Subdivision curves

Reading

Stollnitz, DeRose, and Salesin. *Wavelets for Computer Graphics: Theory and Applications,* 1996, section 6.1-6.3, A. 5.

Subdivision curves

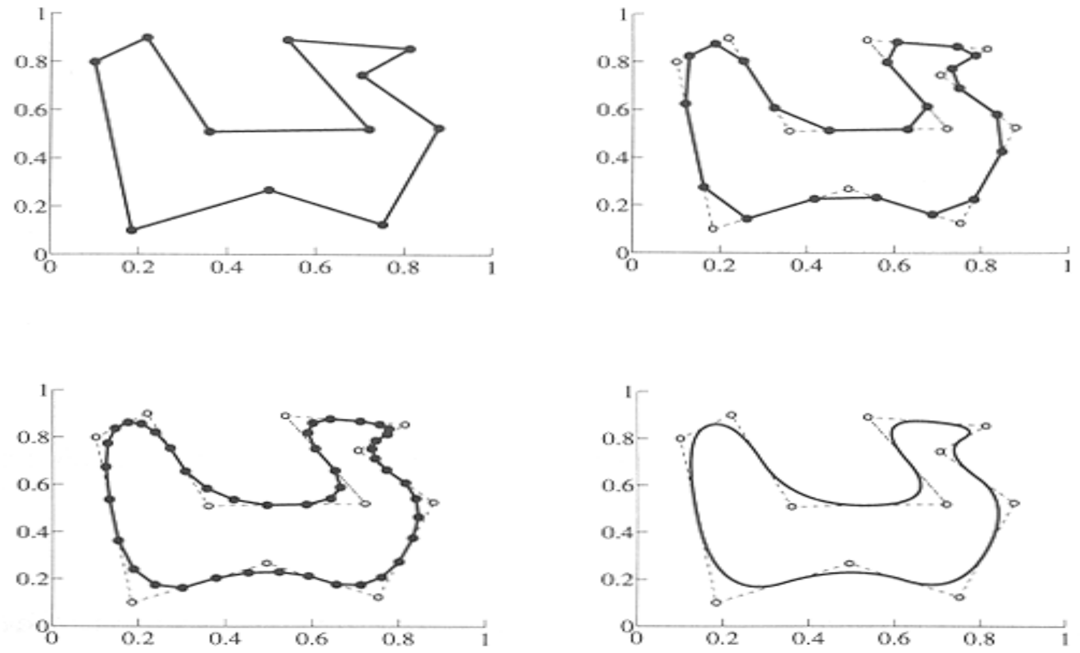
Idea:

- repeatedly refine the control polygon

$$P_0 \longrightarrow P_1 \longrightarrow P_2 \longrightarrow \cdots$$

$$C = \lim_{i\to\infty} P_i$$

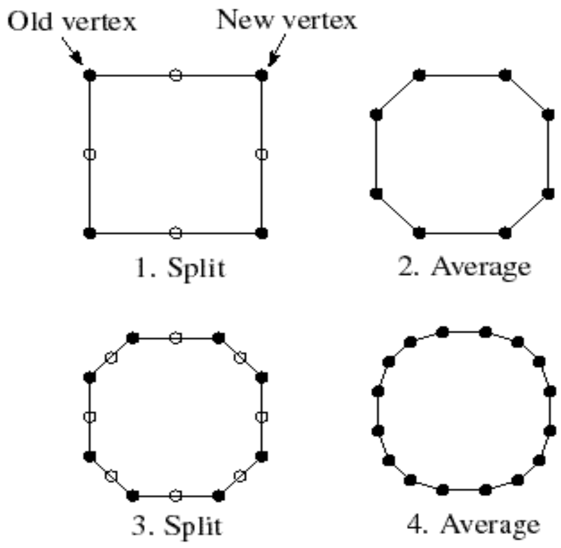
- curve is the limit of an infinite process



Chaikin's algorithm

Chakin introduced the following "corner-cutting" scheme in 1974:

- Start with a piecewise linear curve
- Insert new vertices at the midpoints (the **splitting step**)
- Average each vertex with the "next" neighbor (the **averaging step**)
- Go to the splitting step



Averaging masks

The limit curve is a quadratic B-spline!

Instead of averaging with the nearest neighbor, we can generalize by applying an **averaging mask** during the averaging step:

$$r = (K\ , r_{-1}, r_0, r_1, K\ )$$

In the case of Chaikin's algorithm:

$$r =$$

Lane-Riesenfeld algorithm (1980)

Use averaging masks from Pascal's triangle:

$$r = \frac{1}{2^n}\left(\binom{n}{0},\binom{n}{1},\cdots,\binom{n}{n}\right)$$

Gives B-splines of degree *n*+1.

n=0:

n=1:

n=2:

Subdivide ad nauseum?

After each split-average step, we are closer to the limit surface.

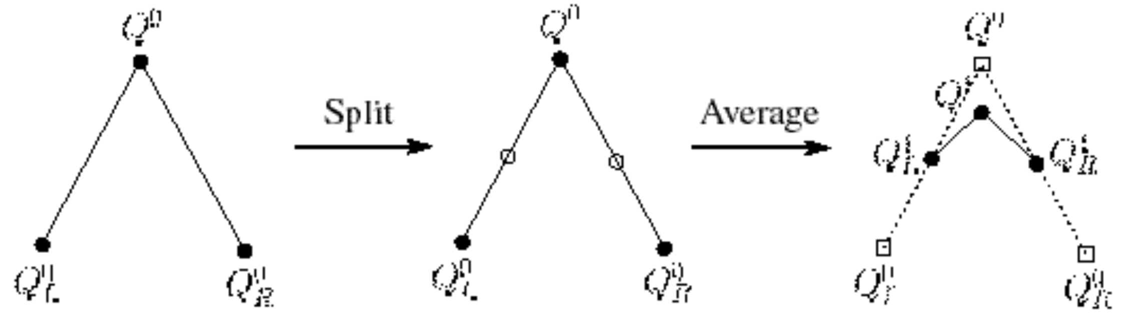
How many steps until we reach the final (limit) position?

Can we push a vertex to its limit position without infinite subdivision? Yes!

Local subdivision matrix

Consider the cubic B-spline subdivision mask: $\frac{1}{4}\begin{pmatrix}1 & 2 & 1\end{pmatrix}$

Now consider what happens during splitting and averaging:



Relating points at one subdivision level to points at the previous:

$$Q_L^1 = \frac{1}{2}\left(Q_L^0 + Q^0\right) = \frac{1}{8}\left(4Q_L^0 + 4Q^0\right)$$

$$Q^1 = \frac{1}{8}\left(Q_L^0 + 6Q^0 + Q_R^0\right)$$

$$Q_R^1 = \frac{1}{2}\left(Q^0 + Q_R^0\right) = \frac{1}{8}\left(4Q^0 + 4Q_R^0\right)$$

Local subdivision matrix

We can write this as a recurrence relation in matrix form:

$$\begin{pmatrix} Q_L^j \\ Q^j \\ Q_R^j \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 4 & 4 & 0 \\ 1 & 6 & 1 \\ 0 & 4 & 4 \end{pmatrix}\begin{pmatrix} Q_L^{j-1} \\ Q^{j-1} \\ Q_R^{j-1} \end{pmatrix}$$

$$\mathbf{Q}^j = \mathbf{S}\mathbf{Q}^{j-1}$$

Q's are row vectors and **S** is the **local subdivision matrix**.

Looking at the x-coordinate independently:

$$\begin{pmatrix} x_L^j \\ x^j \\ x_R^j \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 4 & 4 & 0 \\ 1 & 6 & 1 \\ 0 & 4 & 4 \end{pmatrix}\begin{pmatrix} x_L^{j-1} \\ x^{j-1} \\ x_R^{j-1} \end{pmatrix}$$

$$\mathbf{X}^j = \mathbf{S}\mathbf{X}^{j-1}$$

Local subdivision matrix, cont’d

Tracking just the *x* components through subdivision:

$$\begin{aligned}\mathbf{X}^{j} &= \mathbf{S}\mathbf{X}^{j-1} = \mathbf{S}\cdot\mathbf{S}\mathbf{X}^{j-2} = \mathbf{S}\cdot\mathbf{S}\cdot\mathbf{S}\mathbf{X}^{j-3} = \mathrm{L} \\ &= \mathbf{S}^{j}\mathbf{X}^{0}\end{aligned}$$

The limit position of the x’s is then:

$$X^{\infty} = \lim_{j\to\infty} S^{j} X^{0}$$

OK, so how do we apply a matrix an infinite number of times??

Eigenvectors and eigenvalues

To solve this problem, we need to look at the eigenvectors and eigenvalues of *S*. First, a review…

Let *v* be a vector such that:

$$Sv = \lambda v$$

We say that *v* is an eigenvector with eigenvalue λ.

An *n*x*n* matrix can have *n* eigenvalues and eigenvectors:

$$\begin{gathered} Sv_1 = \lambda_1 v_1 \\ \mathrm{M} \\ Sv_n = \lambda_n v_n \end{gathered} \qquad X = \sum^{n} a_i v_i$$

For *non-defective* matrices, the eigenvectors form a basis, which means we can re-write *X* in terms of the eigenvectors:

To infinity, but not beyond…

Now let’s apply the matrix to the vector X:

$$SX = S\sum^{n} a_i v_i = \sum^{n} a_i S v_i = \sum^{n} a_i \lambda_i v_i$$

Applying it *j* times:

$$S^j X = S^j \sum^{n} a_i v_i = \sum^{n} a_i S^j v_i = \sum^{n} a_i \lambda_i^j v_i$$

Let’s assume the eigenvalues are sorted so that:

$$\lambda_1 > \lambda_2 > \lambda_3 \geq \mathrm{L} \geq \lambda_n$$

Now let *j* go to infinity.

If $\lambda_1 > 1$, then…

$$S^{\infty} X = \sum^{n} a_i \lambda_i^{\infty} v_i = a_1 v_1$$

If $\lambda_1 < 1$, then…

If $\lambda_1 = 1$, then:

Evaluation masks

What are the eigenvalues and eigenvectors of our cubic B-spline subdivision matrix?

$$\lambda_1 = 1 \qquad \lambda_2 = \frac{1}{2} \qquad \lambda_3 = \frac{1}{4}$$

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$

We're OK!

But where did the x-coordinates end up?

Evaluation masks, cont'd

To finish up, we need to compute a_1.

It turns out that, if we call v_i the "right eigenvectors" then there are a corresponding set of "left eigenvectors" with the same eigenvalues such that:

$$u_1^T S = \lambda_1 u_1^T$$

M

$$u_n^T S = \lambda_n u_n^T$$

Using the first left eigenvector, we can compute: $x^\infty = a_1 = u_1^T X^0$

In fact, this works at any subdivision level: $x^\infty = S^\infty X^j = u_1^T X^j$

The same result obtains for the y-coordinate: $y^\infty = S^\infty Y^j = u_1^T Y^j$

We call u_i an **evaluation mask**.

Recipe for subdivision curves

The evaluation mask for the cubic B-spline is:

$$\frac{1}{6}\begin{pmatrix} 1 & 4 & 1 \end{pmatrix}$$

Now we can cook up a simple procedure for creating subdivision curves:

- Subdivide (split+average) the control polygon a few times. Use the averaging mask.
- Push the resulting points to the limit positions. Use the evaluation mask.

Question: what is the tangent to the curve?

Answer: apply the second left eigenvector, u_2, as a tangent mask.

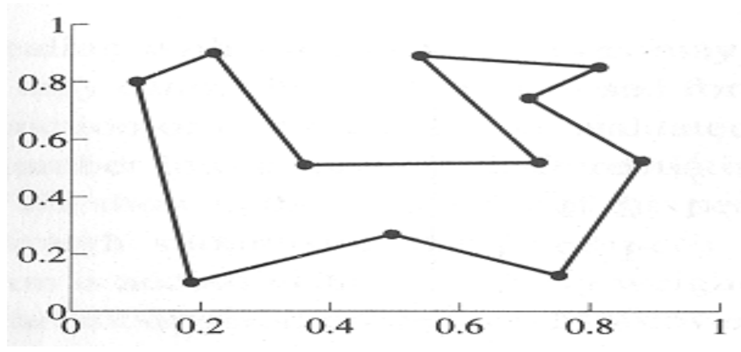
DLG interpolating scheme (1987)

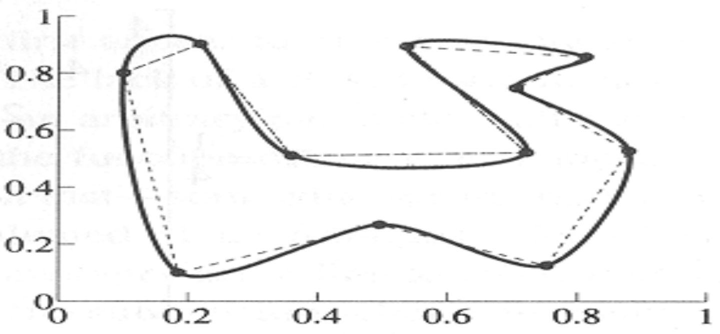
Slight modification to algorithm:

- splitting step introduces midpoints
- averaging step *only changes midpoints*

For DLG (Dyn-Levin-Gregory), use:

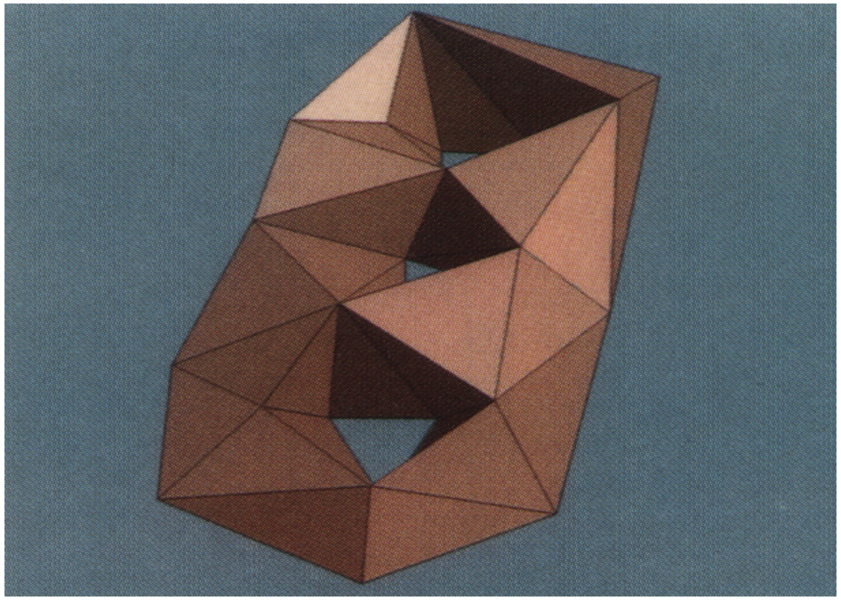
$$r = \frac{1}{16}(-2,6,10,6,-2)$$

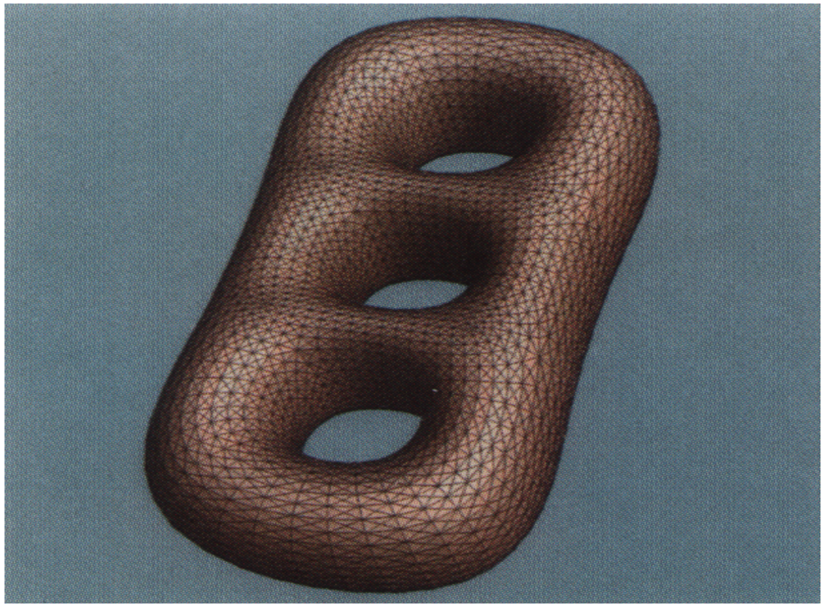




Since we are only changing the midpoints, the points after the averaging step do not move.

Building complex models





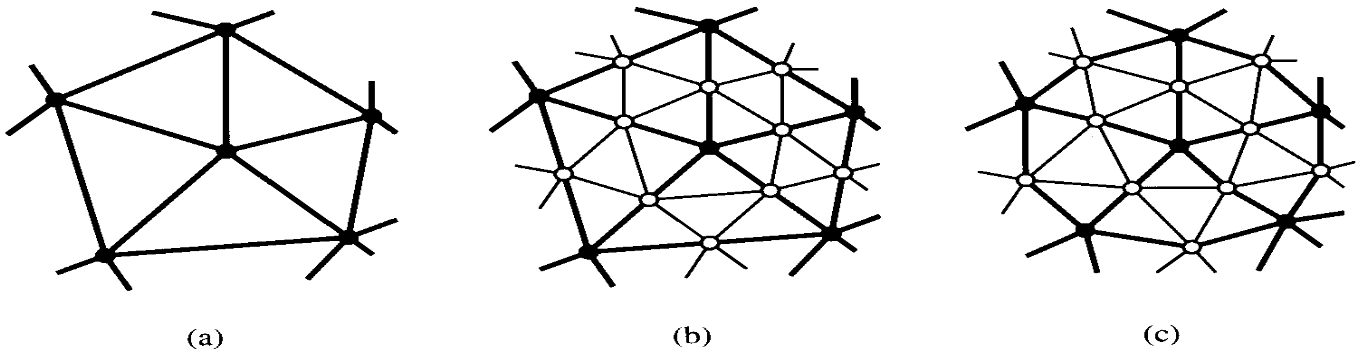
Subdivision surfaces

Chaikin's use of subdivision for curves inspired similar techniques for subdivision.

Iteratively refine a **control polyhedron** (or **control mesh**) to produce the limit surface

$$\sigma = \lim_{j \to \infty} M^j$$

using splitting and averaging steps.

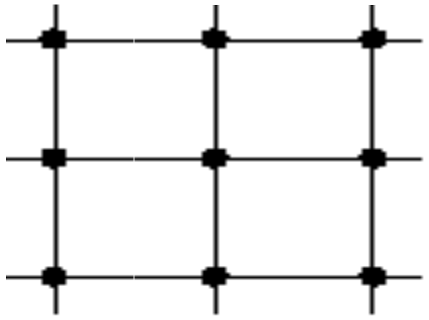


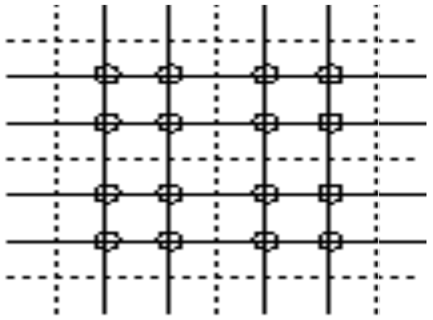
There are two types of splitting steps:

- **vertex schemes**
- **face schemes**

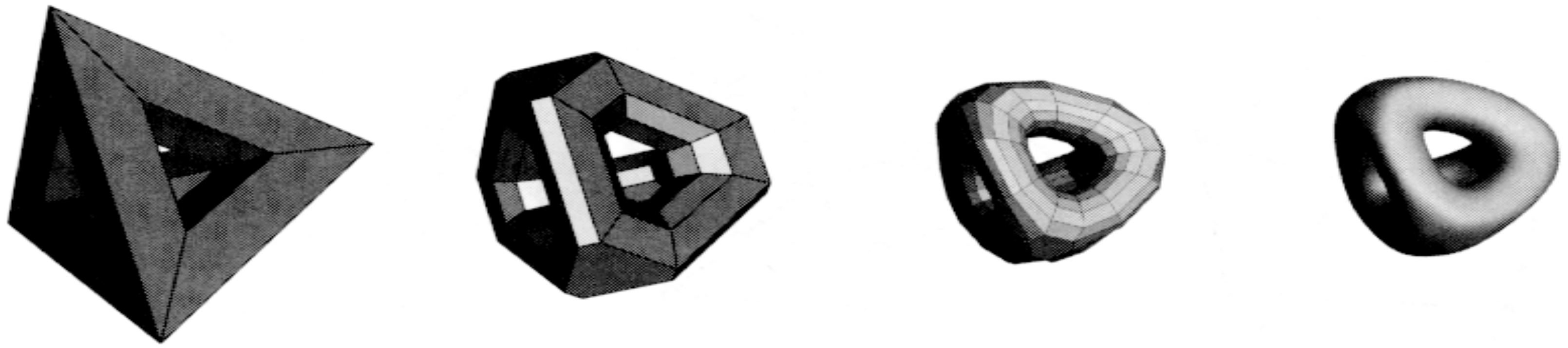
Vertex schemes

A vertex surrounded by *n* faces is split into *n* subvertices, one for each face:


Original

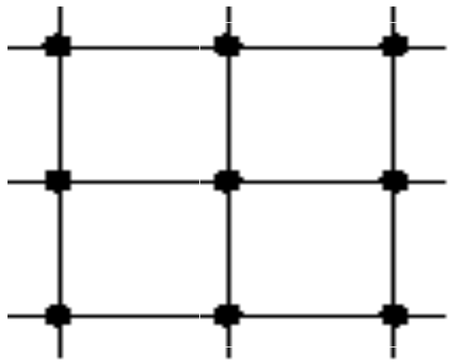

After splitting

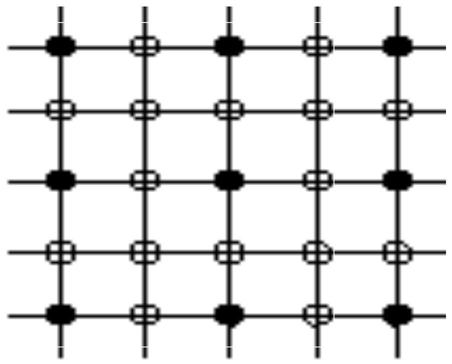
Doo-Sabin subdivision:



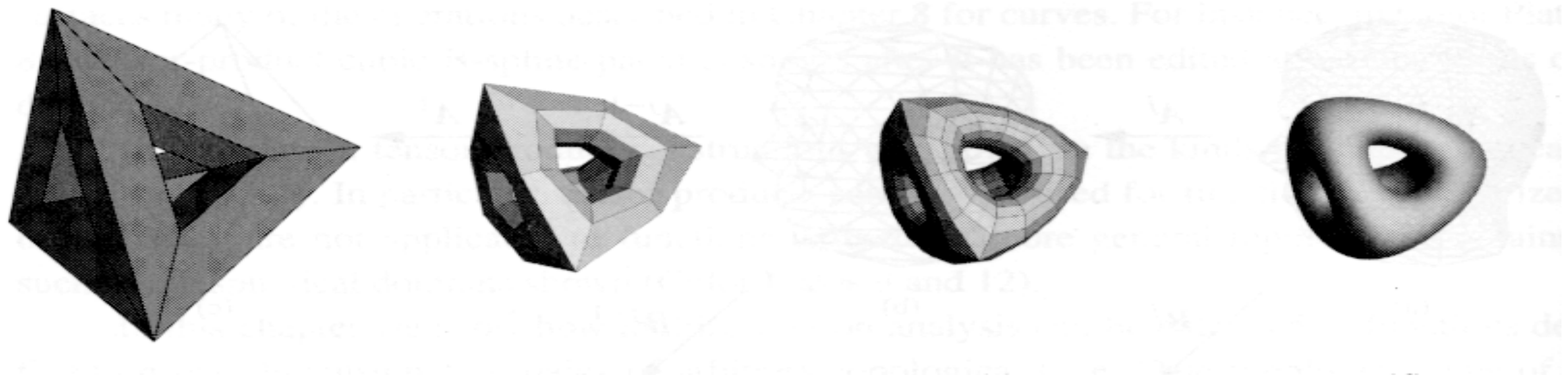
Face schemes

Each quadrilateral face is split into four subfaces:


Original

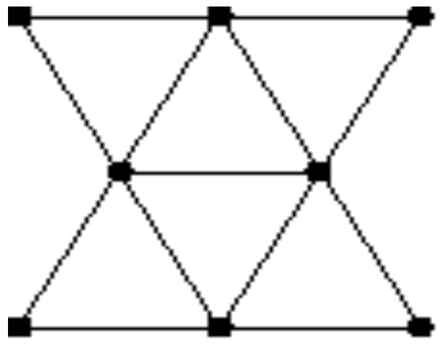

After splitting

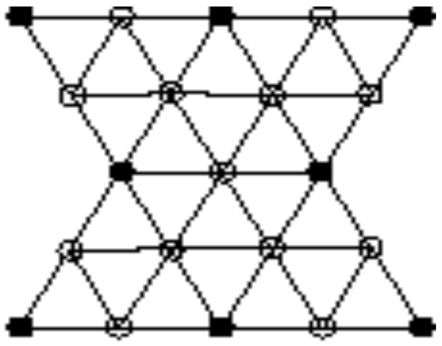
Catmull-Clark subdivision:



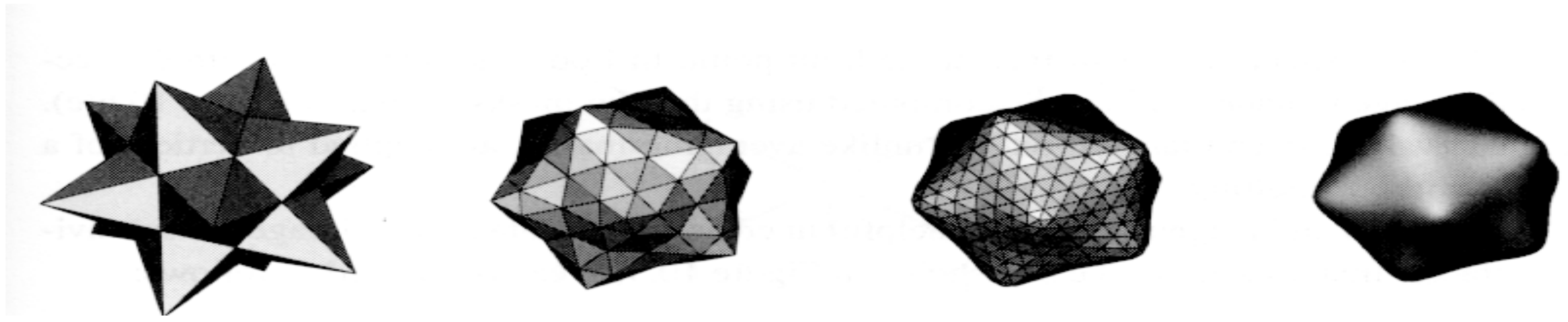
Face schemes, cont.

Each triangular face is split into four subfaces:


Original

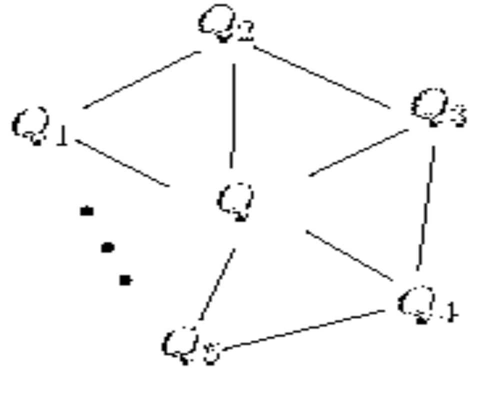

After splitting

Loop subdivision:

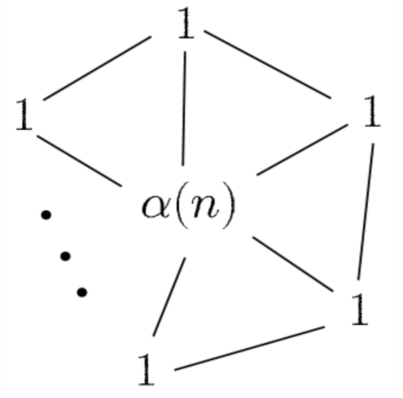


Averaging step

Once again we can use **masks** for the averaging step:



Vertex labeling



Averaging mask

$$Q \leftarrow \frac{\alpha(n) + Q_1 + \mathrm{L} \; + Q_n}{\alpha(n) + n}$$

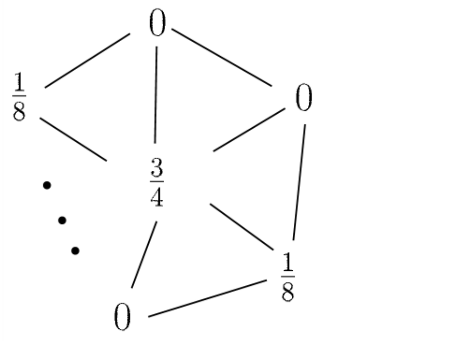
where

$$\alpha(n) = \frac{n(1 - \beta(n))}{\beta(n)} \qquad \beta(n) = \frac{5}{4} - \frac{(3 + 2\cos(2\pi/n))^2}{32}$$

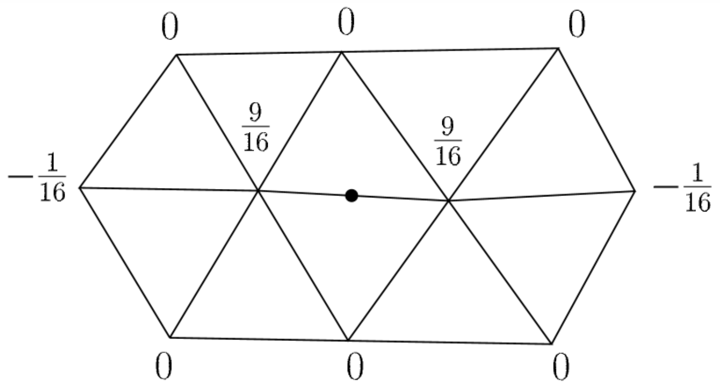
(carefully chosen to ensure smoothness.)

Adding creases without trim curves

Sometimes, particular feature such as a crease should be preserved. With NURBS surfaces, this required the use of trim curves.
For subdivision surfaces, we just modify the subdivision mask:

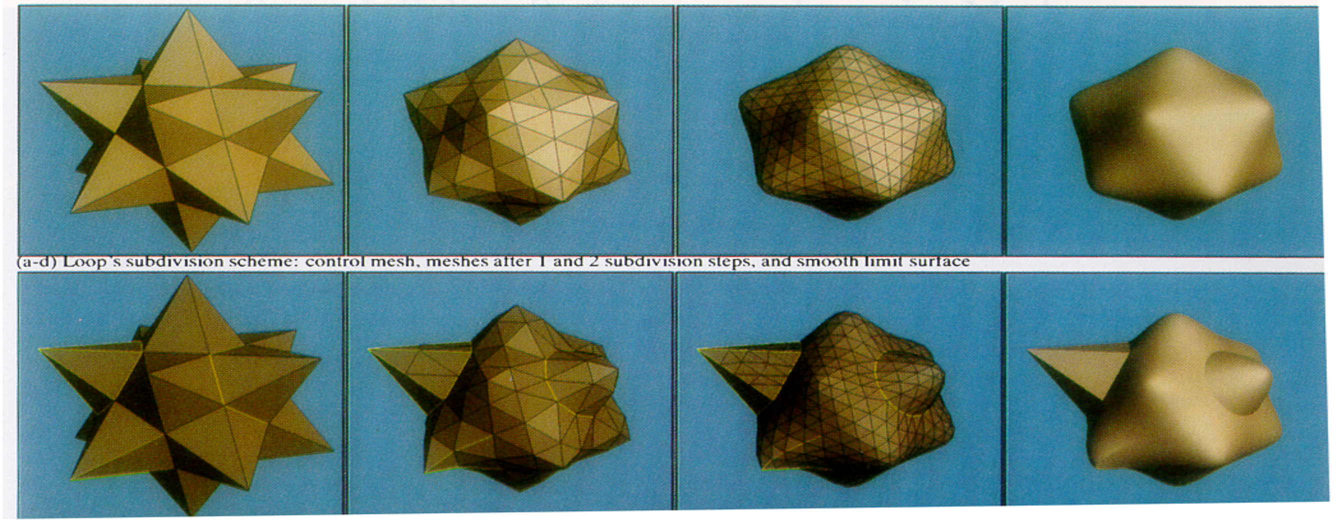


Loop crease/boundary edge



Buttery crease/boundary edge

This gives rise to G^0 continuous surfaces.



(a-d) Loop's subdivision scheme: control mesh, meshes after 1 and 2 subdivision steps, and smooth limit surface

Creases without trim curves, cont.

Here's an example using Catmull-Clark surfaces of the kind found in Geri's Game: