Today

- online algs
- bipartite matching
- mini-probability review

Online algs & competitive analysis

- algorithms make decisions w/ full information
- full knowledge of past, none of future.

Ski rental

\$50 to rent skis
\$500 to buy skis

Optimal $\underbrace{\text{offline}}_{\text{knows the future}}$ alg: rent if ski < 10, buy if ski ≥ 10 times.

- Rent until paid the cost of buying

This algorithm has competitive ratio = 2

An online algorithm has competitive ratio c if $\forall$ input (future) σ

$$\text{Alg cost}(\sigma) \le c \, \text{OPT}(\sigma)$$

online alg — optimal clairvoyant

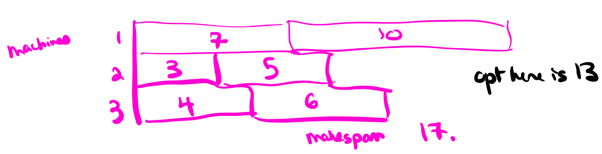
Scheduling

- m identical machines
- sequence of jobs that we see one at a time
- upon arrival of job j, learn processing time p_j of job j

  assign it to one of machines
- Goal: schedule the jobs to minimize makespan of schedule.

  makespan: time at which last job finishes

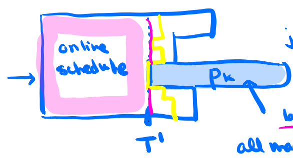
$m = 3$ $\quad p_1 = 7, p_2 = 3, p_3 = 4, 5, 6, 10$



Greedy alg: always assign job to machine that will become available soonest. ($m = \#$ machines, $n = \#$ jobs)

<u>Thm</u> Greedy alg is $2 - \frac{1}{m}$ competitive.

<u>Proof</u> Suppose processing times are $p_1, p_2, \ldots, p_n$



Online makespan: $\boxed{T' + p_k}$ $\qquad T' \cdot m \le \sum_{j \ne k} p_j$

Lower bounds on optimal offline

(1) $OPT \ge p_k$

(2) $OPT \ge \frac{\sum_{i=1}^{n} p_i}{m}$

$$\le \frac{\sum_{j \ne k} p_j}{m} + p_k$$

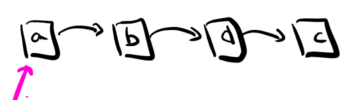
$$= \frac{\sum_j p_j}{m} - \frac{p_k}{m} + p_k$$

$$\overset{(2)}{\le} OPT + p_k\left(1 - \frac{1}{m}\right)$$

$$\overset{(1)}{\le} OPT + OPT\left(1 - \frac{1}{m}\right)$$

$$= OPT\left(2 - \frac{1}{m}\right)$$

<u>List Update</u>



<u>Model-1</u>

- cost to "service" request for item at depth i in list $= i$
- free to move elt forward (towards front) by any amt. ✓
- adjacent items can be swapped at cost of 1 ✓

<u>Model-2</u>

- cost to service req at depth $i = i$
- required to always MTF
- adjacent items can be swapped at cost of 1

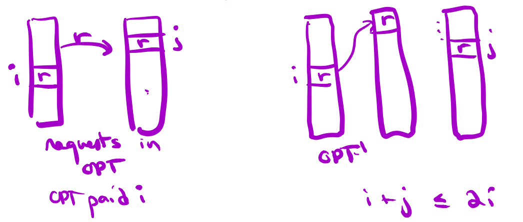
Input: sequence of requests $\sigma = r_1, r_2, r_3, \ldots\ r_T$

<u>Thm</u> Move-to-front is 2-competitive.

3 steps to proof.

- $OPT(\sigma)$ opt offline cost in Model-1.

Let $OPT'(\sigma)$ be "simulation" of $OPT(\sigma)$ in Model-2



$$OPT'(\sigma) \le 2\, OPT(\sigma) \qquad (1)$$

$$OPT\text{-}Model2(\sigma) \le OPT'(\sigma) \qquad (2)$$

$$* \quad OPT\text{-}Model2(\sigma) = MTF(\sigma) \qquad (3)$$

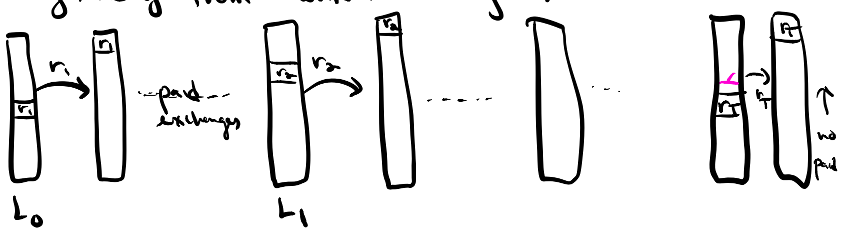
$$(1)\,(2)\,(3) \Rightarrow MTF(\sigma) \le 2\,OPT(\sigma)$$

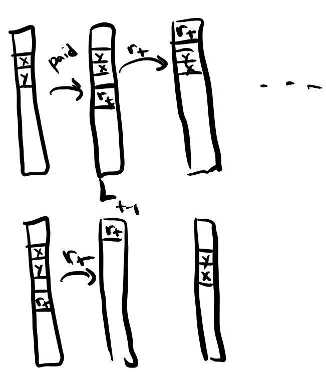
Proof of (3) ≡ in model 2, never any benefit to paid exchanges.

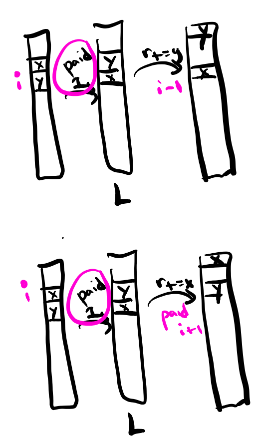
Model-2

- cost to service req at depth i = i
- required to always MTF
- adjacent items can be swapped at cost of 1

suppose by contradiction $OPT\text{-}Model2(\sigma)$ does paid exchanges. Show how to get rid of them without increasing cost.



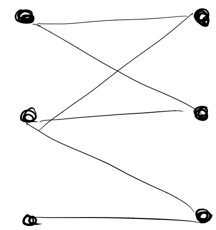




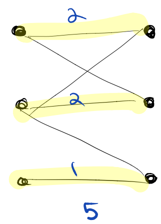
don't do swap. also pay' end up in same state.

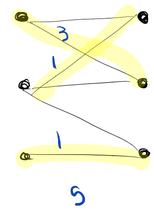
Maximum Weight Matching

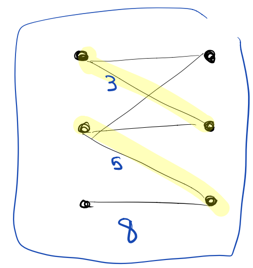
bipartite graphs



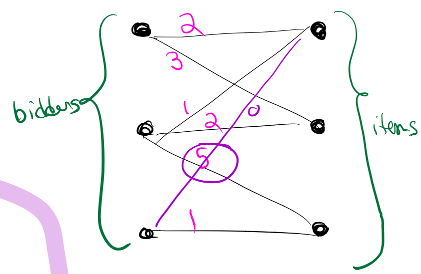
matching: subset of edges
with no common endpts







matching is perfect if all vertices are incident to an edge in matching



Algorithmic problem:

find a maximum weight matching in a weighted n×n bipartite graph. [if ∄ edge (i,j), set $v_{ij}=0$]

v_{ij} - value bidder i has for item j

assume weights (v_{ij}'s) are integers

ascending auction algorithm

Fix bid increment $\delta = \frac{1}{n+1}$

Maintain price vector $(p_1,\dots,p_n)$ $\quad$ p_j is the price of item j

Initially all prices = 0 and matching is empty. $\quad M(i)$ [matching bidder i] $\quad M(i)=\emptyset$.

As long as some bidder is not matched

pick unmatched bidder i

consider $D(i) = \left\{ \text{items } j \;\middle|\; \overline{v_{ij}-p_j} \geq \overline{v_{ik}-p_k} \;\; \forall k \neq j, \;\; v_{ij} \geq p_j \right\}$

Pick some $j \in D(i)$

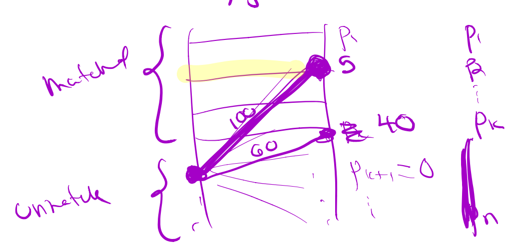
If j unmatched, then $M(i):=j$

else, say $M(\ell)=j$,

$M(\ell):=\emptyset$

$M(i):=j$

increase j by δ.



Probability mini-review

Ω sample space – set of possible outcomes

$p(\cdot)$ tells you prob of each outcome.

3 homeworks to 3 students
random perm

X (# of people that got own hw back)	Ω	$Pr(\cdot)$
3	1 2 3	$\frac{1}{6}$
1	1 3 2	$\frac{1}{6}$
1	2 1 3	$\frac{1}{6}$
0	2 3 1	$\frac{1}{6}$
0	3 1 2	$\frac{1}{6}$
1	3 2 1	$\frac{1}{6}$

$$E(X) = 3\cdot\frac{1}{6} + 1\cdot\frac{3}{6} + 0\cdot\frac{2}{6}$$

$\forall \omega \in \Omega$
$$0 \le Pr(\omega) \le 1$$
$$\sum_{\omega\in\Omega} Pr(\omega) = 1$$

event $E \subseteq \Omega$
$$Pr(E) = \sum_{\omega \in E} Pr(\omega)$$

E: person 1 got their own hw back

$$\rightarrow Pr(E) = Pr(123) + Pr(132) = \frac{2}{6}$$

unif prob space: every outcome has same prob.
$$Pr(\omega) = \frac{1}{|\Omega|}$$

unif: $$Pr(E) = \frac{|E|}{|\Omega|}$$

A random variable X on a prob space
$$X: \Omega \to \mathbb{R}$$

$$\{X=a\} \qquad \{\omega\in\Omega \mid X(\omega)=a\}$$

The expectation (expected value) of r.v. X

$$E(X) = \sum_{k\in Range(X)} k\, Pr(X=k) = \sum_{\omega\in\Omega} X(\omega) Pr(\omega)$$

Linearity of expectation:

$$X = X_1 + X_2 + \dots + X_k$$
$$E(X) = E(X_1) + E(X_2) + \dots + E(X_k)$$

X (# of people that got own hw back)	Ω	$\Pr(\cdot)$	X_1	X_2	X_3
3	1 2 3	$\frac{1}{6}$	1	1	1
1	1 3 2	$\frac{1}{6}$	1	0	0
1	2 1 3	$\frac{1}{6}$	0	0	1
0	2 3 1	$\frac{1}{6}$	0	0	0
0	3 1 2	$\frac{1}{6}$	0	0	0
1	3 2 1	$\frac{1}{6}$	0	1	0

$E(X_i) = \frac{1}{3}$

$$E(X) = 3 \cdot \frac{1}{6} + 1 \cdot \frac{3}{6} + 0 \cdot \frac{2}{6} = 1$$

$$X_i = \begin{cases} 1 & \text{if person } i \text{ got their own hw back} \\ 0 & \text{o.w.} \end{cases}$$

$$E(X) = E(X_1) + E(X_2) + E(X_3)$$

$$E(X_i) = 1 \cdot \Pr(X_i = 1) + 0 \cdot \Pr(X_i = 0)$$
$$= \Pr(X_i = 1)$$