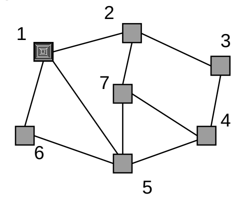
CSE 589
Applied Algorithms
Spring 1999

Minimum Spanning Tree
Disjoint Union / Find

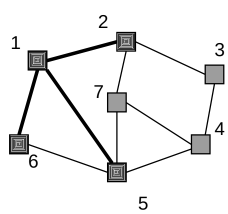
ST using Breadth First Search 1

- Uses a queue to order search



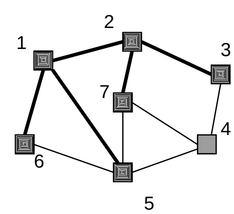
Queue = 1

Breadth First Search 2



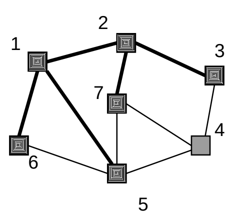
Queue = 2,6,5

Breadth First Search 3



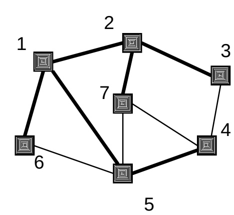
Queue = 6,5,7,3

Breadth First Search 4



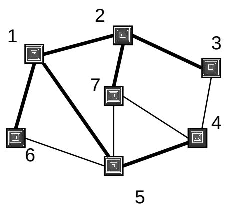
Queue = 5,7,3

Breadth First Search 5



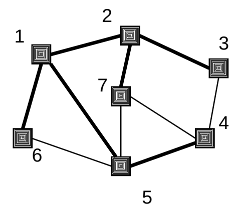
Queue = 7,3,4

Breadth First Search 6



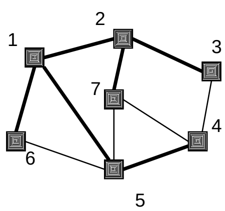
Queue = 3,4

Breadth First Search 7



Queue = 4

Breadth First Search 8



Queue =

Spanning Tree using Breadth First Search

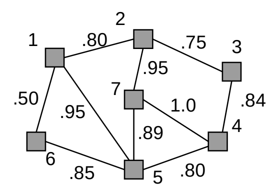
```
Initialize T to be empty;
Initialize Q to be empty;
Enqueue(1,Q) and mark 1;
while Q is not empty do
   i := Dequeue(Q);
   for each j adjacent to i do
      if j is not marked then
         add {i,j} to T;
         Enqueue(j,Q) and mark j
```


Depth First vs Breadth First

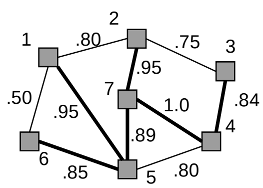
- Depth First
  - Stack or recursion
  - Many applications
- Breadth First
  - Queue (recursion no help)
  - Can be used to find shortest paths from the start vertex

Best Spanning Tree

- Each edge has the probability that it won't fail
- Find the spanning tree that is least likely to fail



Example of a Spanning Tree



Probability of success = .85 x .95 x .89 x .95 x 1.0 x .84
= .5735

Minimum Spanning Tree Problem

- Input: Undirected Graph G = (V,E) and a cost function C from E to the reals. C(e) is the cost of edge e.
- Output: A spanning tree T with minimum total cost. That is: T that minimizes

$$C(T) = \sum_{e \in T} C(e)$$

Reducing Best to Minimum

Let P(e) be the probability that an edge doesn't fail.
Define:

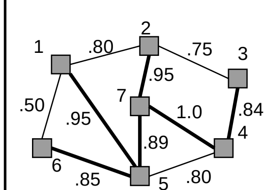
$$C(e) = -\log_{10}(P(e))$$

Minimizing $\sum_{e \in T} C(e)$

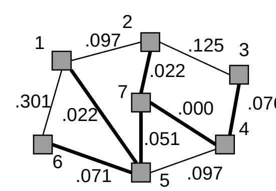
is equivalent to maximizing $\prod_{e \in T} P(e)$

because $\prod_{e \in T} P(e) = 10^{-\sum_{e \in T} C(e)}$

Example of Reduction



Best Spanning Tree Problem



Minimum Spanning Tree Problem

Minimum Spanning Tree

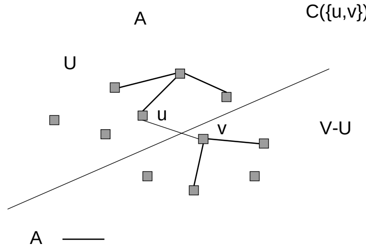
- Boruvka 1926
- Kruskal 1956
- Prim 1957 also by Jarnik 1930
- Karger, Klein, Tarjan 1995
  - Randomized linear time algorithm
  - Probably not practical, but very interesting

MST Optimality Principle

- G = (V,E) with costs C. G connected.
- Let (V,A) be a subgraph of G that is contained in a minimum spanning tree. Let U be a set such that no edge in A has one end in U and one end in V-U. Let C({u,v}) minimal and u in U and v in V-U. Let A' be A with {u,v} added. Then (V,A') is contained in a minimum spanning tree.

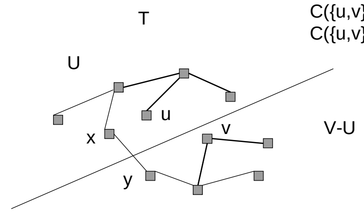
Proof of Optimality Principle 1

C({u,v}) is minimal



Proof of Optimality Principle 2

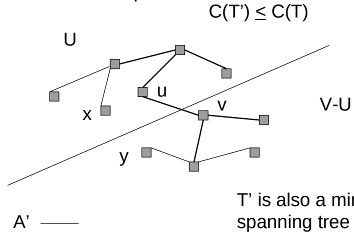
C({u,v}} is minimal
C({u,v}) ≤ C({x,y})



Proof of Optimality Principle 3

T’

C(T’) = C(T) + C({u,v}) - C({x,y})
C(T’) ≤ C(T)



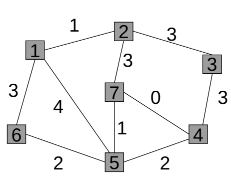
T’ is also a minimum spanning tree

Kruskal’s Greedy Algorithm

```
Sort the edges by increasing cost;
Initialize A to be empty;
For each edge e chosen in increasing order do
    if adding e does not form a cycle then
        add e to A
```

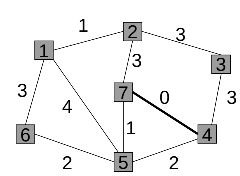
Invariant: A is always contained in some minimum spanning tree

Example of Kruskal 1



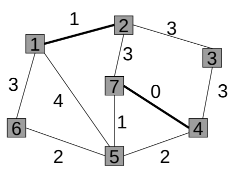
{7,4}	{2,1}	{7,5}	{5,6}	{5,4}	{1,6}	{2,7}	{2,3}	{3,4}	{1,5}
0	1	1	2	2	3	3	3	3	4

Example of Kruskal 2



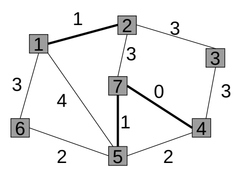
~~{7,4}~~	{2,1}	{7,5}	{5,6}	{5,4}	{1,6}	{2,7}	{2,3}	{3,4}	{1,5}
~~0~~	1	1	2	2	3	3	3	3	4

Example of Kruskal 2



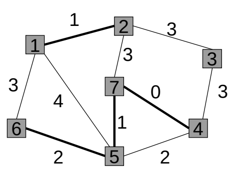
~~{7,4}~~	~~{2,1}~~	{7,5}	{5,6}	{5,4}	{1,6}	{2,7}	{2,3}	{3,4}	{1,5}
~~0~~	~~1~~	1	2	2	3	3	3	3	4

Example of Kruskal 3



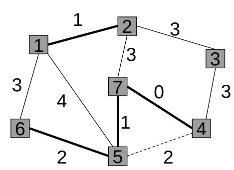
~~{7,4}~~	~~{2,1}~~	~~{7,5}~~	{5,6}	{5,4}	{1,6}	{2,7}	{2,3}	{3,4}	{1,5}
~~0~~	~~1~~	~~1~~	2	2	3	3	3	3	4

Example of Kruskal 4



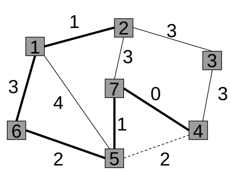
~~{7,4}~~	~~{2,1}~~	~~{7,5}~~	~~{5,6}~~	{5,4}	{1,6}	{2,7}	{2,3}	{3,4}	{1,5}
~~0~~	~~1~~	~~1~~	~~2~~	2	3	3	3	3	4

Example of Kruskal 5



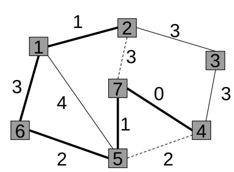
~~{7,4}~~	~~{2,1}~~	~~{7,5}~~	~~{5,6}~~	~~{5,4}~~	{1,6}	{2,7}	{2,3}	{3,4}	{1,5}
~~0~~	~~1~~	~~1~~	~~2~~	~~2~~	3	3	3	3	4

Example of Kruskal 6



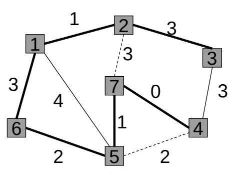
~~{7,4}~~	~~{2,1}~~	~~{7,5}~~	~~{5,6}~~	~~{5,4}~~	~~{1,6}~~	{2,7}	{2,3}	{3,4}	{1,5}
~~0~~	~~1~~	~~1~~	~~2~~	~~2~~	~~3~~	3	3	3	4

Example of Kruskal 7



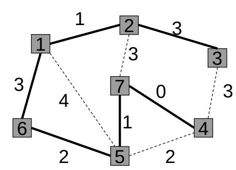
~~{7,4}~~	~~{2,1}~~	~~{7,5}~~	~~{5,6}~~	~~{5,4}~~	~~{1,6}~~	~~{2,7}~~	{2,3}	{3,4}	{1,5}
~~0~~	~~1~~	~~1~~	~~2~~	~~2~~	~~3~~	~~3~~	3	3	4

Example of Kruskal 7



{7,4} {2,1} {7,5} {5,6} {5,4} {1,6} {2,7} {2,3} {3,4} {1,5}
0 1 1 2 2 3 3 3 3 4

Example of Kruskal 8,9



{7,4} {2,1} {7,5} {5,6} {5,4} {1,6} {2,7} {2,3} {3,4} {1,5}
0 1 1 2 2 3 3 3 3 4

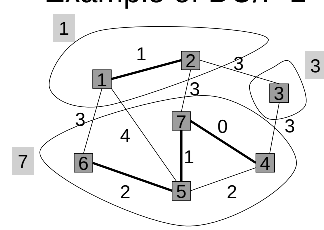
Data Structures for Kruskal

- Sorted edge list

  {7,4} {2,1} {7,5} {5,6} {5,4} {1,6} {2,7} {2,3} {3,4} {1,5}
  0 1 1 2 2 3 3 3 3 4

- Disjoint Union / Find
  - Union(a,b) - union the disjoint sets named by a and b
  - Find(a) returns the name of the set containing a

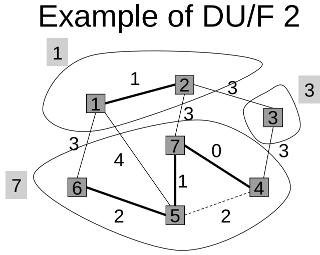
Example of DU/F 1



Find(5) = 7
Find(4) = 7

{7,4} {2,1} {7,5} {5,6} {5,4} {1,6} {2,7} {2,3} {3,4} {1,5}
0 1 1 2 2 3 3 3 3 4

Example of DU/F 2

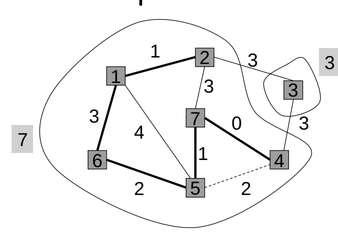


Find(1) = 1
Find(6) = 7

{7,4} {2,1} {7,5} {5,6} {5,4} {1,6} {2,7} {2,3} {3,4} {1,5}
0 1 1 2 2 3 3 3 3 4

Example of DU/F 3

Union(1,7)

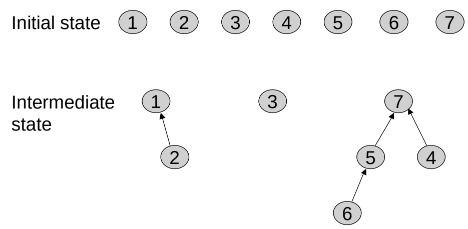


{7,4} {2,1} {7,5} {5,6} {5,4} {1,6} {2,7} {2,3} {3,4} {1,5}
0 1 1 2 2 3 3 3 3 4

Kruskal's Algorithm with DU / F

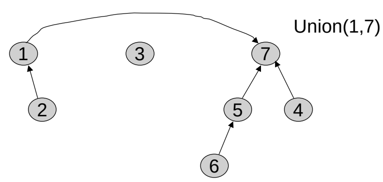
```
Sort the edges by increasing cost;
Initialize A to be empty;
for each edge {i,j} chosen in increasing order do
   u := Find(i);
   v := Find(j);
   if not(u = v) then
      add {i,j} to A;
      Union(u,v);
```


Up Tree for DU/F



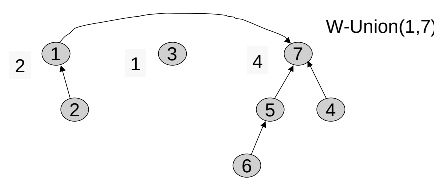
DU/F Operation

- Find(i) - follow pointer to root and return the root.
- Union(i,j) - assuming i and j roots, point i to j.



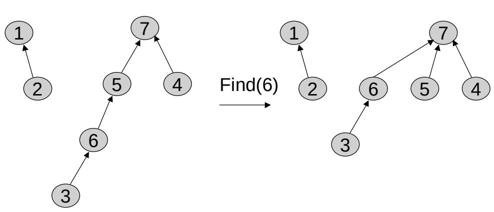
Weighted Union

- Weighted Union
  – Always point the smaller tree to the root of the larger tree

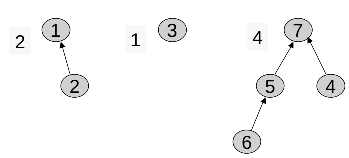


Path Compression

- On a Find operation point all the nodes on the search path directly to the root.



Elegant Array Implementation



	1	2	3	4	5	6	7
up	0	1	0	7	7	5	0
weight	2		1				4

Up Tree Pseudo-Code

```
PC-Find(i : index)
   r := i;
   while not(up[r] = 0) do
      r := up[r]
   k := up[i];
   while not(k = r) do
      up[i] := r;
      i := k;
      k := up[k]
   return(r)
end{Find}
```

```
W-Union(i,j : index)
   // i and j are roots
   wi := weight[i];
   wj := weight[j];
   if wi < wj then
      up[i] := j;
      weight[i] := wi + wj;
   else
       up[j] :=i;
       weight[j] := wi +wj;
end{W-Union}
```


Disjoint Union / Find Notes

- Weighted union and path compression analyzed by Tarjan in 1975
  - Worst case time complexity for a W-Union is O(1) and for a PC-Find is O(log n).
  - Time complexity for m operations on n elements is O(m α(m,n)) where α is a very slow growing function $\alpha(m,n) \leq 4$ for all practical m and n. α is called inverse Ackermann's function. Essentially constant time per operation!