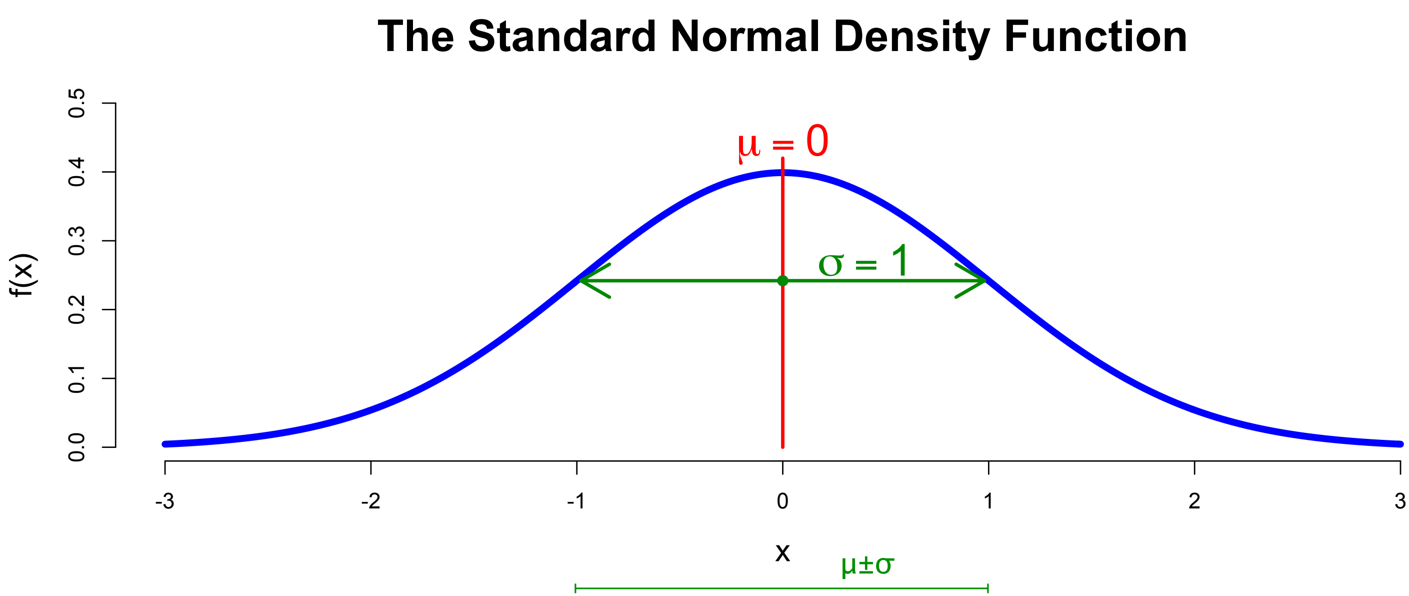
X is a normal (aka Gaussian) random variable X ~ N(μ, σ²)

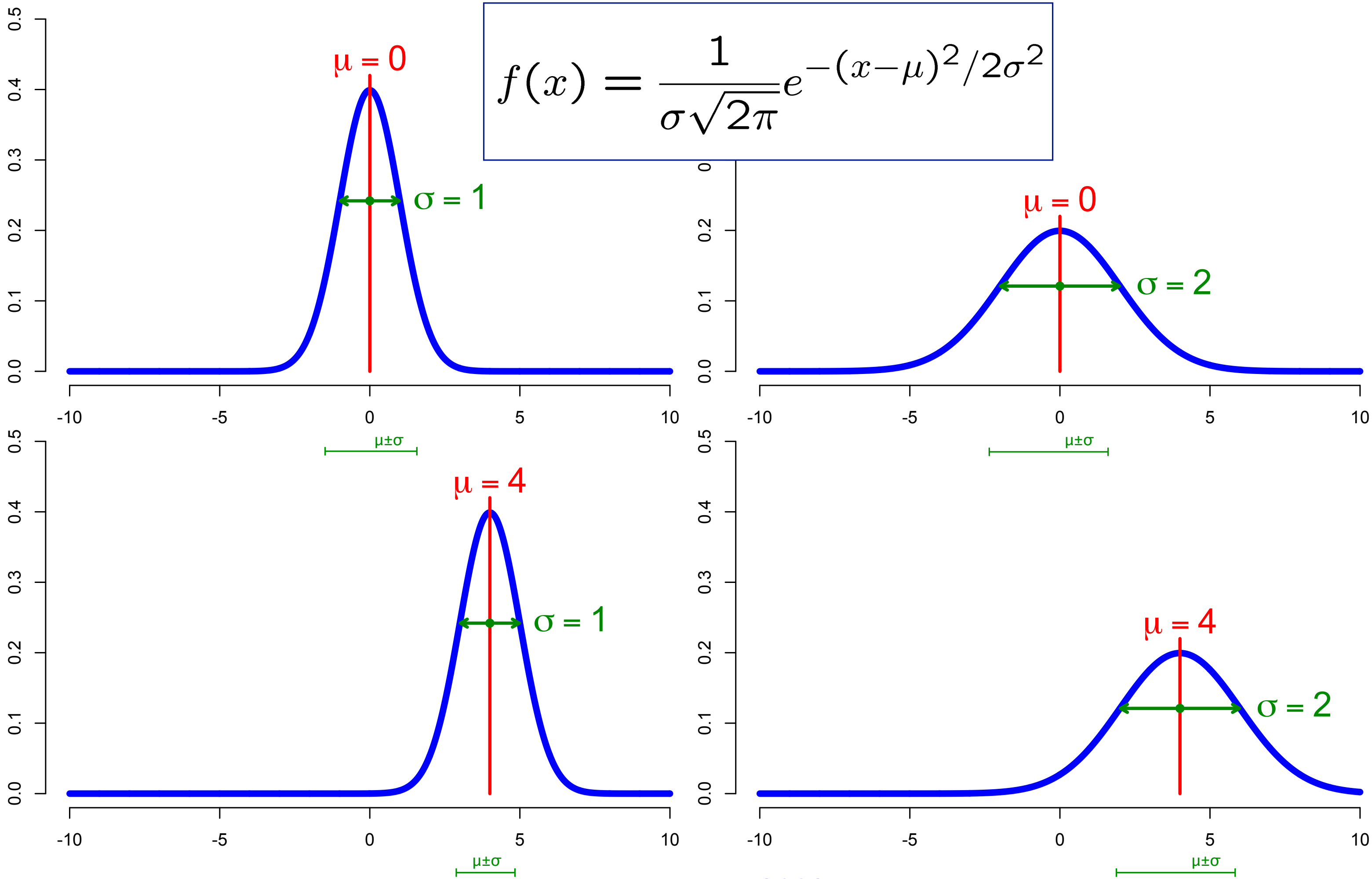
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

$$E[X] = \mu \qquad \mathrm{Var}[X] = \sigma^2$$



changing μ, σ

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$



density at μ is ≈ .399/σ

X is a normal random variable $X \sim N(\mu,\sigma^2)$

$Y = aX + b$

$E[Y] = E[aX+b] =$

$Var[Y] = Var[aX+b] =$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

$X = a_1X_1 + a_2X_2 + \dots + a_nX_n$

a_i's consts

$X_i \sim N(\mu_i, \sigma_i^2)$

X_i's are indep.

$E(X) = a_1\mu_1 + a_2\mu_2 + \dots + a_n\mu_n$

$Var(X) = a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + \dots + a_n^2\sigma_n^2$

$X \sim N\left(\sum_{i=1}^{n} a_i\mu_i, \sum_{i=1}^{n} a_i^2\sigma_i^2\right)$

X is a normal random variable $X \sim N(\mu,\sigma^2)$

$Y = aX + b$

$E[Y] = E[aX+b] = a\mu + b$

$Var[Y] = Var[aX+b] = a^2\sigma^2$

$Y \sim N(a\mu + b, a^2\sigma^2)$

E[ ], Var[ ] as expected; "normality" is the surprise

Important special case: $Z = (X-\mu)/\sigma \sim N(0,1)$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

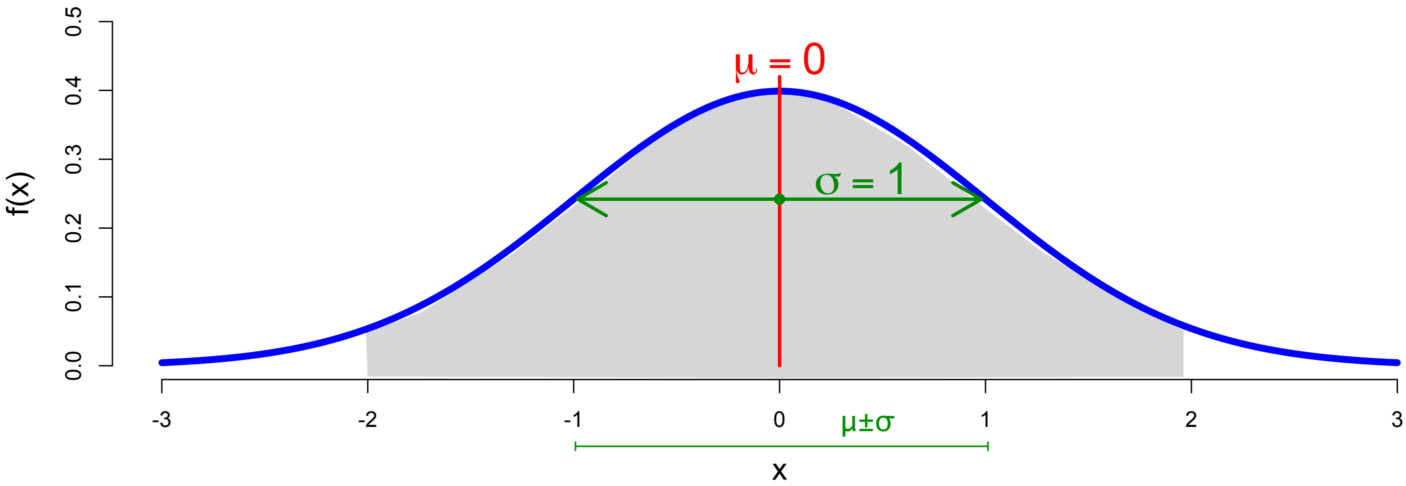
$Z \sim N(0,1)$ "*standard* (or *unit*) normal"

Use $\Phi(z)$ to denote CDF, i.e.

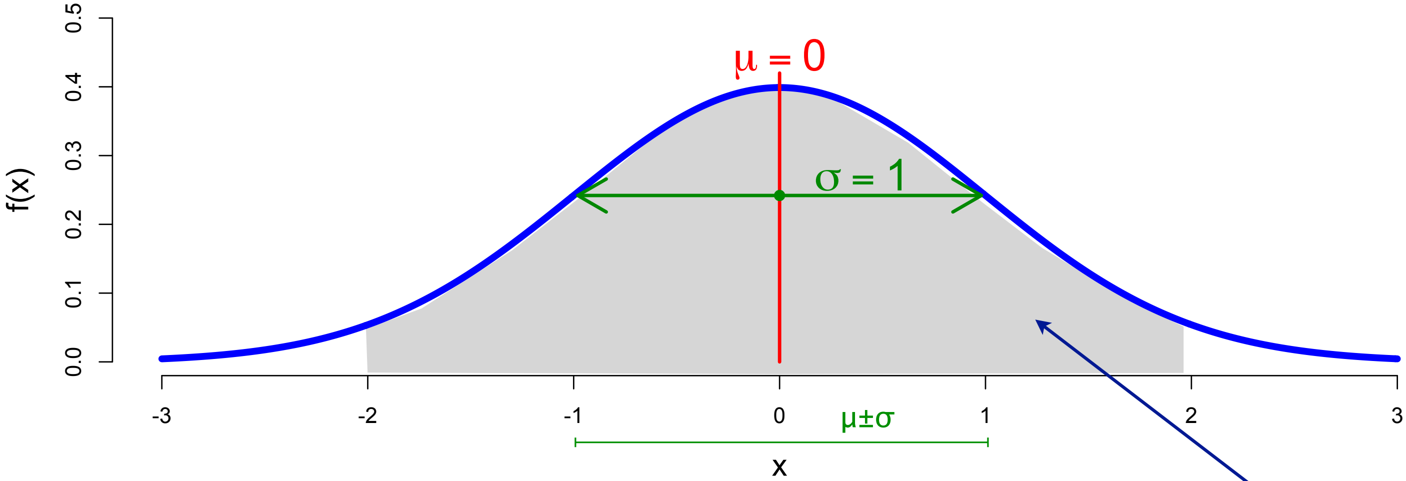
$$\Phi(z) = \Pr(Z \le z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$

no closed form ☹

The Standard Normal Density Function



The Standard Normal Density Function



If $Z \sim N(\mu,\sigma^2)$ what is $P(\mu-\sigma < Z < \mu+\sigma)$?

$$P(\mu - \sigma < Z < \mu + \sigma) = \Phi(1) - \Phi(-1) \approx 68\%$$

$$P(\mu - 2\sigma < Z < \mu + 2\sigma) = \Phi(2) - \Phi(-2) \approx 95\%$$

$$P(\mu - 3\sigma < Z < \mu + 3\sigma) = \Phi(3) - \Phi(-3) \approx 99\%$$

Why?

$N(\mu,\sigma^2)$ $N(0,1)$

$$\mu - k\sigma < \boxed{Z} < \mu + k\sigma \iff -k < \boxed{(Z-\mu)/\sigma} < +k$$

Consider i.i.d. (independent, identically distributed) random vars $X_1, X_2, X_3, \ldots$

X_i has $\mu = E[X_i]$ and $\sigma^2 = Var[X_i]$

Consider random variables

$$X_1 + X_2 + \ldots + X_n$$

and

$$\frac{1}{n}\sum_{i=1}^{n} X_i$$

Consider i.i.d. (independent, identically distributed) random vars $X_1, X_2, X_3, \dots$

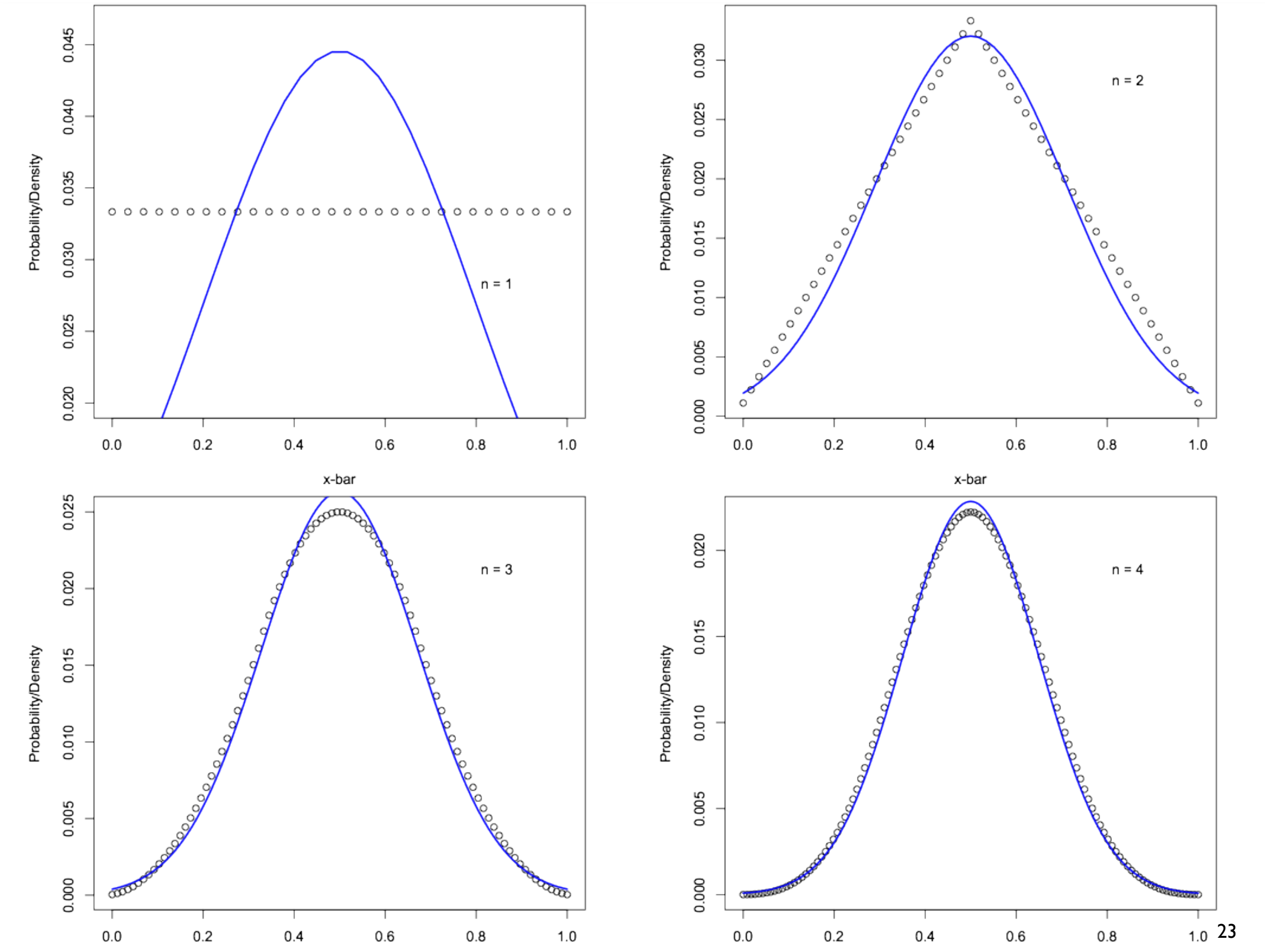
X_i has $\mu = E[X_i]$ and $\sigma^2 = Var[X_i]$

As $n \to \infty$,

$$\frac{X_1 + X_2 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \longrightarrow N(0,1)$$

Restated: As $n \to \infty$,

$$M_n = \frac{1}{n}\sum_{i=1}^{n} X_i \to N\left(\mu, \frac{\sigma^2}{n}\right)$$


Probability/Density
x-bar
n = 1
n = 2
n = 3
n = 4
0.0 0.2 0.4 0.6 0.8 1.0
0.020 0.025 0.030 0.035 0.040 0.045
0.000 0.005 0.010 0.015 0.020 0.025 0.030
0.000 0.005 0.010 0.015 0.020 0.025
0.000 0.005 0.010 0.015 0.020
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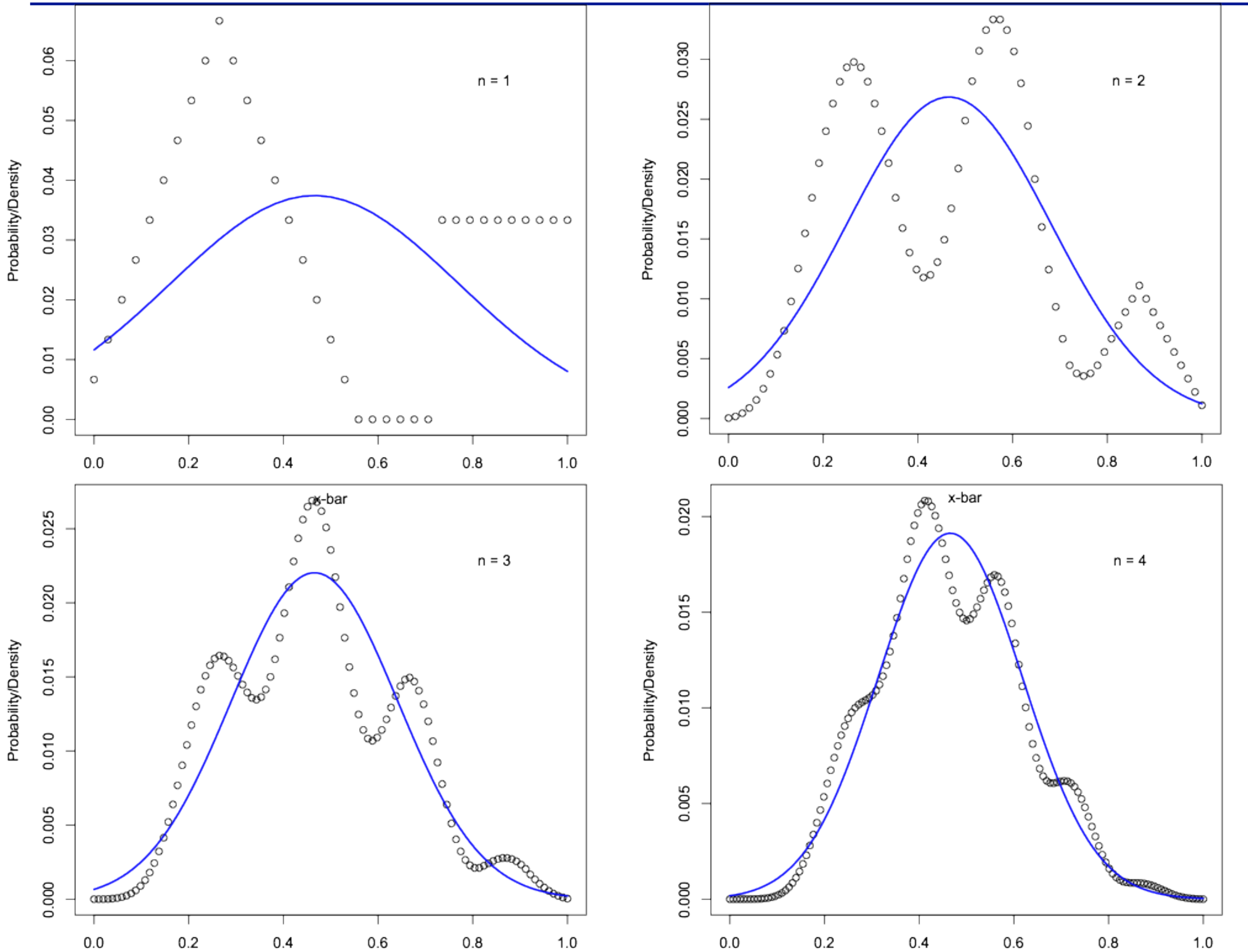
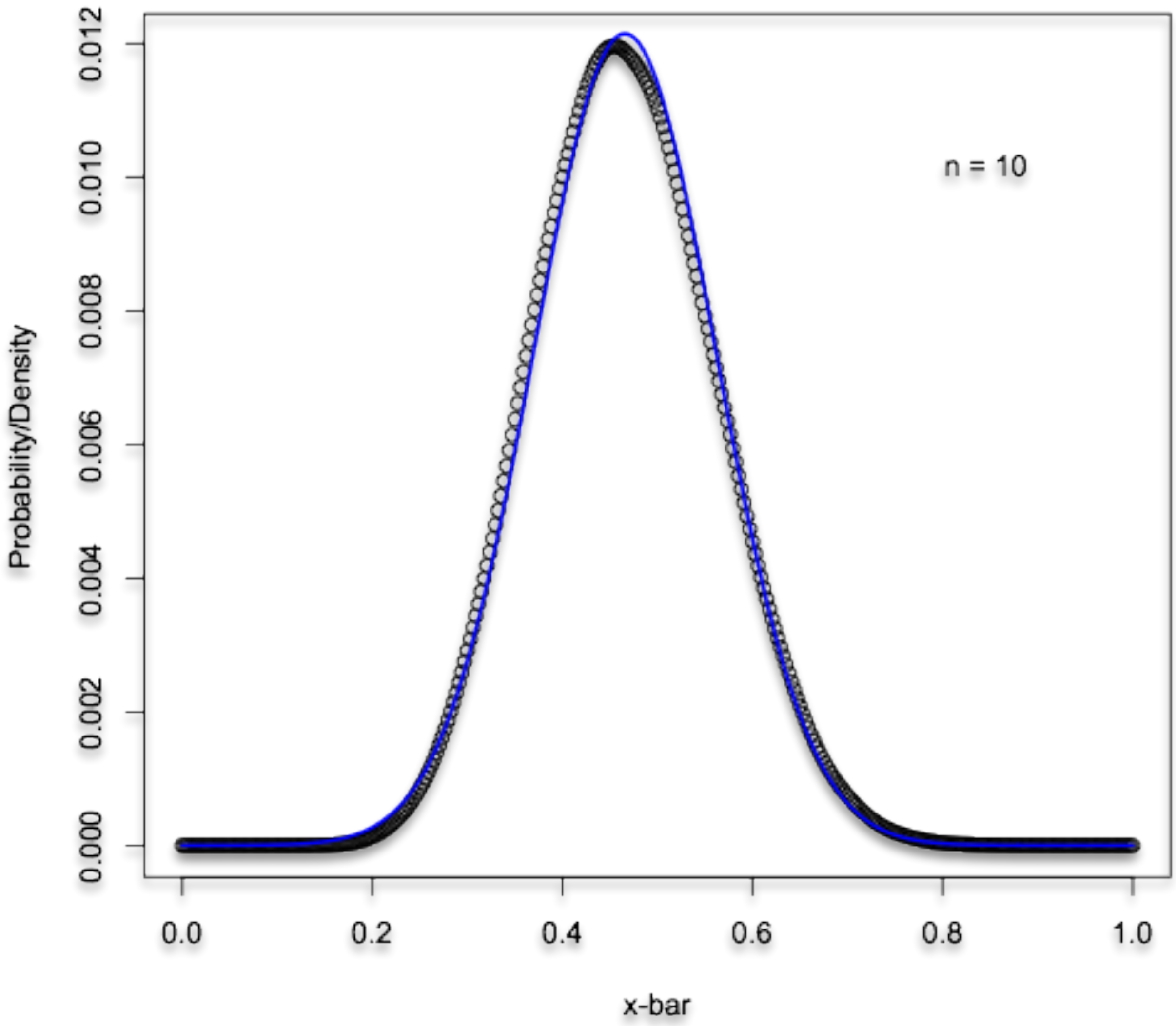
CLT applies even to even wacky distributions




Probability/Density
0.000
0.002
0.004
0.006
0.008
0.010
0.012
n = 10
0.0
0.2
0.4
0.6
0.8
1.0
x-bar

CLT in the real world

CLT is the reason many things appear normally distributed
Many quantities = sums of (roughly) independent random vars

Exam scores: sums of individual problems
People's heights: sum of many genetic & environmental factors
Measurements: sums of various small instrument errors
...