CSE 589
Applied Algorithms
Spring 1999

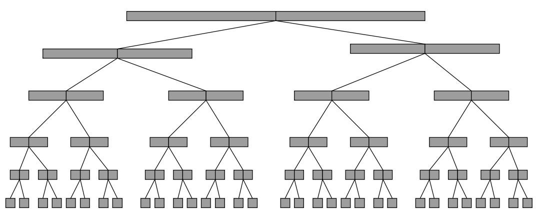
Cache Performance
Mergesort
Heapsort

Recursive Mergesort

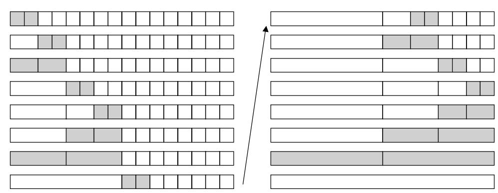
```
A[1..n] is to be sorted;
B[1..n] is an auxiliary array;

Mergesort(i,j) {sorts the subarray A[i..j] }
   if i < j then
      k := (i+j)/2;
      Mergesort(i,k);
      Mergesort(k+1,j);
      Merge A[i..k] with A[k+1..j] into B[i..j];
      Copy B[i..j] into A[i..j];
```


Mergesort Call Tree



Merging Pattern of Recursive Mergesort

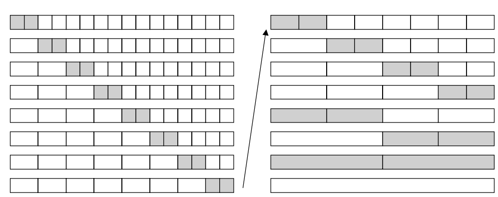


1/2 cache size

Notes on Recursive Mergesort

- Oblivious recursion.
  – The subarrays that are merged do not depend on the particular keys, just on the number of keys.
- Lots of copying from the auxiliary array to the source arrays.
- Recursion is elegant, but is it really needed?
- Sorting very small arrays should be done in-place.

Reorder the Merging Steps



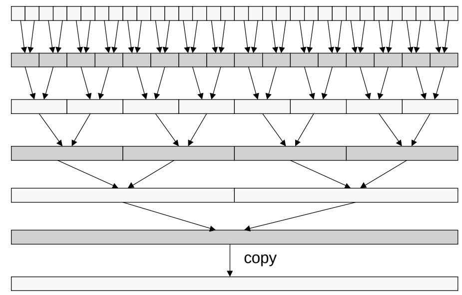
Interative Mergesort

- Sort small groups in-place.
- Alternate the roles of A and B as the source of the merging passes.
- Copy B to A if needed at the end.

```
in-place sort groups of 4;
merge sorted groups of 4 in A into sorted groups of 8 in B;
merge sorted groups of 8 in B into sorted groups of 16 in A;
merge sorted groups of 16 in A into sorted groups of 32 in B;
.

in the end if the sorted array is B then copy it to A;
```

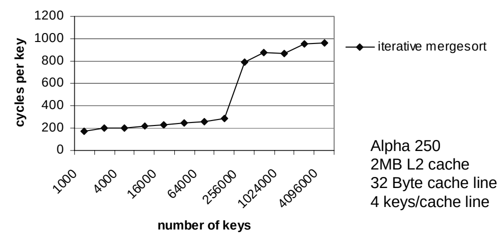

Interative Mergesort Access Pattern



Analysis of Access Pattern

- one pass to sort into groups of 4.
  - Pass touches n key locations.
- $\log_2(n/4)$ merge passes.
  - Each pass touches 2n key locations, n in the source array and n in the destination array.
- One copy pass if $\log_2(n/4)$ is odd.
  - Pass touches 2n key locations.

Performance of Iterative Mergesort

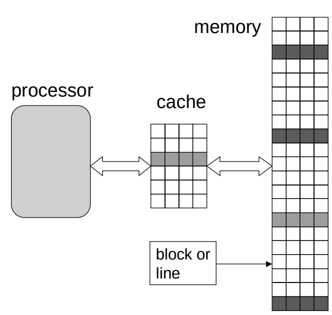


Alpha 250
2MB L2 cache
32 Byte cache line
4 keys/cache line

Cache Performance Matters

- Processor speeds increasing faster than memory speeds.
- Cache miss penalties can be 100 cycles and are growing.
- Algorithm design can be used to reduce cache misses and improve overall performance.

Cache Model



Direct mapped cache

Cache line

Cache hit

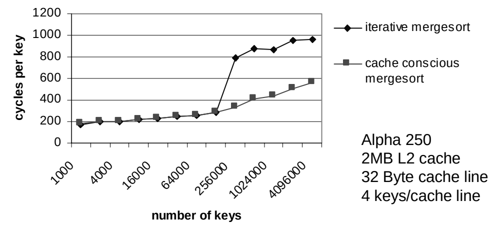
Cache miss

Cache parameters
- Cache capacity
- Cache line size
- Set associativity

Cache Miss Terminolgy

- Types of misses
  - Compulsory miss: first time a memory block is read.
  - Capacity miss: accessed data does not fit in cache.
  - Conflict miss: several active memory blocks map to the same place in the cache.
- Locality reduces cache misses
  - temporal locality: a location that was recently accessed is accessed again.
  - Spatial locality: data on the same block are accessed together.

Cache Conscious Mergesort Execution Performance

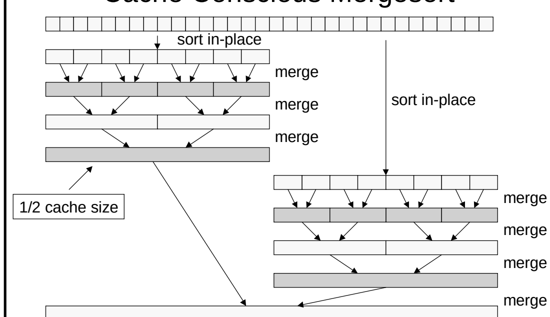


Alpha 250
2MB L2 cache
32 Byte cache line
4 keys/cache line

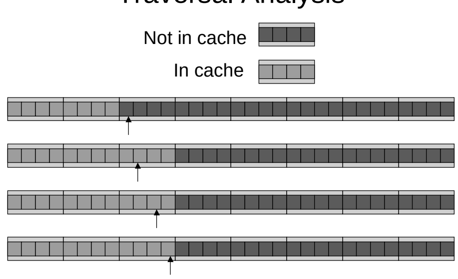
Cache Conscious Mergesort

- Partition problem into “tiles” that fit in the cache.
- Mergesort the tiles.
- Merge the tiles.
- Avoid copying by sorting in-place into groups of 2 or 4 depending on whether $\log_2(n/4)$ is odd or even.

Cache Conscious Mergesort

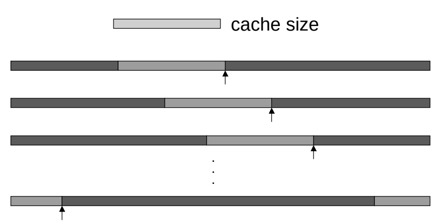


Traversal Analysis



1/B misses per access where B is number of access per line

Traversal Longer than Cache



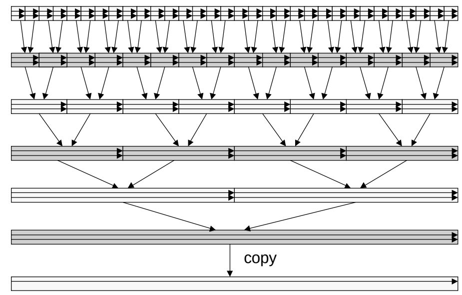
Analysis of Cache Misses

- Parameters
  - B keys per cache line
  - C cache lines in the cache
  - n keys with n >> BC
- Iterative Mergesort

$$\frac{1}{B}+\frac{2}{B}\left\lceil \log_2 \frac{n}{4}\right\rceil+\frac{2}{B}\left(\left\lceil \log_2 \frac{n}{4}\right\rceil \bmod 2\right)$$ cache misses per key

in-place sort　merge passes　copy

Iterative Mergesort Cache Misses



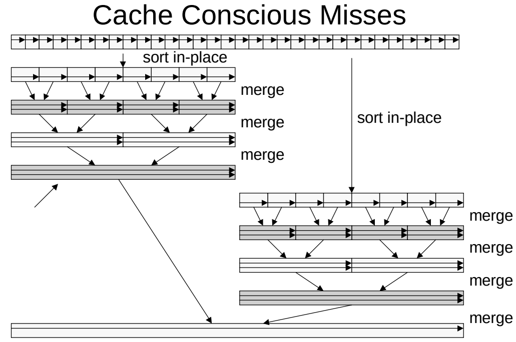
Cache Conscious Merge Sort Analysis

$$\frac{2}{B}+\frac{2}{B}\left\lceil \log_2 \frac{2n}{BC}\right\rceil$$ cache misses per key

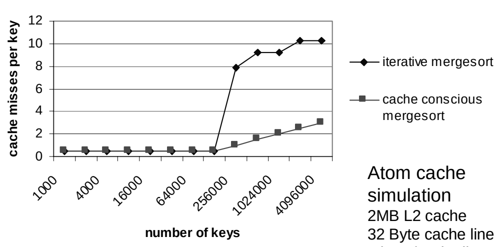
sort each tile　final merge passes

Tile size is BC/2.
n/(BC/2) tiles to be merged in the end.
This take $\log_2(n/(BC/2))$ passes.

Cache Conscious Misses

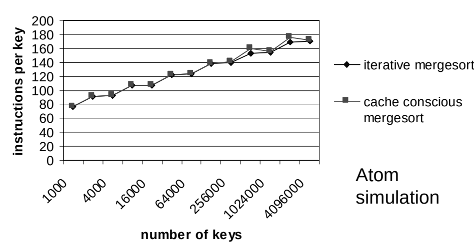


Simulated Cache Performance



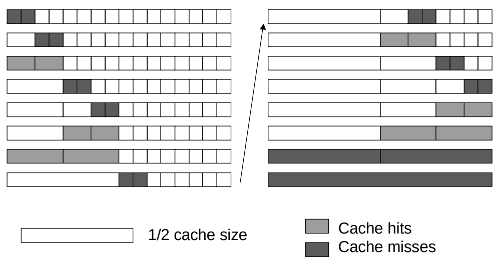
Atom cache simulation
2MB L2 cache
32 Byte cache line
4 keys/cache line

Instruction Counts



Atom simulation

What About Recursive Mergesort?



Notes on Cache Performance

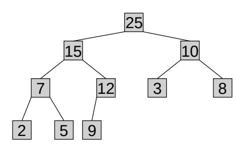
- Before trying cache conscious algorithm design you should ask if performance is really a problem.
  - if not, then don't tinker
  - if so, then check out the algorithm and data structures first. Going from an n^2 algorithm to a n log n algorithm can make a world of difference.
  - if the algorithm and data structures are basically good then consider a cache conscious design.

Some Guiding Principles

- Sacrifice instructions for better cache performance.
- Knowing architectural constants can lead to better algorithms. Cache capacity, line size.
- Small memory footprints are good.
  - Reduces capacity misses
- Block data into cache size pieces.
  - Reduces capacity misses
- Fully utilize cache lines.
  - Improves spatial locality

Heapsort

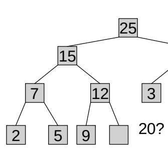
- Classic in-place, O(n log n) sorting algorithm.
- Uses the binary heap, an elegant priority queue data structure (insert and delete-max)



Perfectly balanced tree with the heap property. Each node is larger than its children.

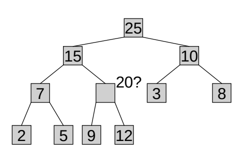
Insert

- Add a new leaf and percolate up.



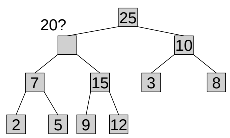
Insert (2)

- Add a new leaf and percolate up.



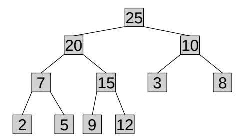
Insert (3)

- Add a new leaf and percolate up.



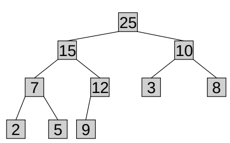
Insert (3)

- Add a new leaf and percolate up.



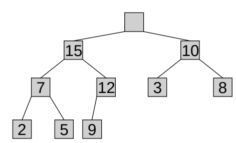
Delete-Max

- Remove the root then percolate last leaf down.



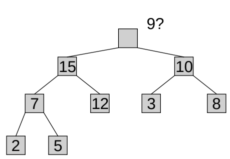
Delete-Max (2)

- Remove the root then percolate last leaf down.



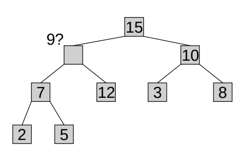
Delete-Max (3)

- Remove the root then percolate last leaf down.



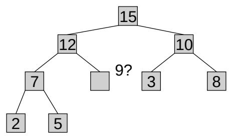
Delete-Max (4)

- Remove the root then percolate last leaf down.



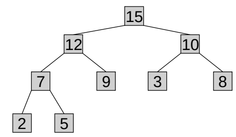
Delete-Max (5)

- Remove the root then percolate last leaf down.



Delete-Max (5)

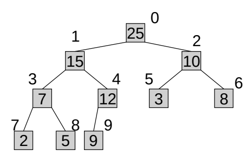
- Remove the root then percolate last leaf down.



Analysis of the Heap Operation

- Insert - O(log n) worst case.
  - Each percolate up goes up at most log n levels.
  - Often O(1) in practice because keys do not percolate far.
- Delete-Max - O(log n) worst case.
  - Percolates down tend to go close to the leaves of the heap.

Implicit Pointers



parent of i is (i-1)/2

children of i are 2i+1, 2i+2

0	1	2	3	4	5	6	7	8	9	10	11							
25	15	10	7	12	3	8	2	5	9									

Heapsort

Williams 1964

```
We will sort the array A[0..n-1] in-place
Build a heap in-place
For i = n-1 to 1
   A[i] := delete-max;
```

0	1	2	3	4	5	6	7	8	9
25	15	10	7	12	3	8	2	5	9
15	12	10	7	9	3	8	2	5	25
12	9	10	7	5	3	8	2	15	25
10	9	8	7	5	3	2	12	15	25
9	7	8	2	5	3	10	12	15	25

Invariants
Heap
Sorted

≤