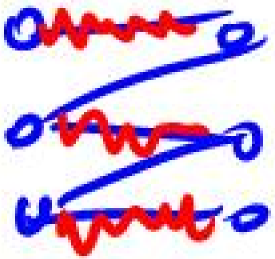
# Multi-source network flow

- Multi-source network flow
  - Sources $s_1, s_2, \ldots, s_k$
  - Sinks $t_1, t_2, \ldots, t_j$
- Solve with Single source network flow



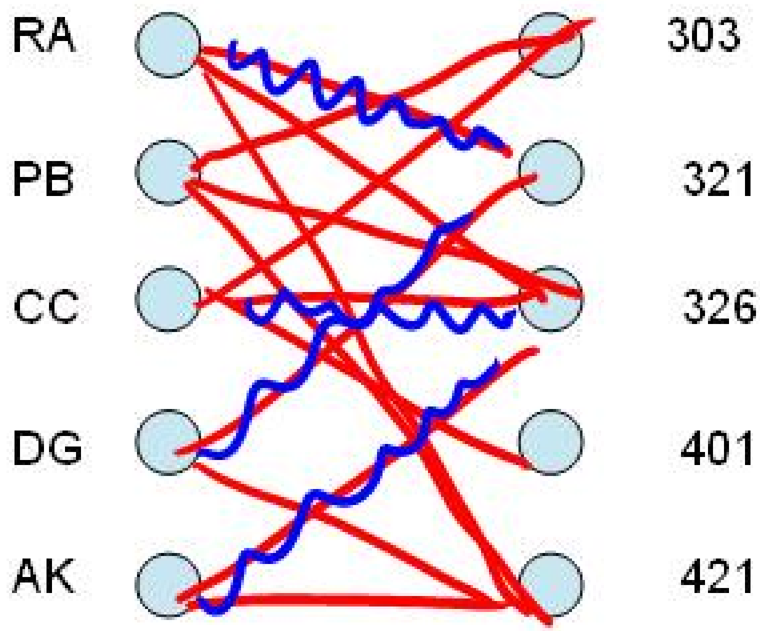
# Bipartite Matching

- A graph G=(V,E) is bipartite if the vertices can be partitioned into disjoints sets X,Y
- A matching M is a subset of the edges that does not share any vertices
- Find a matching as large as possible

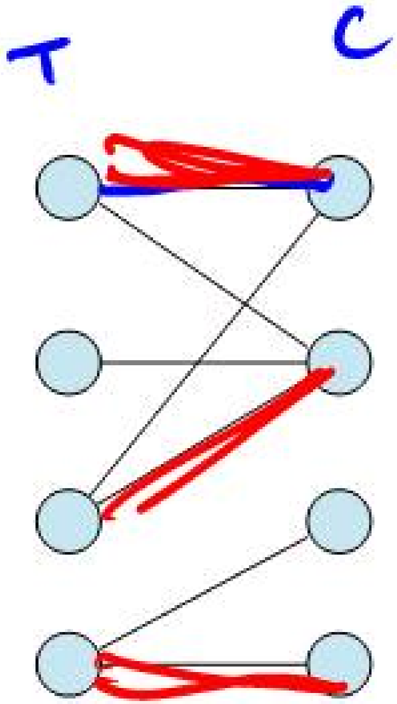


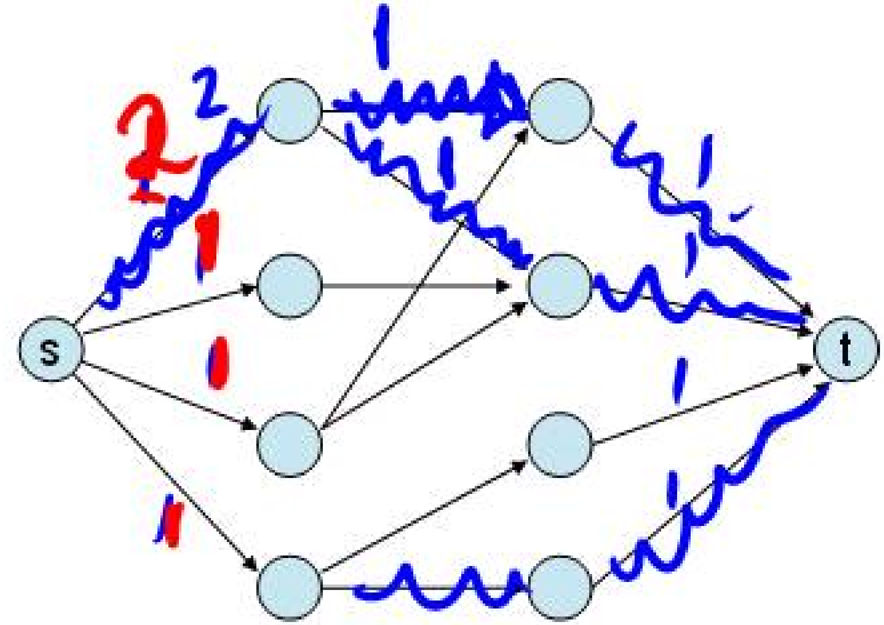
# Application

- A collection of teachers
- A collection of courses
- And a graph showing which teachers can teach which courses

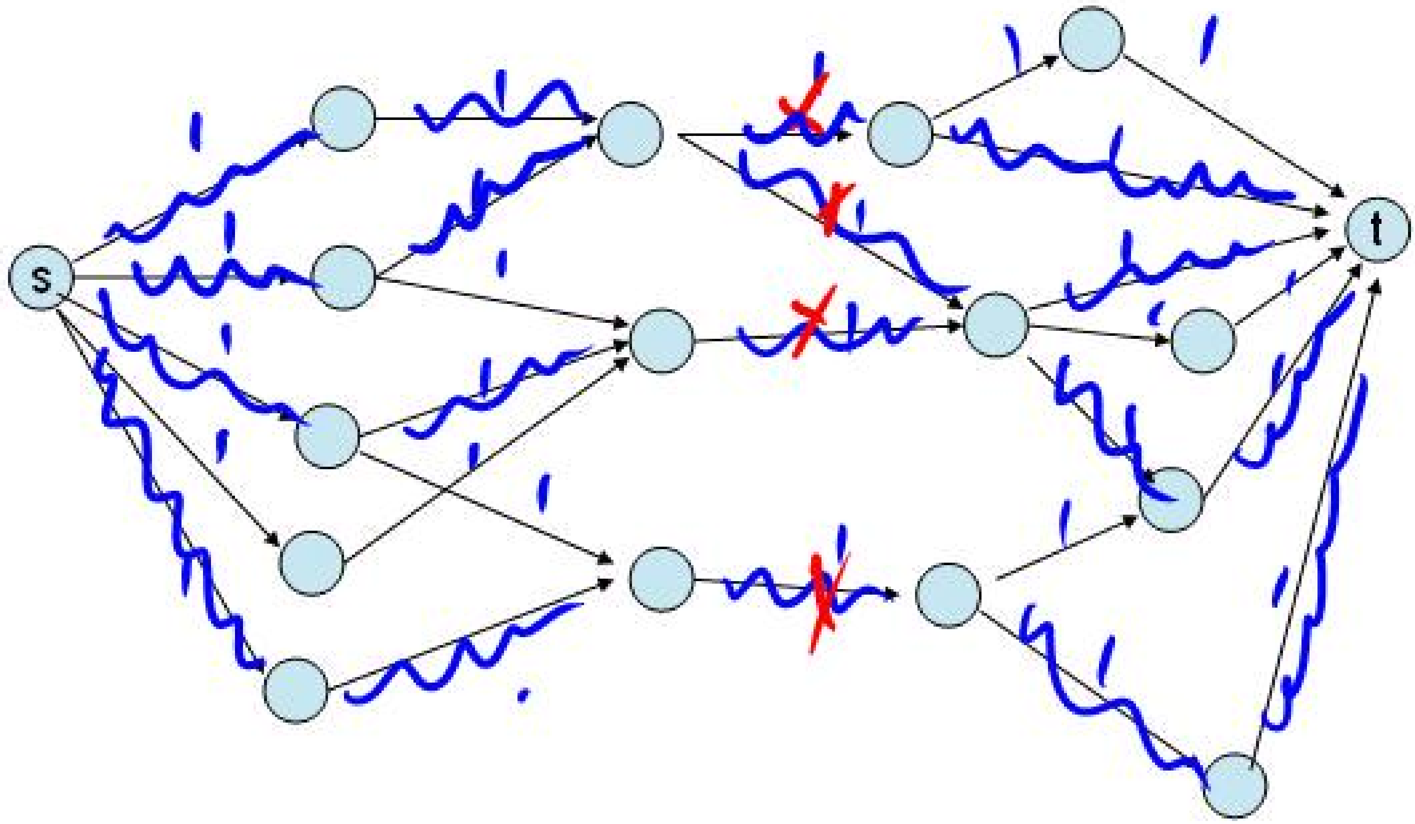


# Converting Matching to Network Flow



# Finding edge disjoint paths

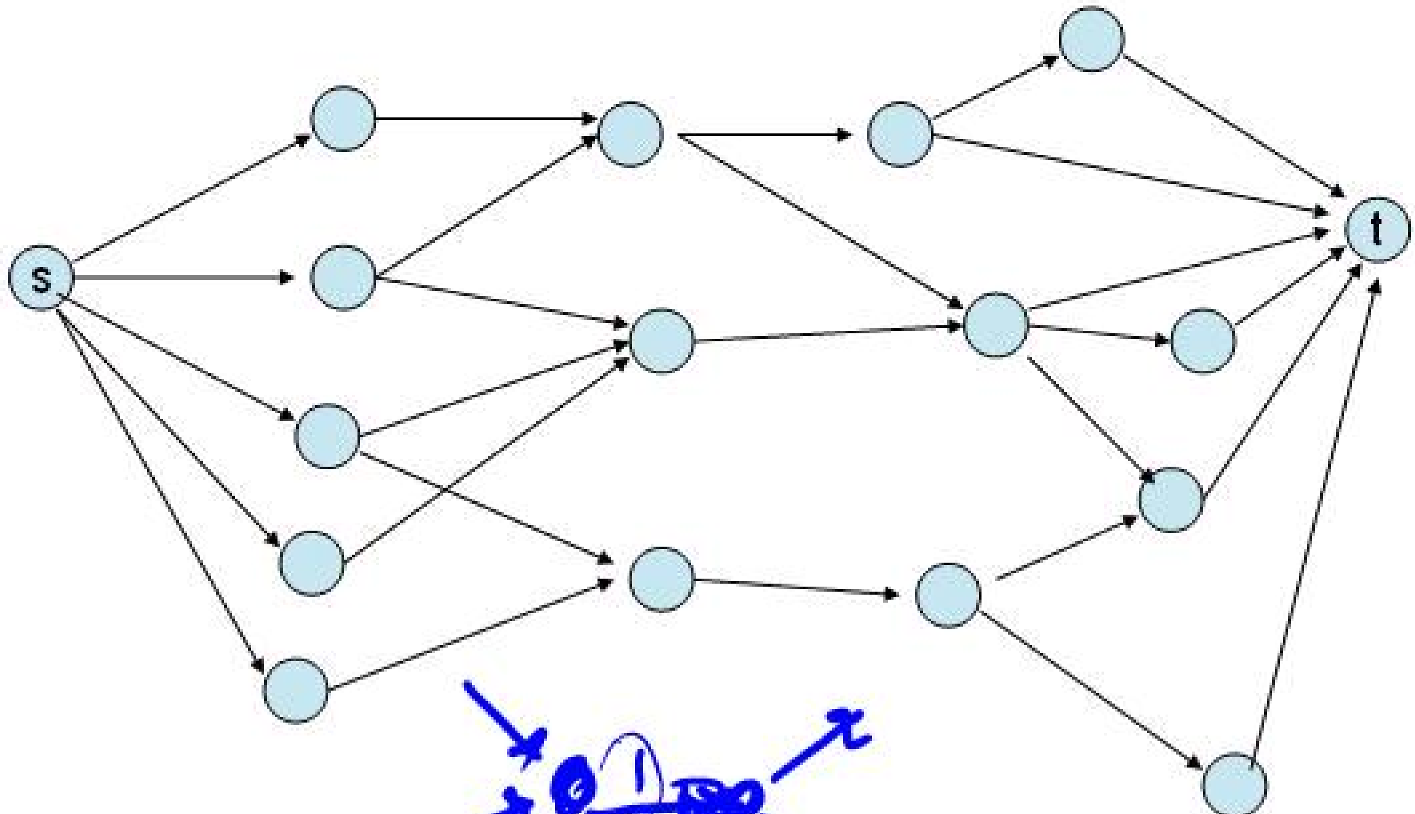


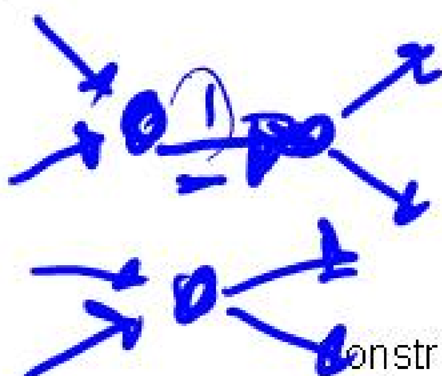

Construct a maximum cardinality set of edge disjoint paths

# Theorem

- The maximum number of edge disjoint paths equals the minimum number of edges whose removal separates s from t

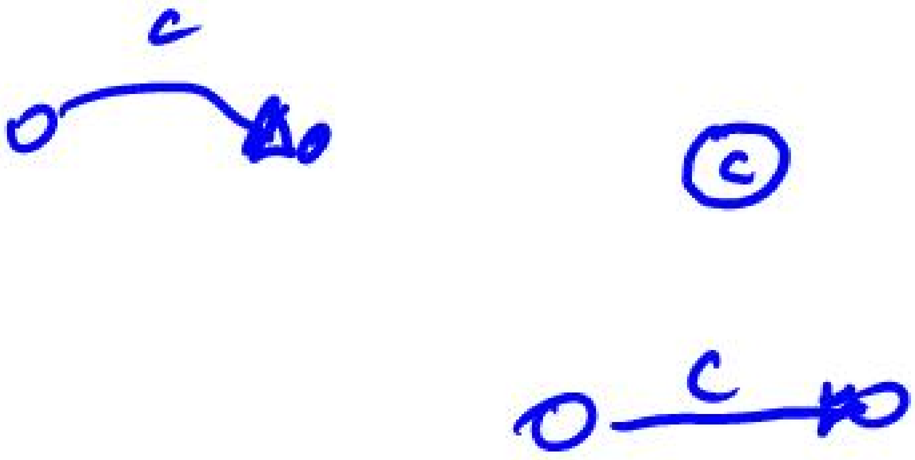
# Finding vertex disjoint paths



Construct a maximum cardinality set of vertiex disjoint paths

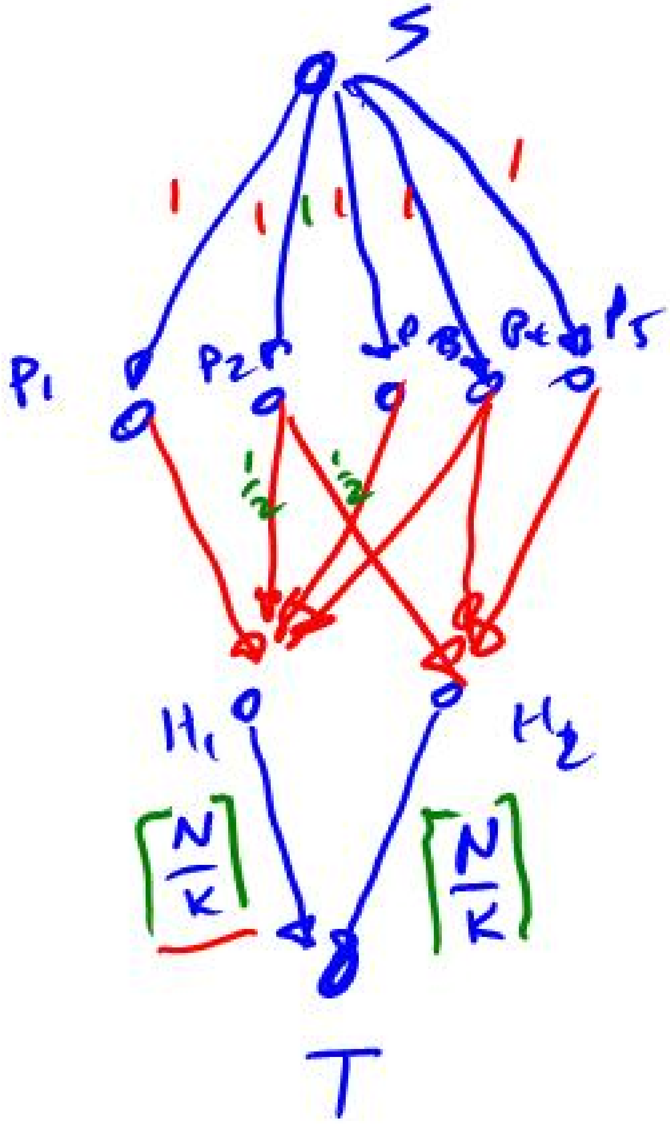
# Network flow with vertex capacities



# Balanced allocation
# Problem 9, Page 419

- To make a long story short:
  - N injured people
  - K hospitals
  - Assign each person to a hospital with 30 minutes drive
  - Assign N/K patients to each hospital

$\lceil x \rceil$



# Baseball elimination

- Can the Dinosaurs win the league?
- Remaining games:
  - AB, AC, AD, AD, AD, BC, BC, BC, BD, CD

| | | W | L |
|---|---|---|---|
| Ants | 1 | 4 | 2 5 |
| Bees | 1 | 4 | 2 3 |
| Cockroaches | 2 | 3 | 2 4 |
| Dinosaurs | | 1 6 | 5 |

A team wins the league if it has strictly more wins than any other team at the end of the season
A team ties for first place if no team has more wins, and there is some other team with the same number of wins

# Baseball elimination

- Can the Fruit Flies win or tie the league?
- Remaining games:
  - AC, AD, AD, AD, AF, BC, BC, BC, BC, BC, BD, BE, BE, BE, BE, BF, CE, CE, CE, CF, CF, DE, DF, EF, EF

| | | W | L |
|---|---|---|---|
| Ants | 2 | 17 | 12 |
| Bees | 3 | 16 | 7 |
| Cockroaches | 3 | 16 | 7 |
| Dinosaurs | 5 | 14 | 13 |
| Earthworms | 5 | 14 | 10 |
| Fruit Flies | | 12 | 15 |
| | | 19 | |

# Assume Fruit Flies win remaining games

- Fruit Flies are tied for first place if no team wins more than 19 games
- Allowable wins
  - Ants (2)
  - Bees (3)
  - Cockroaches (3)
  - Dinosaurs (5)
  - Earthworms (5)
- 18 games to play
  - AC, AD, AD, AD, BC, BC, BC, BC, BC, BD, BE, BE, BE, BE, CE, CE, CE, DE

| | W | L |
|---|---|---|
| Ants | 17 | 13 |
| Bees | 16 | 8 |
| Cockroaches | 16 | 9 |
| Dinosaurs | 14 | 14 |
| Earthworms | 14 | 12 |
| Fruit Flies | 19 | 15 |

# Remaining games

AC, AD, AD, AD, BC, BC, BC, BC, BC, BD, BE, BE, BE, BE, CE, CE, CE, DE

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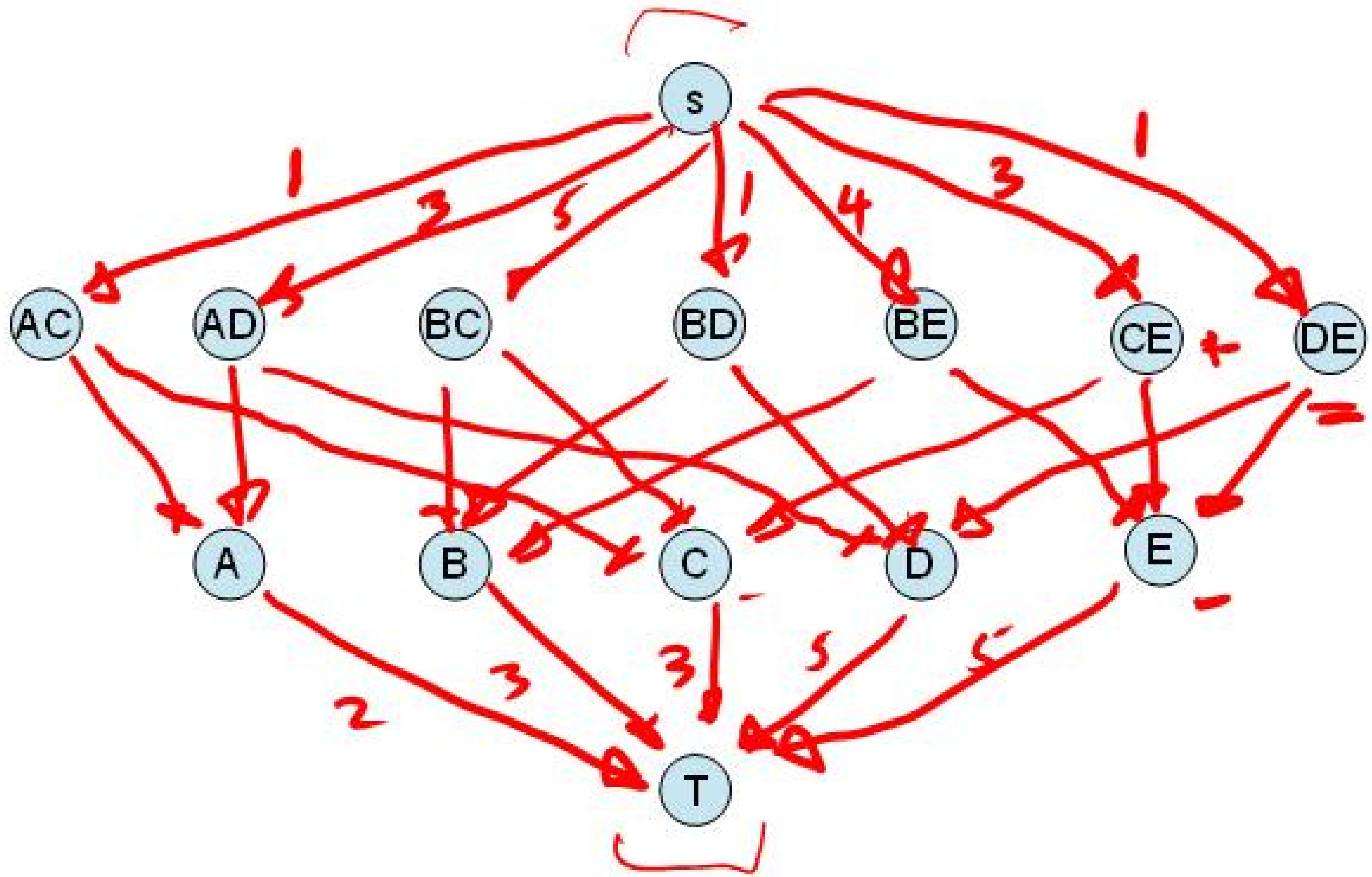


# Solving problems with a minimum cut

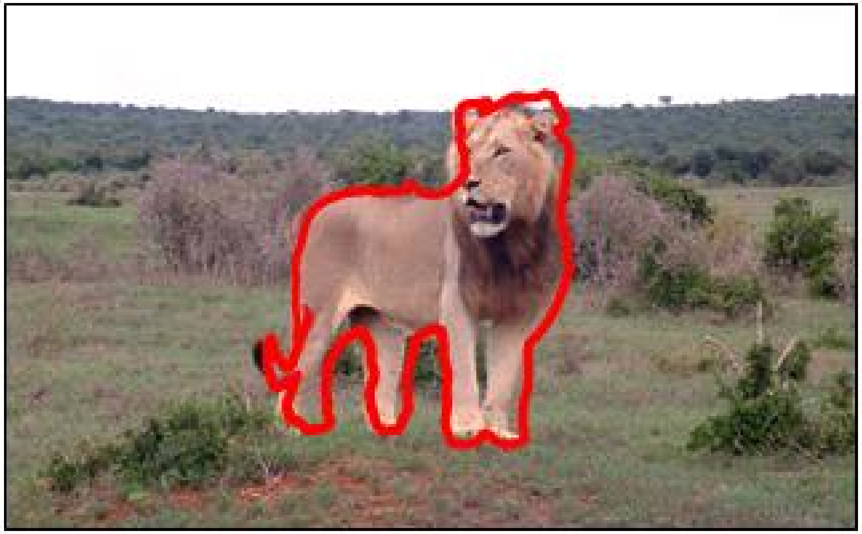
- Image Segmentation
- Open Pit Mining / Task Selection Problem

$S$, $T$ is a cut if $S$, $T$ is a partition of the vertices with $s$ in $S$ and $t$ in $T$

The capacity of an $S$, $T$ cut is the sum of the capacities of all edges going from $S$ to $T$

# Image Segmentation

- Separate foreground from background
- Reduction to min-cut problem



*S*, T is a cut if *S*, T is a partition of the vertices with
s in *S* and t in T

The capacity of an *S*, T cut is the sum of the capacities of all edges going from *S* to T

E - edges cut in pixel graph

# Image analysis

$A^* = \sum_i a_i$

$B^* = \sum_j b_j$

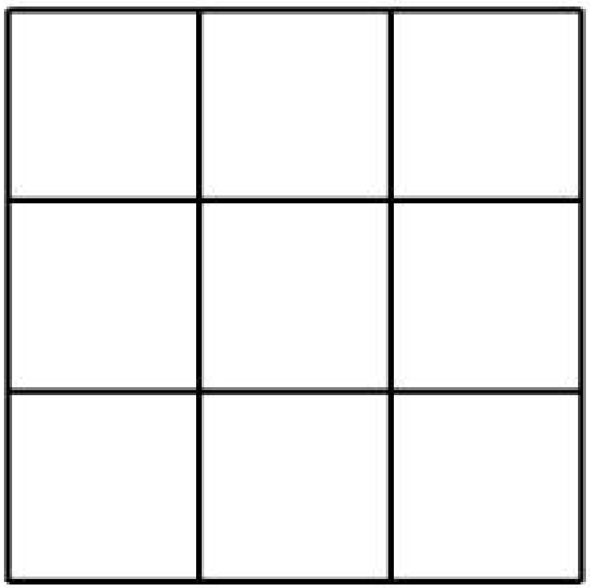
- $a_i$: value of assigning pixel i to the foreground
- $b_i$: value of assigning pixel i to the background
- $p_{ij}$: penalty for assigning i to the foreground, j to the background or vice versa
- A: foreground, B: background
- $Q(A,B) = \Sigma_{\{i \text{ in } A\}} a_i + \Sigma_{\{j \text{ in } B\}} b_j - \Sigma_{\{(i,j) \text{ in } E,\ i \text{ in } A,\ j \text{ in } B\}} p_{ij}$

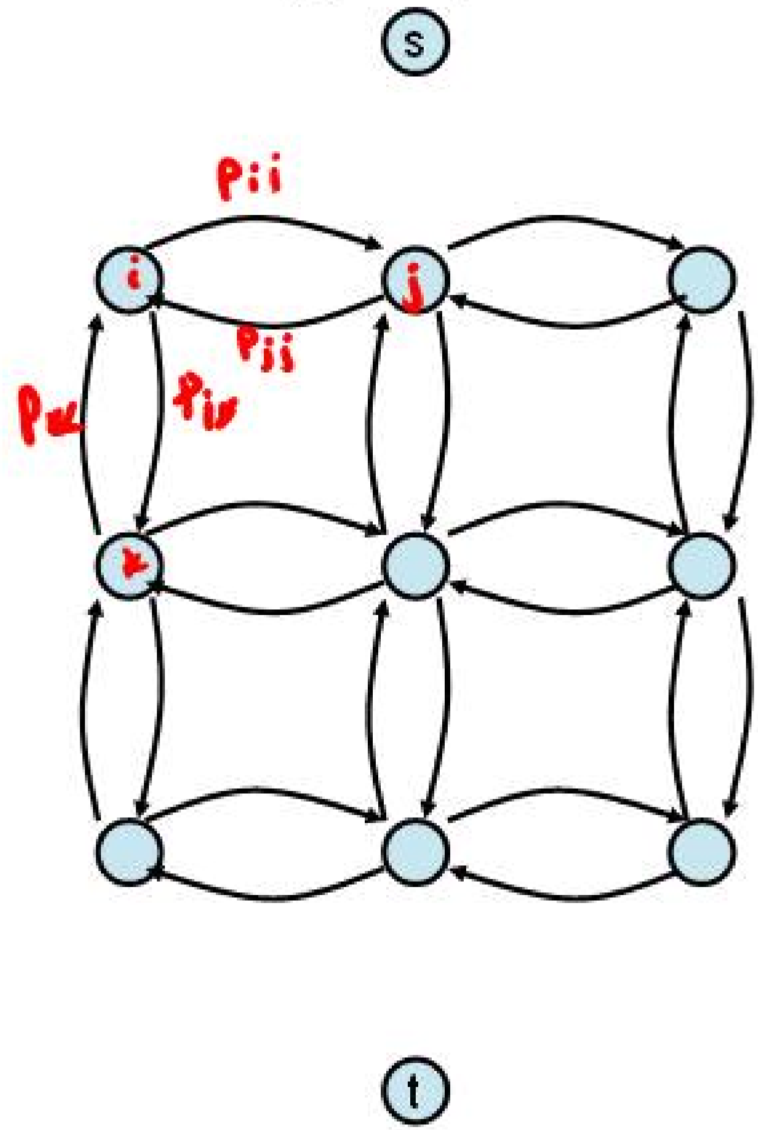
$Q(A,B) = A^* + B^* - CAP(A,B)$

$$= \sum_{i \in A} a_i + \sum_{j \in B} b_j - \sum_{(i,j) \in E} p_{ij} + \sum_{i \notin A} a_i - \sum_{i \notin A} a_i + \sum_{j \notin B} b_j - \sum_{j \notin B} b_j$$

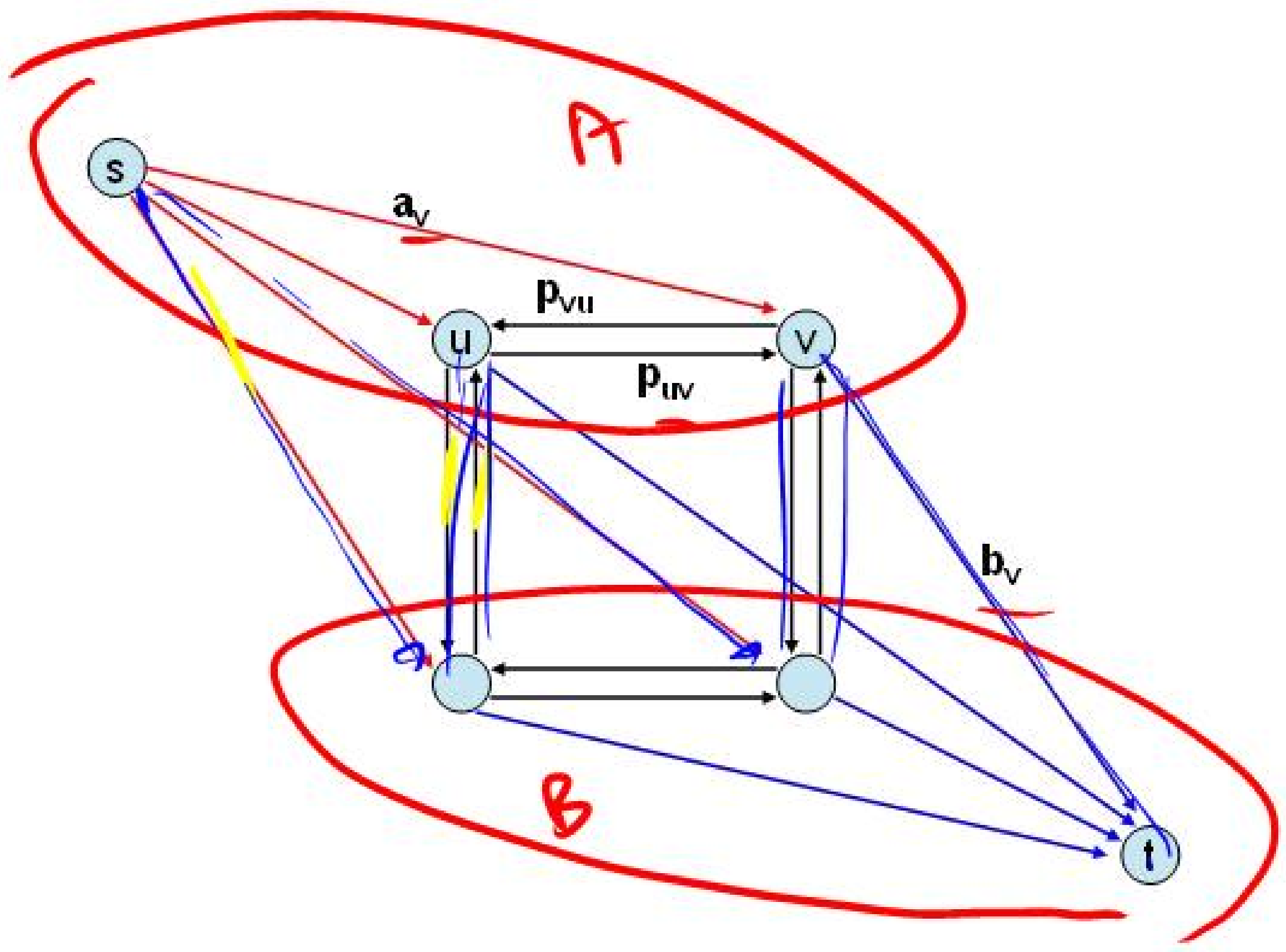
$$= A^* + B^* - \sum_{i \notin A} a_i - \sum_{j \notin B} b_j - \sum_{(i,j) \in E} p_{ij}$$

# Pixel graph to flow graph





# Mincut Construction



# Open Pit Mining

# Application of Min-cut

- Open Pit Mining Problem
- Task Selection Problem
- Reduction to Min Cut problem

*S*, T is a cut if *S*, T is a partition of the vertices with s in *S* and t in T

The capacity of an *S*, T cut is the sum of the capacities of all edges going from *S* to T

# Open Pit Mining

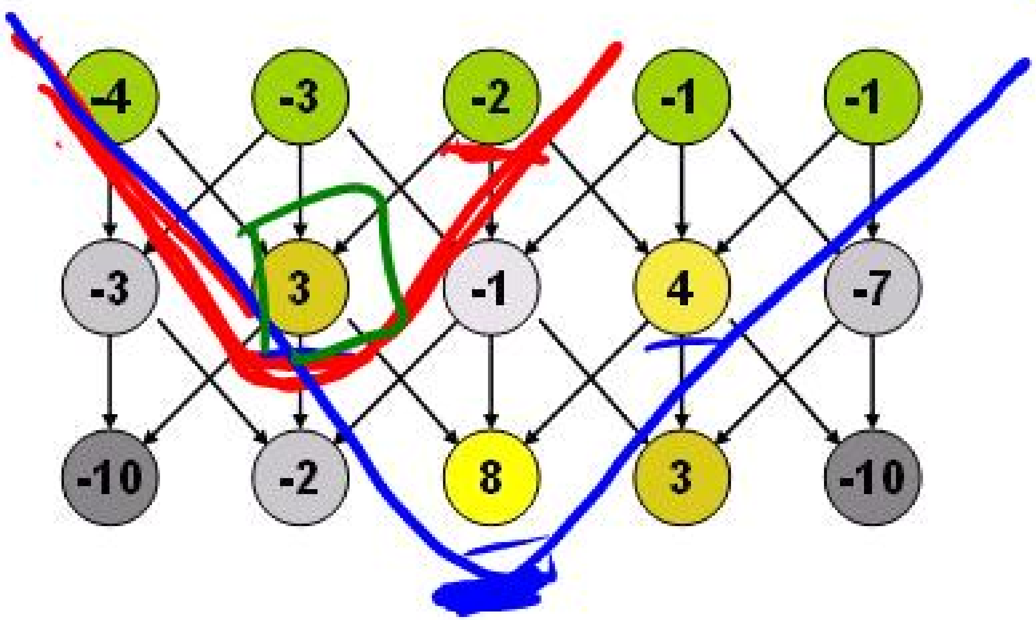
- Each unit of earth has a profit (possibly negative)
- Getting to the ore below the surface requires removing the dirt above
- Test drilling gives reasonable estimates of costs
- Plan an optimal mining operation

$-4 - 3 - 2 + 3 = -6$

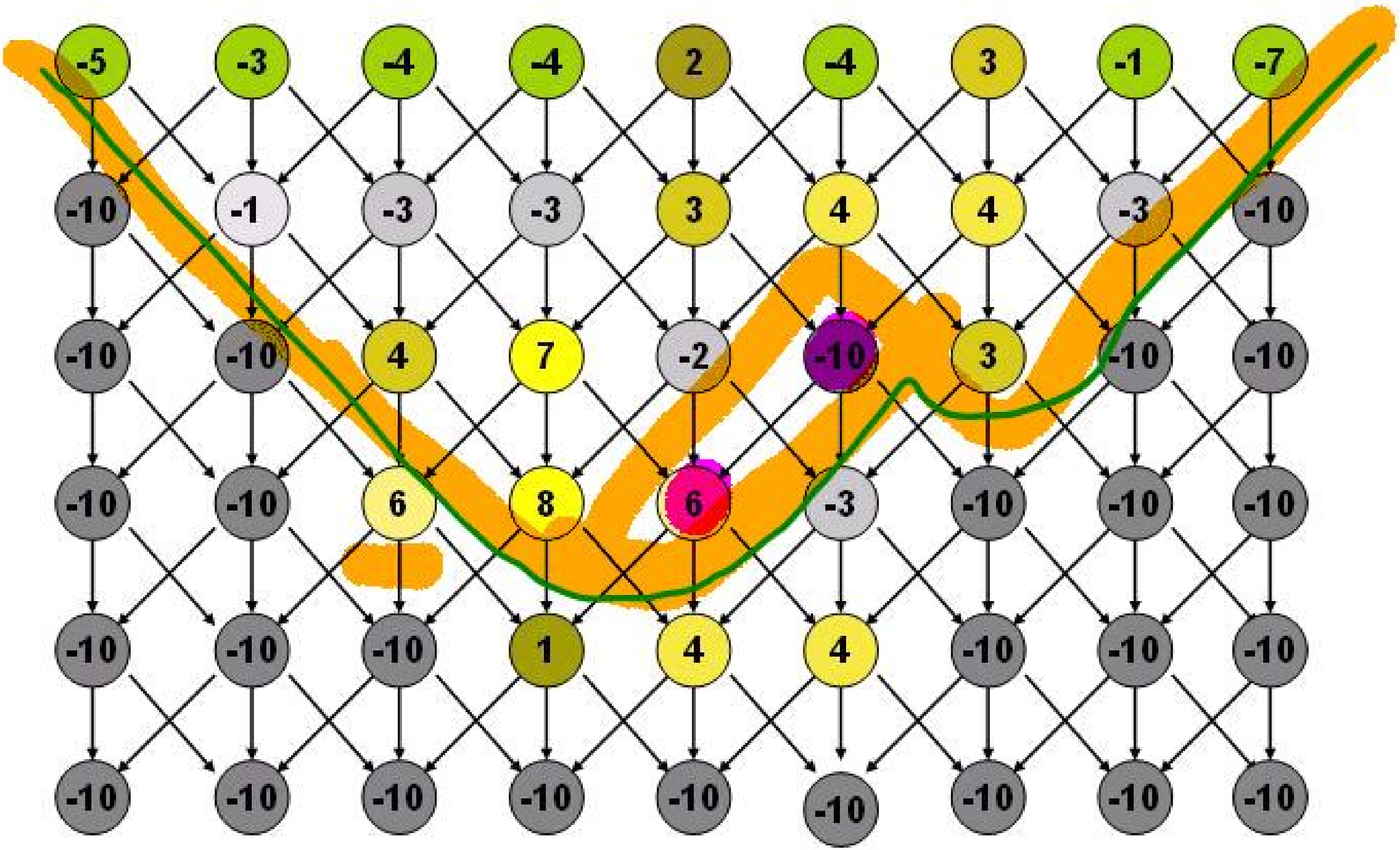
# Mine Graph

+15 −12

Profit +3



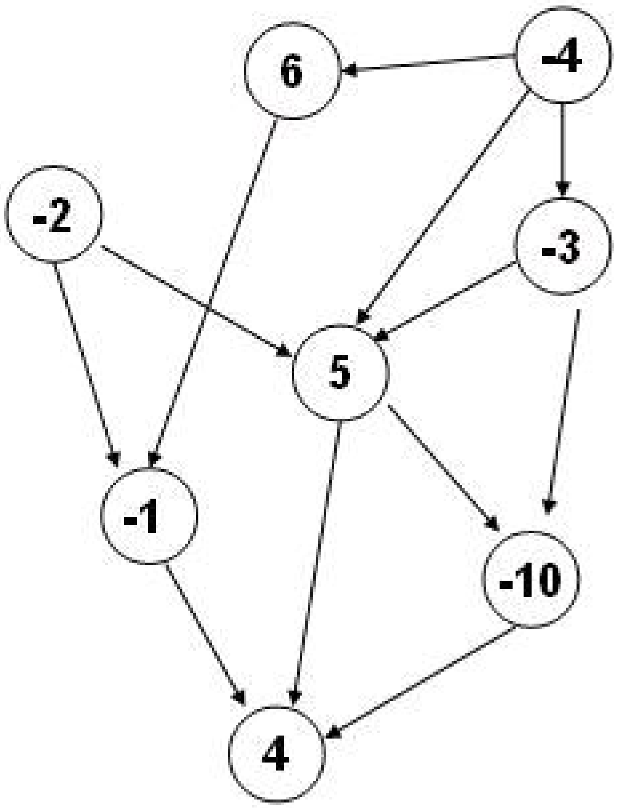
# Determine an optimal mine



Directed Acyclic Graph.

# Generalization

- Precedence graph G=(V,E)
- Each v in V has a profit p(v)
- A set F if *feasible* if when w in F, and (v,w) in E, then v in F.
- Find a feasible set to maximize the profit

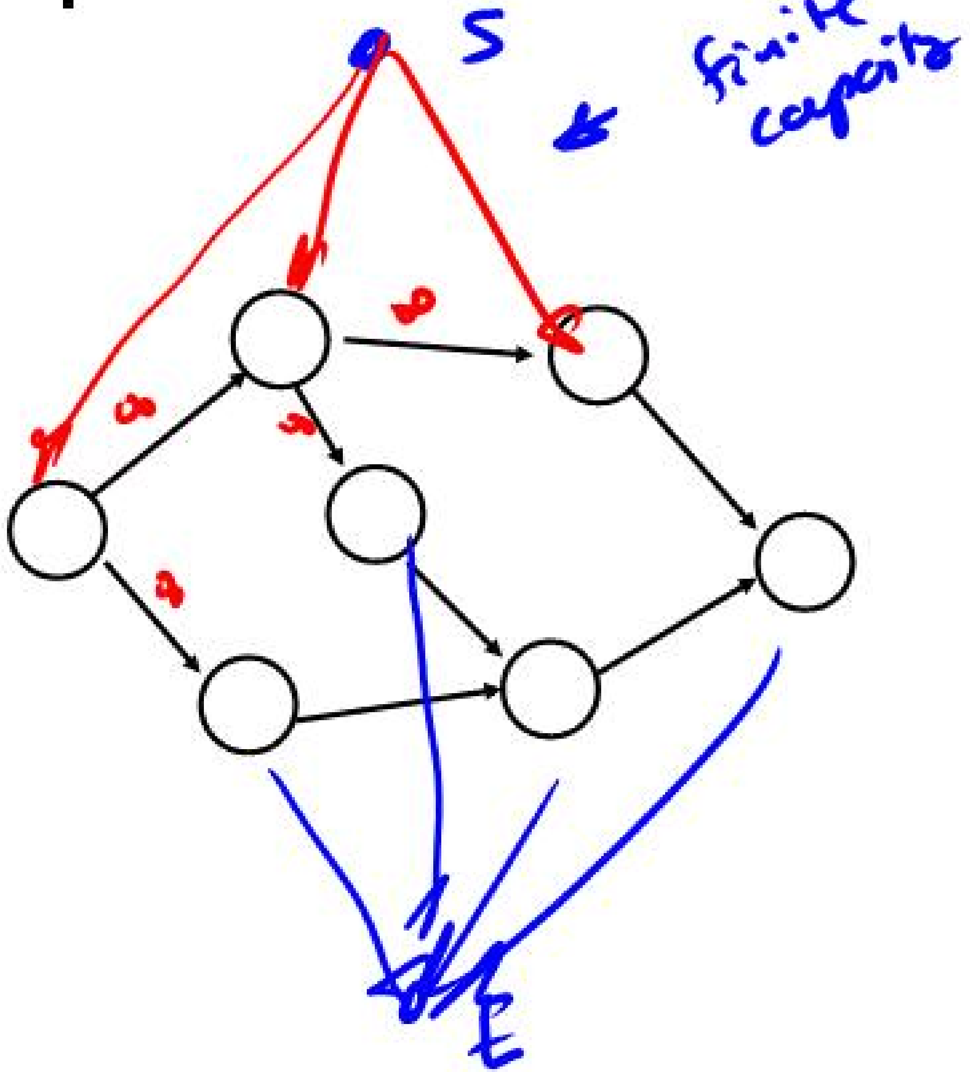


# Min cut algorithm for profit maximization

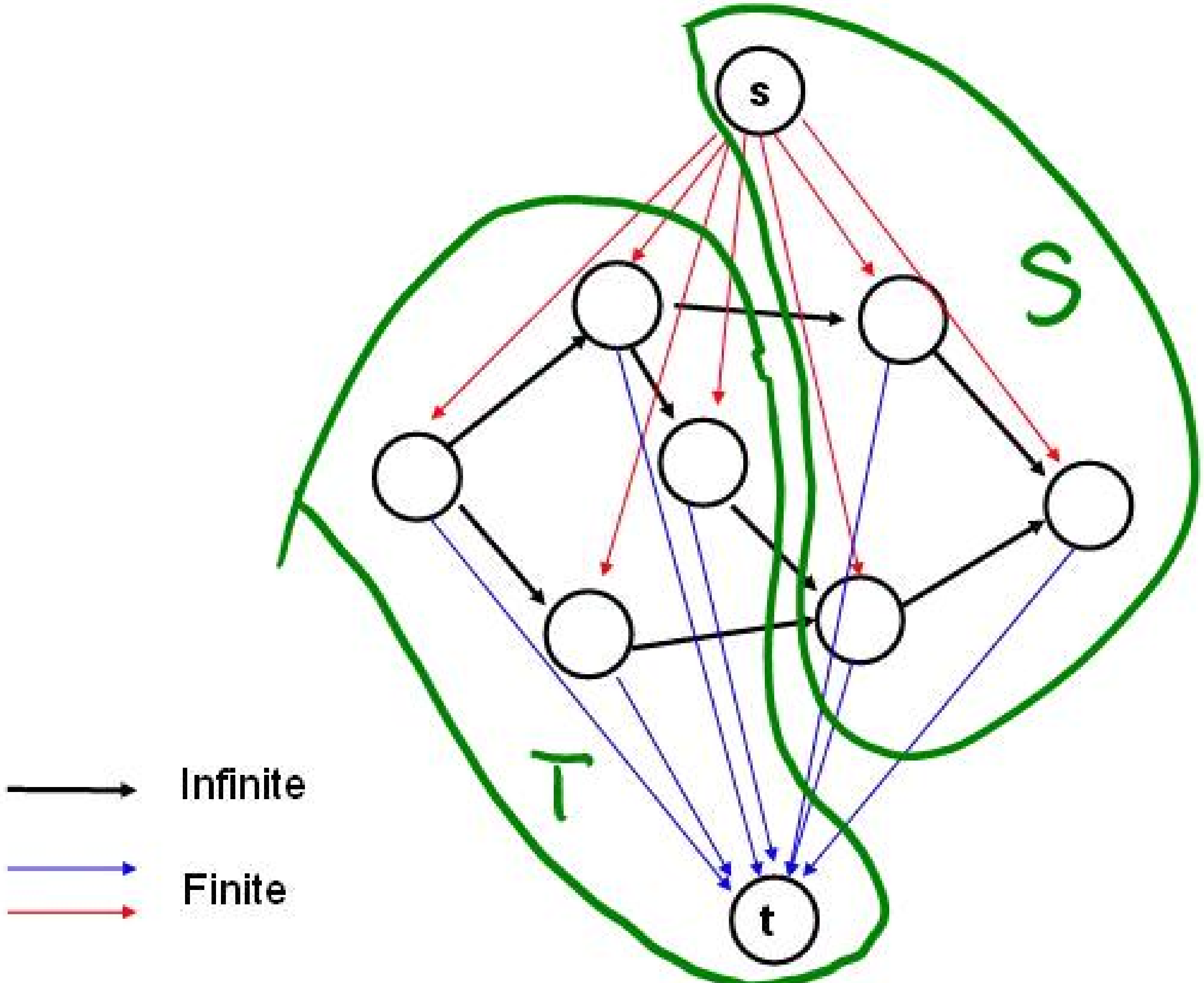
- Construct a flow graph where the minimum cut identifies a feasible set that maximizes profit

# Precedence graph construction

- Precedence graph G=(V,E)
- Each edge in E has infinite capacity
- Add vertices s, t
- Each vertex in V is attached to s and t with finite capacity edges

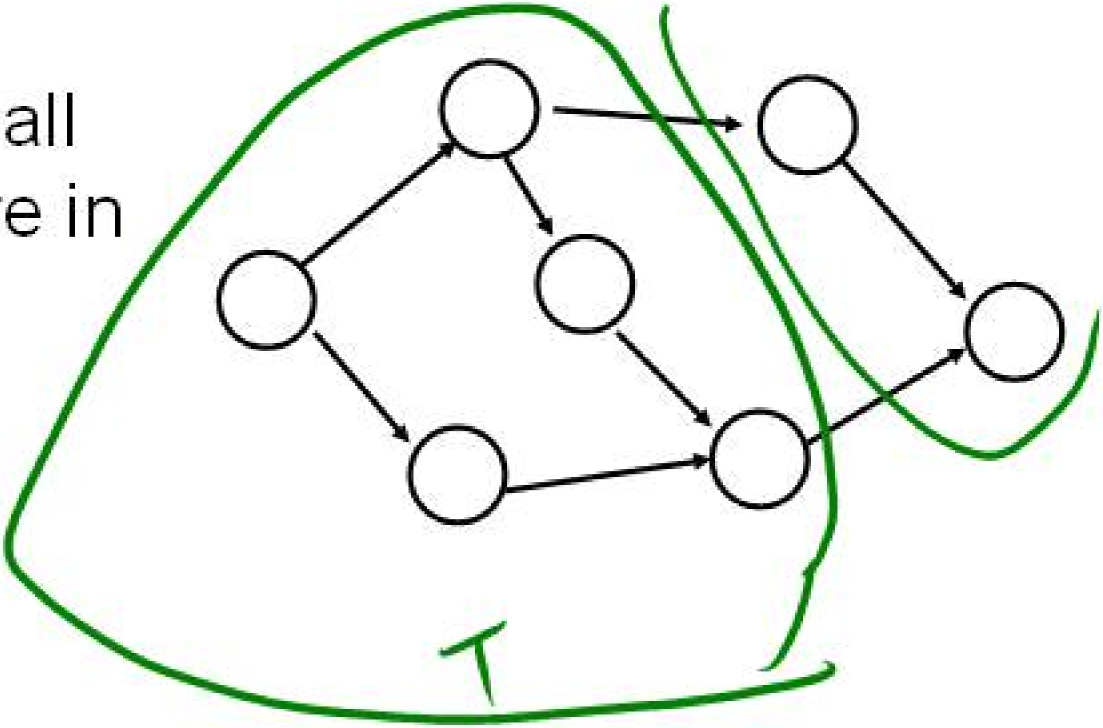


# Show a finite value cut with at least two vertices on each side of the cut



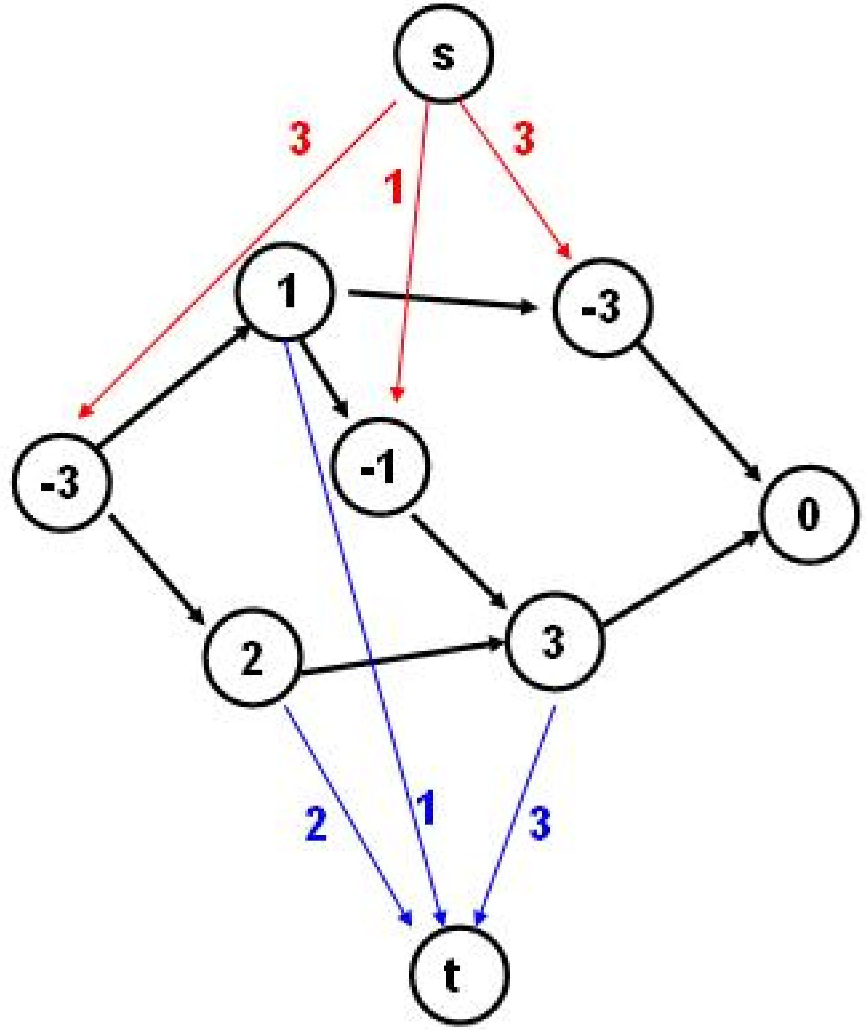
# The sink side of a finite cut is a feasible set

- No edges permitted from S to T
- If a vertex is in T, all of its ancestors are in T



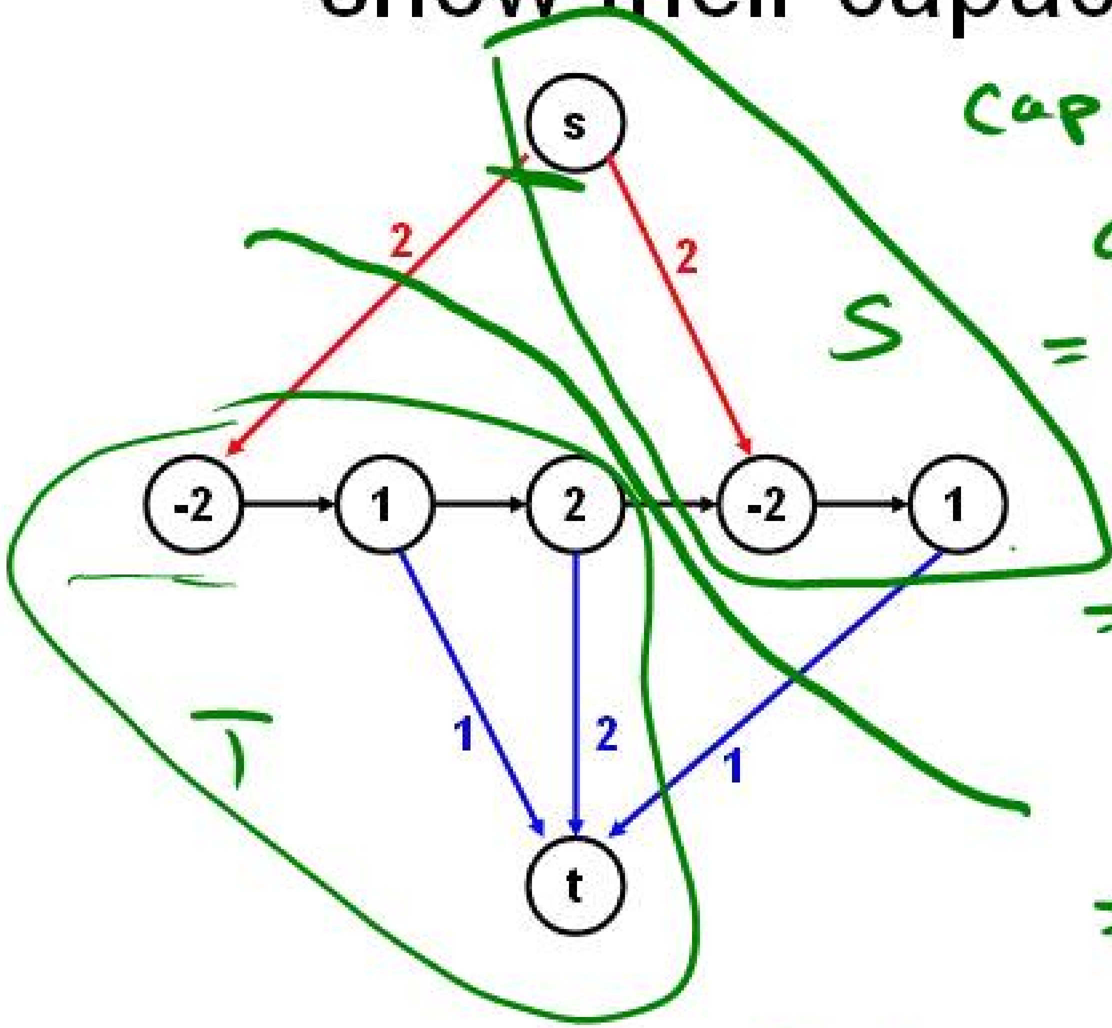
# Setting the costs

- If $p(v) > 0$,
  - $cap(v,t) = p(v)$
  - $cap(s,v) = 0$
- If $p(v) < 0$
  - $cap(s,v) = -p(v)$
  - $cap(v,t) = 0$
- If $p(v) = 0$
  - $cap(s,v) = 0$
  - $cap(v,t) = 0$



$B = \text{Ben}(S) + \text{Ben}(T)$

# Enumerate all finite s,t cuts and show their capacities




$\text{cap}(S,T) =$

$\text{cost}(T) + \text{ben}(S)$

$= \text{cost}(T) + \text{ben}(S) + \text{ben}(T) - \text{ben}(T)$

$= \text{cost}(T) - \text{ben}(T) + B$

$= B - \text{Profit}(T)$

$\text{Profit}(T) = B - \text{cap}(S,T)$

# Summary

- Construct flow graph
  - Infinite capacity for precedence edges
  - Capacities to source/sink based on cost/benefit
- Finite cut gives a feasible set of tasks
- Minimizing the cut corresponds to maximizing the profit
- Find minimum cut with a network flow algorithm