CSEP531 Homework 3 Solution

Disclaimer: *The solution is incomplete, in the sense that the numerical answers to Question 1 are not presented. I post this solution hoping that the answers to Questions 2 and 3 will be of some help dealing with homework 4. I will add the numerical answers to question 1 later.*

1. (a) $f(n) = \Theta(g(n))$.
   (b) $f(n) = o(g(n))$.
   (c) $f(n) = o(g(n))$.
   (d) $g(n) = o(f(n))$.
   (e) $f(n) = o(g(n))$.
   (f) $f(n) = o(g(n))$.

2. - First, we show that Efficient Recruiting (ER) is in NP. A list of t ($t \leq k$) counselors who are qualified for all the sports is a polynomial size certificate for a "yes" answer. We can check the such a certificate in polynomial time as follows. First, check if $t \leq k$. Then, for each sport, search through all the t counselors to see if any of them is qualified for it. If we find a counselor for all the jobs, accept the certificate; otherwise, reject it. This algorithm runs in $O(tn) = O(kn)$ time. Thus, ER is in NP.
   - Next, we show that ER is NP-hard by a reduction from Set Cover (SC). Given an instance $\mathcal{A}$ of SC consisting of a collection of m subsets $S_1, S_2, \ldots S_m$ of a set $S = \{e_1, e_2, \ldots e_n\}$ and a number k, we construct an instance $\mathcal{B}$ of ER as follows. For each elements $e_i \in S$, we create a sport b_i; and for each subset S_j, we create a counselor c_j. The counselor c_j is qualified for the sport b_i iff $e_i \in S_j$. The number k is copied from $\mathcal{A}$ to $\mathcal{B}$. It is clear that this reduction can be performed in polynomial time.

     Now, we show that there are k subsets in $\mathcal{A}$ that covers S iff there are k counselors in $\mathcal{B}$ that covers all the sports.

     On one hand, assume that the k sets $S_{j_1}, S_{j_2}, \ldots S_{j_k}$ cover S; we claim that the corresponding k counselors $c_{j_1}, c_{j_2}, \ldots c_{j_k}$ cover all the sports. Consider any sport b_i, the corresponding element e_i must belong to some subset S_{j_t} that contains e_i. Then, the counselor c_{j_t} is qualified for b_i.

     On the other hand, assume that the k counselors $c_{j_1}, c_{j_2}, \ldots c_{j_2}$ that cover all the sports; we claim that the corresponding k subsets $S_{j_1}, S_{j_2}, \ldots S_{j_k}$ cover S. Consider any element $e_i \in S$, the corresponding sport b_i must be covered by some counselor c_{j_t}. Then, the subset S_{j_t} contains e_i.

     Thus, ER is NP-complete.

3. - First, we show that Multiple Interval Scheduling (MIS) is in NP. A non-conflicting schedule of at least k jobs is a polynomial size certificate. To verify such certificate, we first check the number of job in the schedule. Then, we check if at most one job running in each interval. Finally, we check if all the jobs in the schedule get all the intervals they require. This verfication can be implemented in $O(nm)$ time, where n is the number of jobs and m is the number of intervals, in the obvious way. Thus, MIS is in NP.
   - Next, we prove that MIS is NP-hard by a reduction from Independent Set (IS). Given an instance $\mathcal{C}$ of IS consisting of a graph $G(V, E)$ and a number k, we construct an instance $\mathcal{D}$ of MIS as

follows. For each vertex $v_i \in V$, we construct a job b_i. For each edge $v_i v_j \in E$, we construct an interval c_{ij} and let b_i and b_j require c_{ij}. Finally, the number k is copied from $\mathcal{A}$ to $\mathcal{B}$. It is clear that this reduction can be performed in polynomial time.

Now, we prove that G has an independent set of size k iff there is a non conflicting schedule that completes k jobs.

On one hand, assume that G contains an independent set $\{v_{i_1}, v_{i_2}, \ldots v_{i_k}\}$. Then, we can schedule the corresponding k jobs $b_{i_1}, b_{i_2}, \ldots b_{i_k}$, since no two of them require a common interval.

On the other hand, assume that there is a non conflicting schedule of k jobs $b_{i_1}, b_{i_2}, \ldots b_{i_k}$. Then the set $\{v_{i_1}, v_{i_2}, \ldots v_{i_k}\}$ is an independent set of G; for otherwise, there must be some h and l such that $v_{i_h} v_{i_l} \in E$, which means both b_{i_h} and b_{i_l} require the interval $c_{i_h i_l}$, contradicting the fact that b_{i_h} and b_{i_l} are not conflicting.

Thus, MIS is NP-complete.