6. Affine transformations

Reading

Required:

- Watt, Chapter 1.

Supplemental:

- Foley *et al.*, Chapter 5.1–5.5
- David F. Rogers and J. Alan Adams, *Mathematical Elements for Computer Graphics*, Second edition, McGraw-Hill, New York, 1990, Chapter 2.

Geometric transformations

Geometric transformations will map points in one space to points in another: $(x', y', z') = \mathbf{f}(x, y, z)$.

These tranformations can be very simple, such as scaling each coordinate, or complex, such as non-linear twists and bends.

We'll focus on transformations that can be represented easily with matrix operations.

We'll start in 2D...

Representation

We can represent a **point** $p = (x, y)$ in the plane

- as a column vector $\begin{bmatrix} x \\ y \end{bmatrix}$
- as a row vector $\begin{bmatrix} x & y \end{bmatrix}$

Representation, cont.

We can represent a **2-D transformation** M by a matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

If p is a column vector, M goes on the left:

$$p' = M\,p$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

If p is a row vector, M^{T} goes on the right:

$$p' = p\,T$$

$$\begin{bmatrix} x' & y' \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

We will use **column vectors**.

Two-dimensional transformations

Here's all you get with a 2×2 transformation matrix M:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

So

$$x' = ax + by$$

$$y' = cx + dy$$

We will develop some intimacy with the elements a, b, c, d. . . .

Identity

Suppose we choose $a = d = 1$, $b = c = 0$:

- Gives the **identity matrix**

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- **Doesn't move the points at all**

Scaling

Suppose we set $b = c = 0$, but let a and d take on any *positive* value:

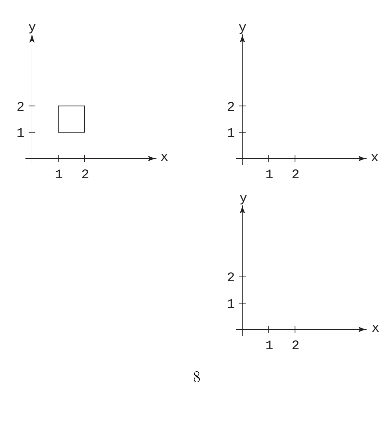
- Gives a **scaling matrix**

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$$

- **Provides differential scaling in x and y:**

$$x' = ax$$

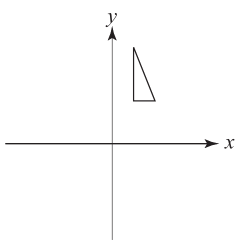
$$y' = dy$$

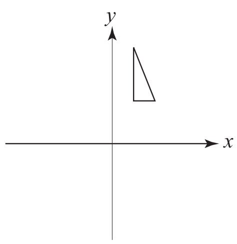


Suppose we keep $b = c = 0$, but let a *or* d go negative.

Examples:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$





Now let's leave $a = d = 1$ and experiment with c. . . .

The matrix

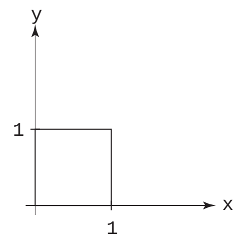
$$\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$$

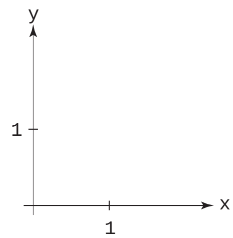
gives:

$$x' = x$$

$$y' = cx + y$$

Effect is called a "________."

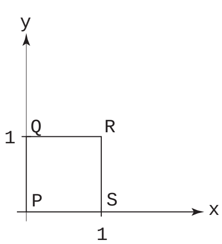


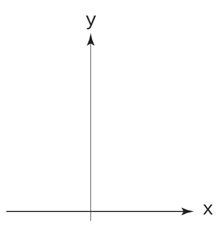


Effect on unit square

Let's see how a general 2×2 transformation M affects the unit square:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & a & a+b & b \\ 0 & c & c+d & d \end{bmatrix}$$





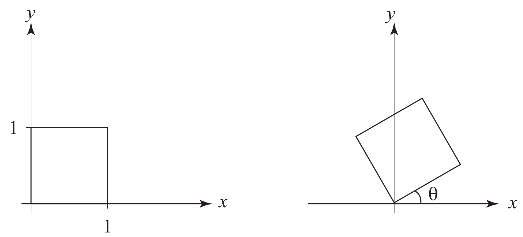
Effect on unit square, cont.

Observe:

- Origin invariant under M
- M can be determined just by knowing how the corners $(1, 0)$ and $(0, 1)$ are mapped
- a and d give x- and y-scaling
- b and c give x- and y-shearing

Rotation

From our observations of the effect on the unit square, it should be easy to write down a matrix for “rotation about the origin”:



- $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow$
- $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow$

Limitations of the 2×2 matrix

A 2×2 matrix allows

- Scaling
- Rotation
- Reflection
- Shearing

Q: What important operation does that leave out?

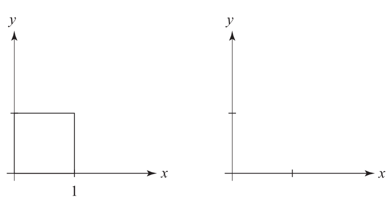
Homogeneous coordinates

Idea is to loft the problem up into 3-space, adding a third component to every point:

$$\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

And then transform with a 3×3 matrix:

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

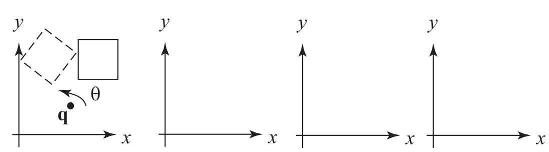


. . . Gives **translation**!

Rotation about arbitrary points

Until now, we have only considered rotation about the origin.

With homogeneous coordinates, you can specify a rotation, θ, about any point $\mathbf{q} = [q_x q_y]^T$ with a matrix:



1. Translate q to origin
2. Rotate
3. Translate back

Note: Transformation order is important!

Mathematical properties of affine transformations

All of the transformations we've looked at so far are examples of "affine transformations."

Here are some useful properties of affine transformations:

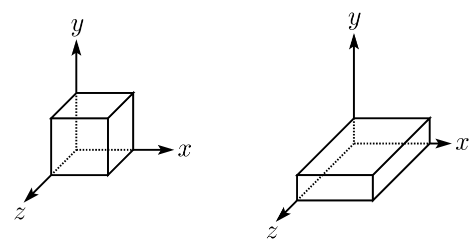
- Lines map to lines
- Parallel lines remain parallel
- Midpoints map to midpoints
  (in fact, ratios are always preserved)

Basic 3-D transformations: scaling

Some of the 3-D transformations are just like the 2-D ones. For example, scaling:

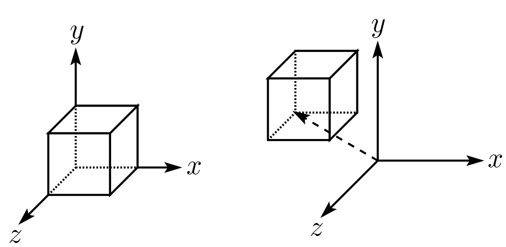
Scaling

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



Translation in 3D

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



Rotation in 3D

Rotation now has more possibilities in 3D:

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

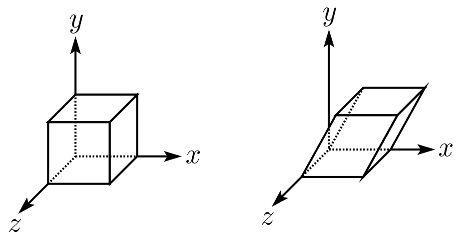
$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Shearing in 3D

Shearing is also more complicated. Here is one example:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & b & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



Summary

What to take away from this lecture:

- All the names in boldface.
- How points and transformations are represented.
- What all the elements of a 2×2 transformation matrix do and how these generalize to 3×3 transformations.
- What homogeneous coordinates are and how they work for affine transformations.
- How to concatenate transformations.
- The mathematical properties of affine trasnformations.