CSEP 573: Artificial Intelligence

Markov Decision Processes (MDP)

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Many slides over the course adapted from Luke Zettlemoyer, Dan Klein, Pieter Abbeel, Stuart Russell or Andrew Moore

Outline (roughly next two weeks)

- Markov Decision Processes (MDP)
  - MDP formalism
  - Value Iteration
  - Policy Iteration

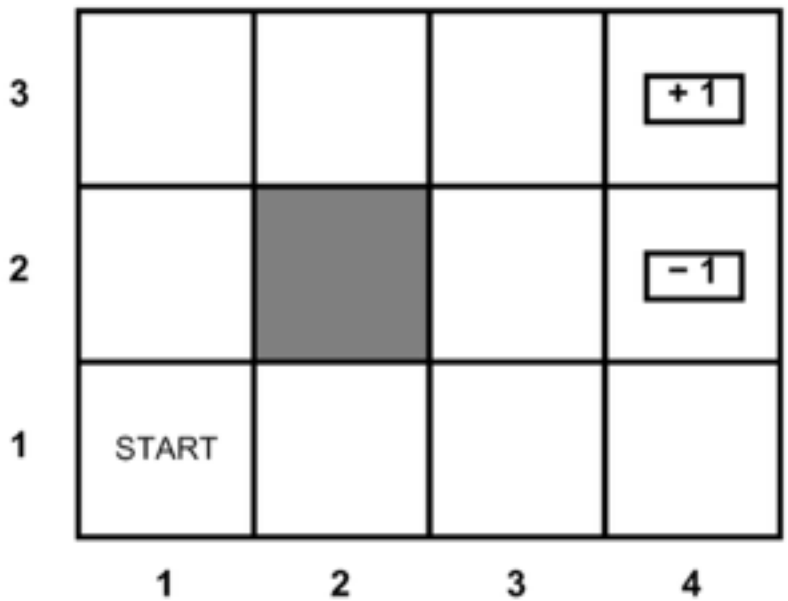
- Reinforcement Learning (RL)
  - Relationship to MDPs
  - Several learning algorithms

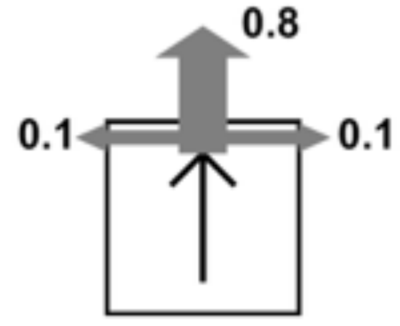
Non-deterministic Search

- Noisy execution of actions
  - Deterministic grid world vs. non-deterministic grid world

Example: Grid World

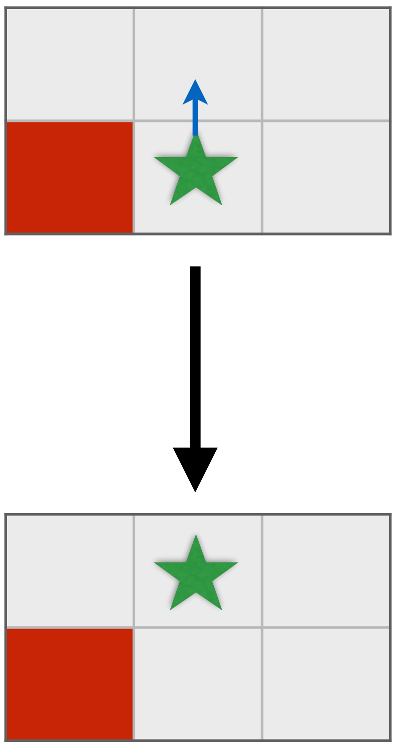
- A maze-like problem:
  - The agent lives in a grid
  - Walls block the agent's path
- The agent's actions do not always go as planned:
  - 80% of the time, the action North takes the agent North
    (if there is no wall there)
  - 10% of the time, North takes the agent West; 10% East
  - If there is a wall in the direction the agent would have been taken, the agent stays put
- Agent receives rewards each time step:
  - Small "living" reward each step
  - Big rewards come at the end
- Goal: maximize sum of rewards



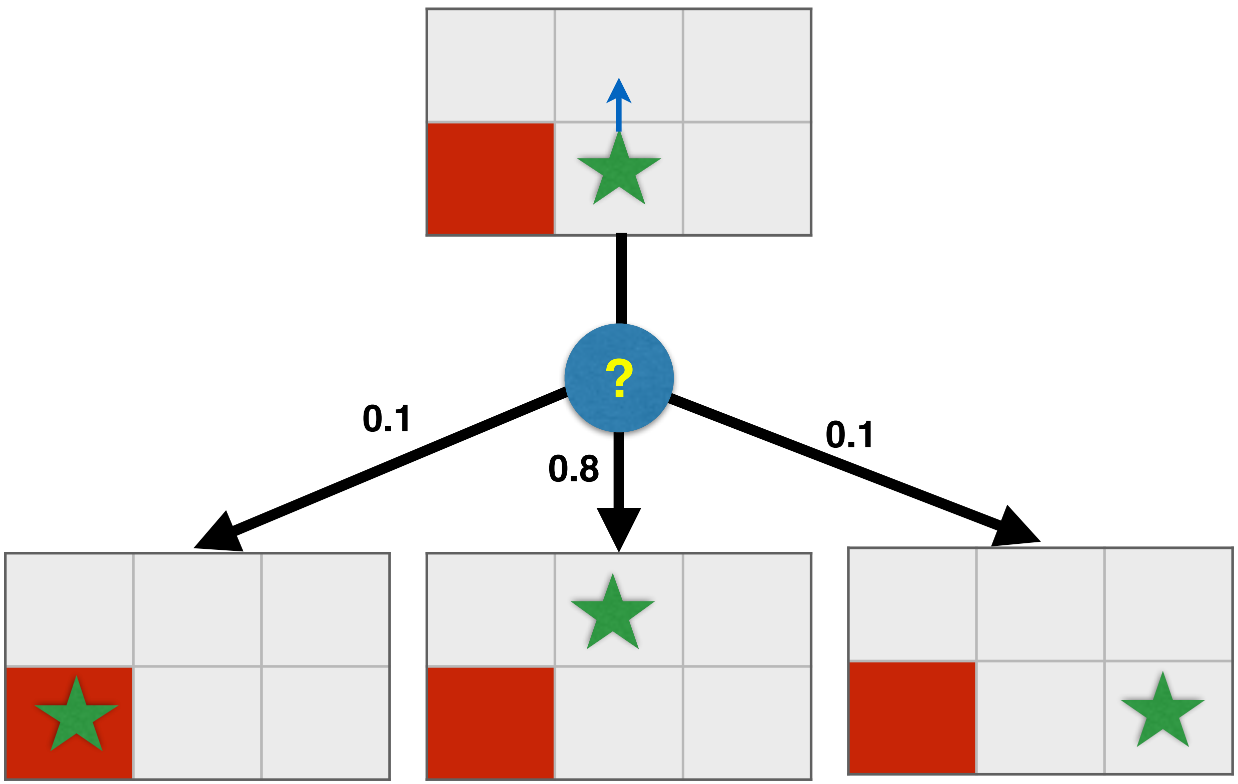


Grid World Actions

Deterministic

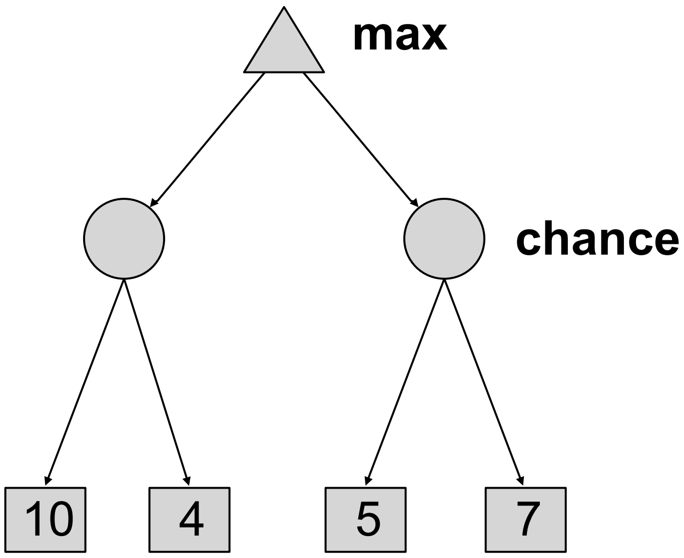


Stochastic



Review: Expectimax

- What if we don't know what the result of an action will be? E.g.,
  - In solitaire, next card is unknown
  - In minesweeper, mine locations
  - In pacman, the ghosts act randomly
- Can do **expectimax search**
  - Chance nodes, like min nodes, except the outcome is uncertain
  - Calculate expected utilities
  - Max nodes as in minimax search
  - Chance nodes take average (expectation) of value of children

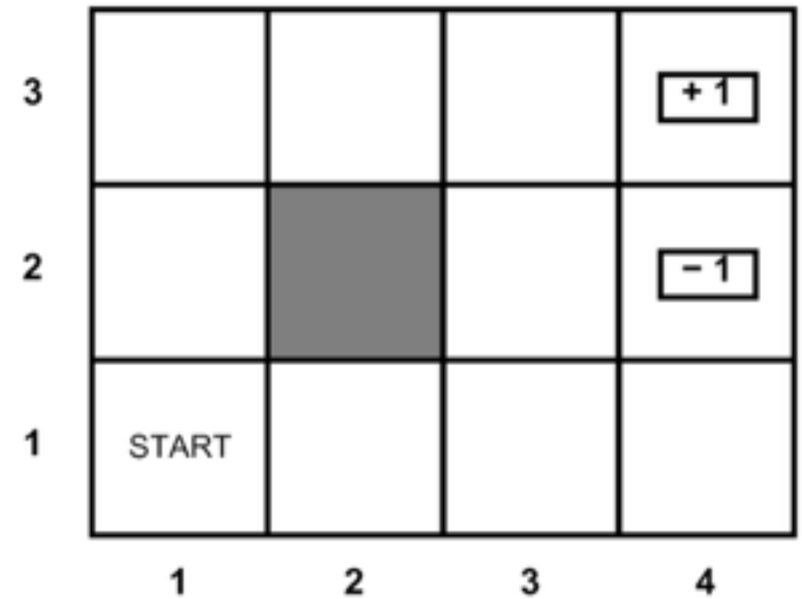


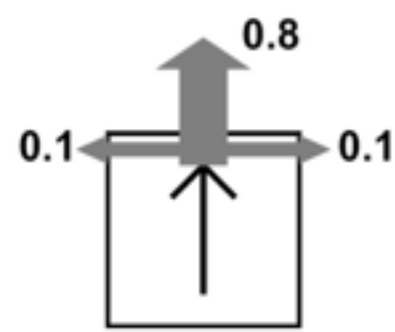
- Today, we'll learn how to formalize the underlying problem as a **Markov Decision Process**

Markov Decision Processes

- An MDP is defined by:
  - A set of states $s \in S$
  - A set of actions $a \in A$
  - A transition function T(s,a,s')
    - Prob that a from s leads to s'
    - i.e., P(s' | s,a)
    - Also called the model
  - A reward function R(s, a, s')
    - Sometimes just R(s) or R(s')
  - A start state (or distribution)
  - Maybe a terminal state

- MDPs: non-deterministic search problems
  - Reinforcement learning: MDPs where we don't know the transition or reward functions





What is Markov about MDPs?

- Andrey Markov (1856-1922)
- "Markov" generally means that given the present state, the future and the past are independent
- For Markov decision processes, "Markov" means:

$$P(S_{t+1} = s' | S_t = s_t, A_t = a_t, S_{t-1} = s_{t-1}, A_{t-1}, \dots S_0 = s_0)$$

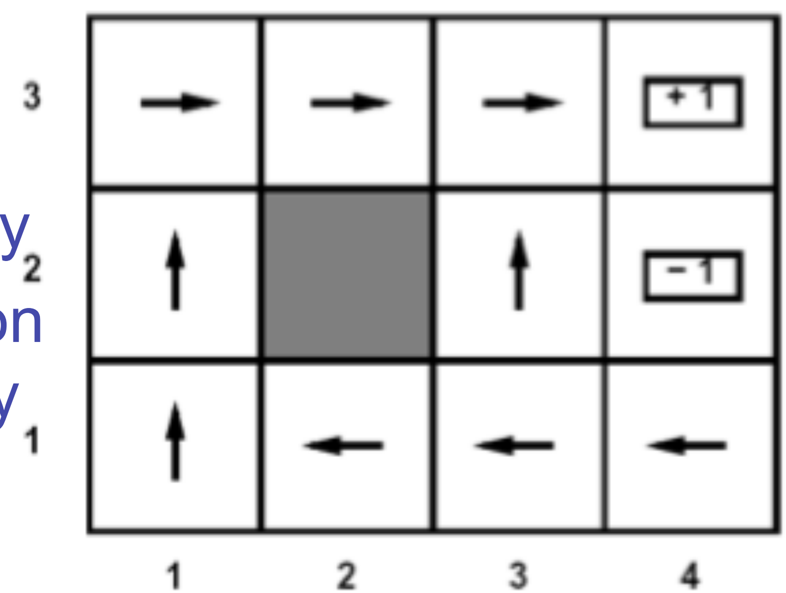
$$=$$

$$P(S_{t+1} = s' | S_t = s_t, A_t = a_t)$$

- This is just like search where the successor function only depends on the current state (not the history)

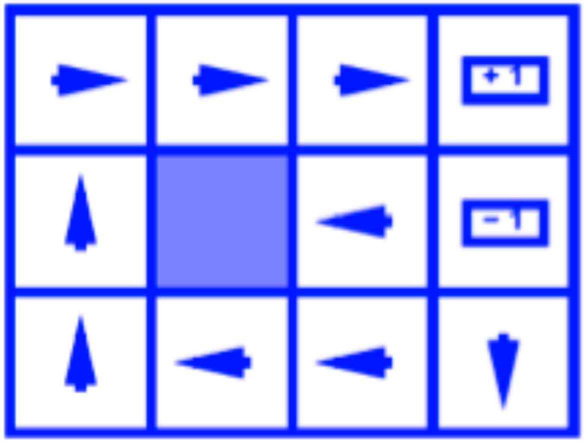
Solving MDPs

- In deterministic single-agent search problems, want an optimal plan, or sequence of actions, from start to a goal
- In an MDP, we want an optimal policy $\pi^*: S \rightarrow A$
  - A policy π gives an action for each state
  - An optimal policy maximizes expected utility if followed
  - Defines a reflex agent
- Expectimax didn't compute the entire policy
  - It computed the action for a single state only

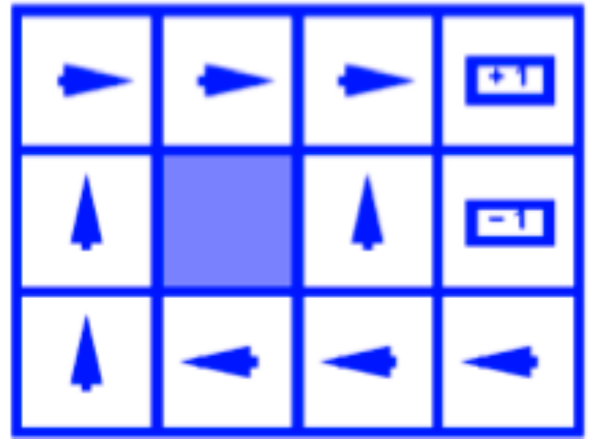


Optimal policy when R(s, a, s') = -0.03 for all non-terminals s

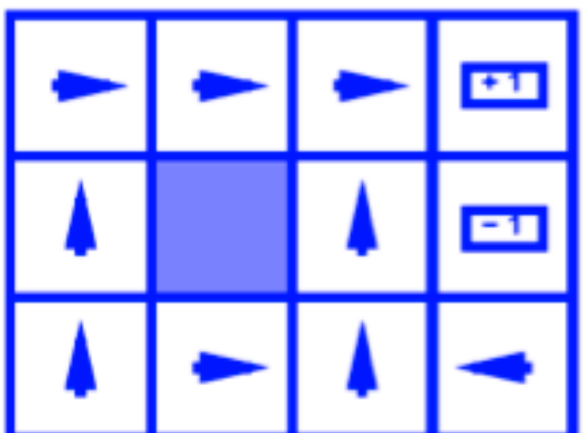
Example Optimal Policies



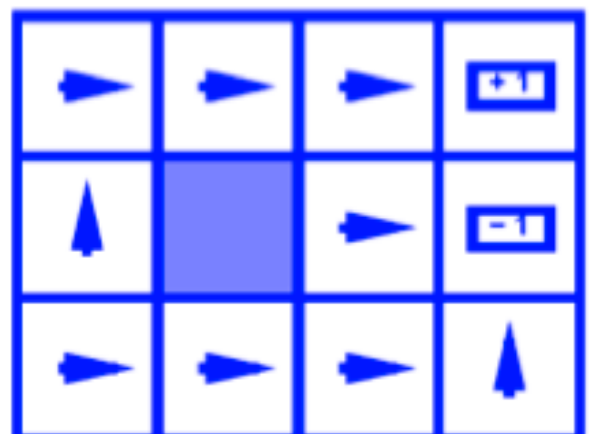
R(s) = -0.01



R(s) = -0.03



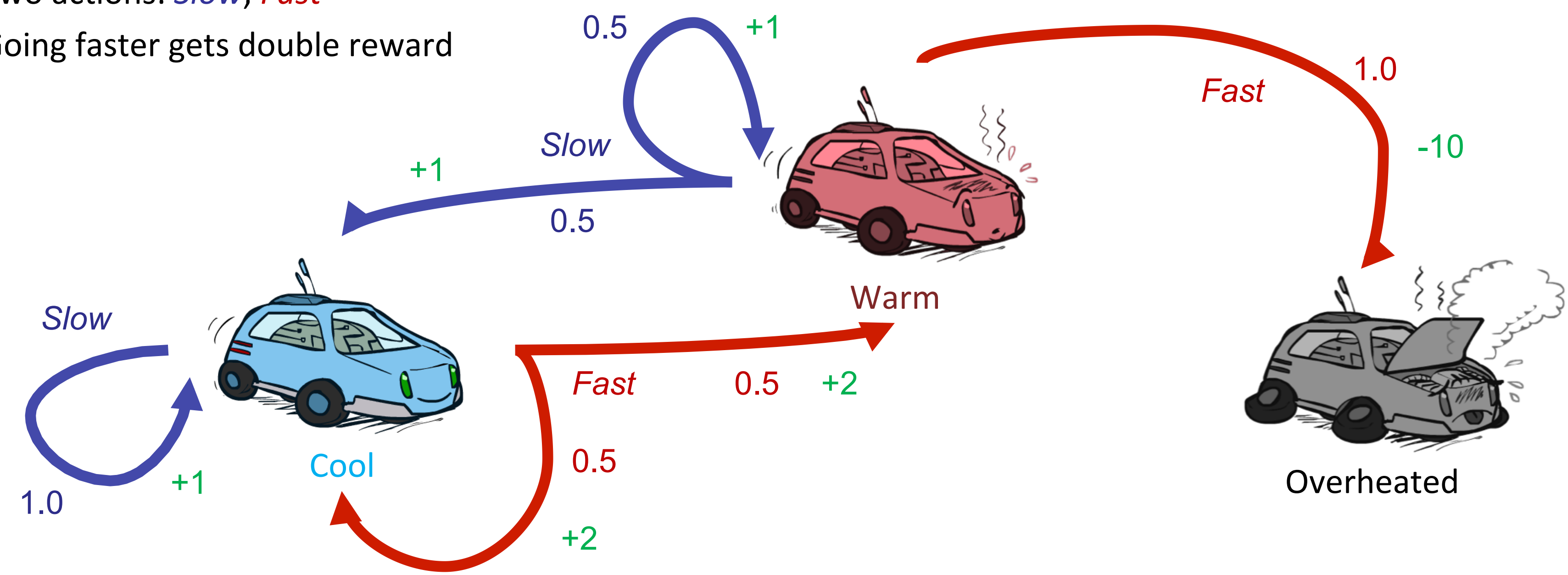
R(s) = -0.4



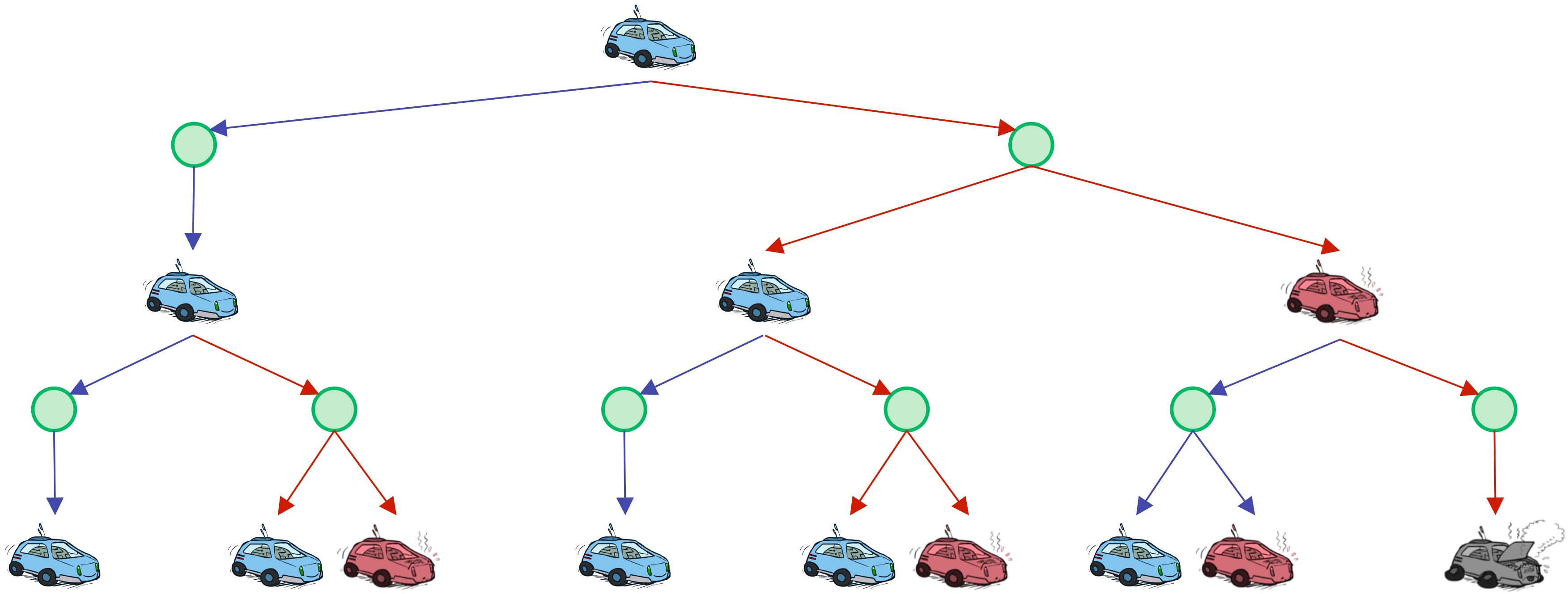
R(s) = -2.0

Another Example: Racing Car

- A robot car wants to travel far, quickly
- Three states: Cool, Warm, Overheated
- Two actions: *Slow*, *Fast*
- Going faster gets double reward

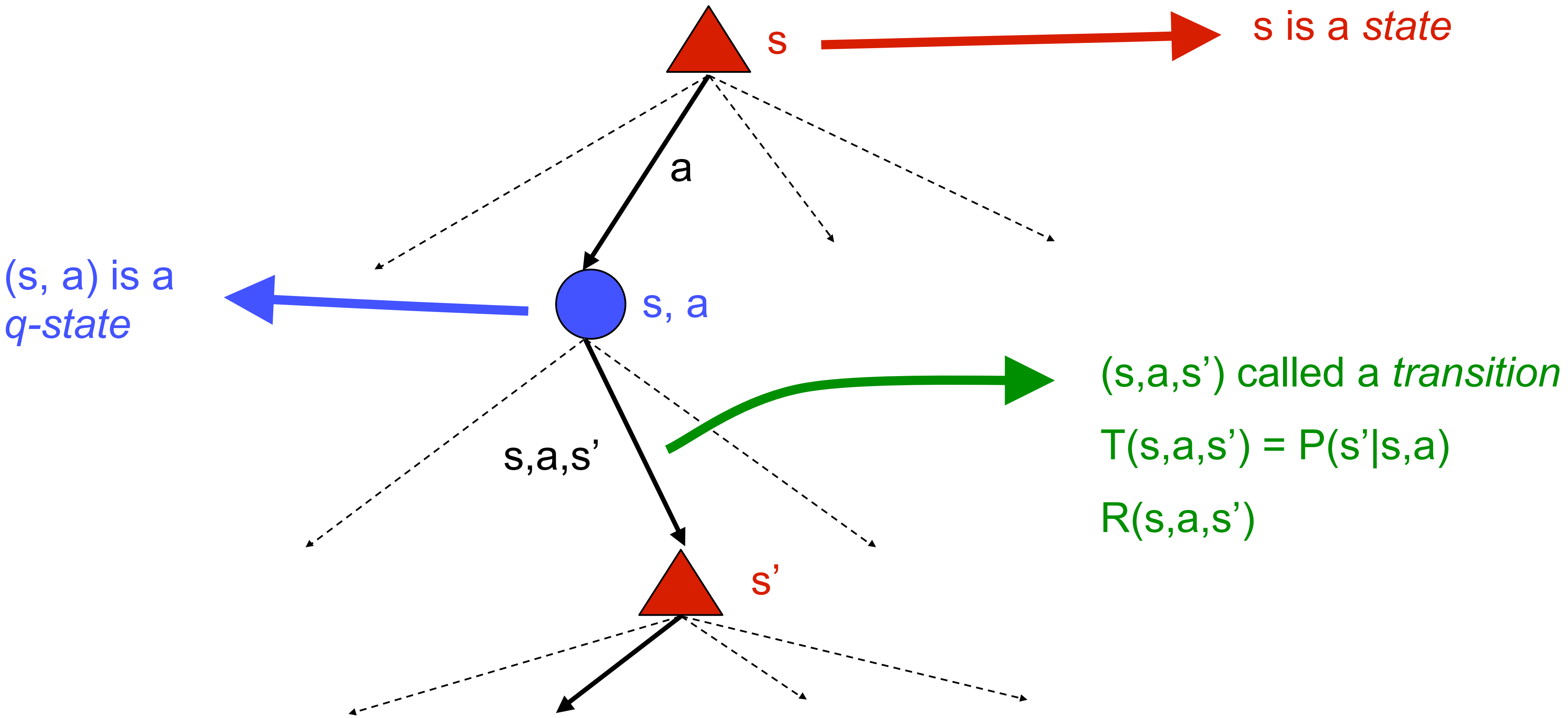


Racing Car Search Tree



MDP Search Trees

- Each MDP state gives an expectimax-like search tree

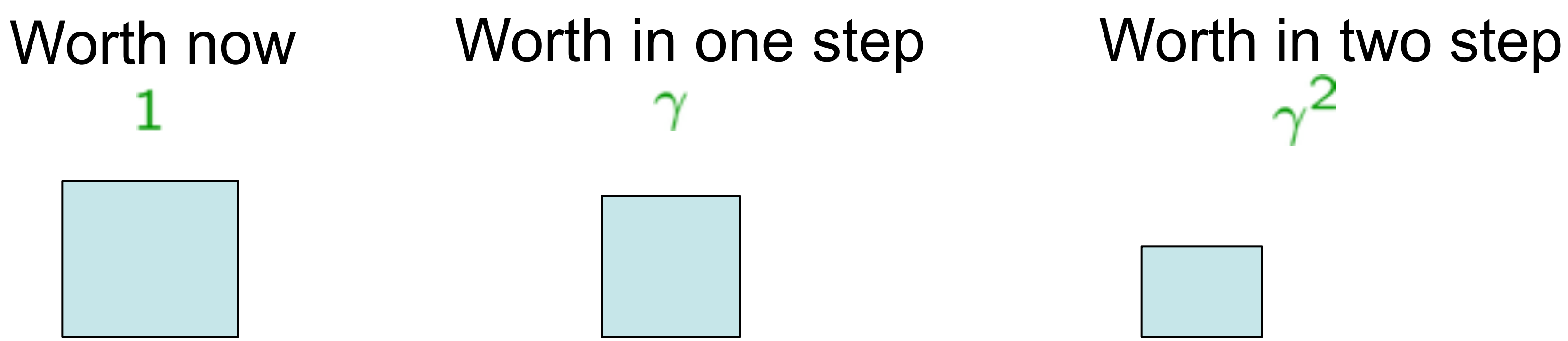


Utilities of Sequences

- What preference should an agent have over reward sequences?
- More or less:
  - [1, 2, 2] or [2, 3, 4]
- Now or later:
  - [0, 0, 1] or [1, 0, 0]

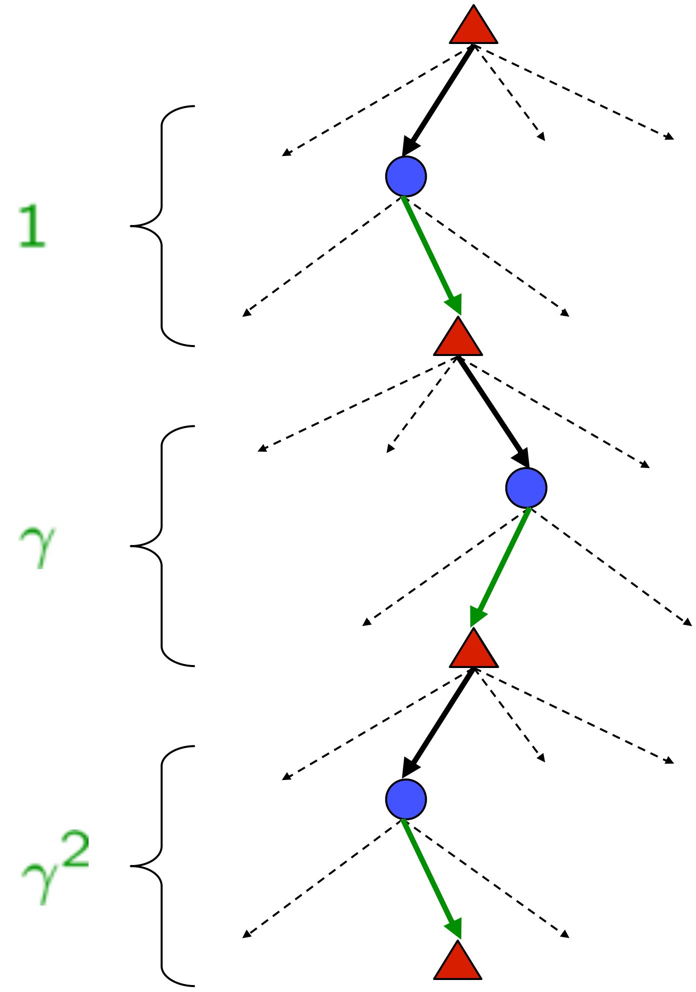
Discounting

- It is reasonable to maximize the sum of rewards
- It also makes sense to prefer rewards now to rewards later
- One solution: value of rewards decay exponentially



Discounting

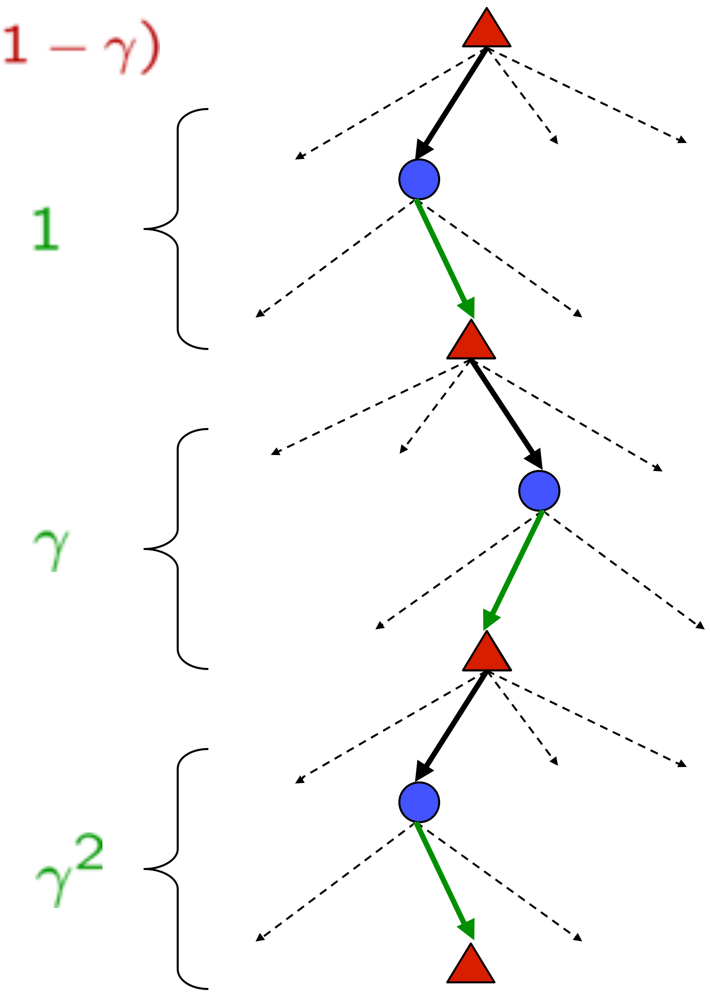
- How to discount?
  - Each time we descend, we multiply in the discount once
- Why discount?
  - Sooner rewards probably do have higher utility than later rewards
  - Also helps our algorithms converge
- Example: discount of 0.5
  - U([1, 2, 3]) = 1*1+.5*2 + .25*3
  - U([1,2,3])<U([3,2,1])



Discounting

$$U([r_0, \ldots r_\infty]) = \sum_{t=0}^{\infty} \gamma^t r_t \leq R_{\max}/(1-\gamma)$$

- Typically discount rewards by $\gamma < 1$ each time step
  - Sooner rewards have higher utility than later rewards
  - Also helps the algorithms converge



Quiz: Discounting

- Given:

10				1
a	b	c	d	e

  - Actions: East, West, and Exit (only available in exit states a, e)
  - Transitions: deterministic

- Quiz 1: For $\gamma = 1$, what is the optimal policy?

10				1

- Quiz 2: For $\gamma = 0.1$, what is the optimal policy?

10				1

- Quiz 3: For which ° are West and East equally good when in state d?

Utilities of Sequences

- In order to formalize optimality of a policy, need to understand utilities of sequences of rewards
- Typically consider stationary preferences:

$$[r, r_0, r_1, r_2, \ldots] \succ [r, r'_0, r'_1, r'_2, \ldots]$$
$$\Leftrightarrow$$
$$[r_0, r_1, r_2, \ldots] \succ [r'_0, r'_1, r'_2, \ldots]$$

- Two ways to define stationary utilities
  - Additive utility:

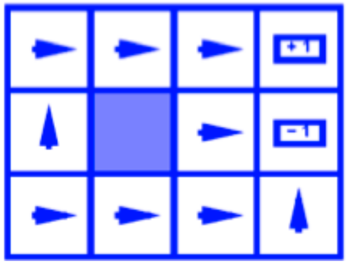
    $$U([r_0, r_1, r_2, \ldots]) = r_0 + r_1 + r_2 + \cdots$$

  - Discounted utility:

    $$U([r_0, r_1, r_2, \ldots]) = r_0 + \gamma r_1 + \gamma^2 r_2 \cdots$$

Infinite Utilities?!

- Problem: what if the game lasts forever?
  - Infinite state sequences have infinite rewards



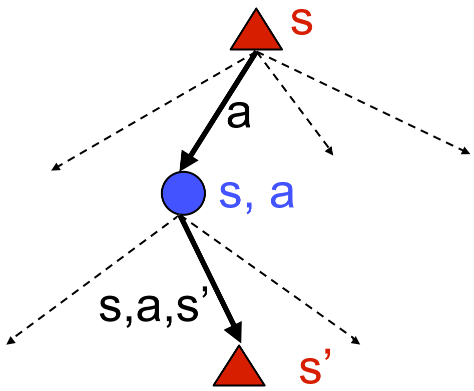
- Solutions:
  - Finite horizon:
    - Terminate episodes after a fixed T steps (e.g. life)
    - Gives nonstationary policies (π depends on time left)
  - Absorbing state: guarantee that for every policy, a terminal state will eventually be reached (like “overheated” for racing)
  - Discounting: for $0 < \gamma < 1$

$$U([r_0, \ldots r_\infty]) = \sum_{t=0}^{\infty} \gamma^t r_t \leq R_{\max}/(1-\gamma)$$

    - Smaller γ means smaller “horizon” – shorter term focus

Recap: Defining MDPs

- Markov decision processes:
  - States S
  - Start state s_0
  - Actions A
  - Transitions P(s'|s,a) (or T(s,a,s'))
  - Rewards R(s,a,s') (and discount γ)



- MDP quantities so far:
  - Policy = Choice of action for each state
  - Utility (or return) = sum of discounted rewards

Solving MDPs

- We want to find the optimal policy π^*:
  - Find best action for each state such that it maximizes Utility (or return) = sum of discounted rewards

Optimal Utilities

- Define the value of a state s:

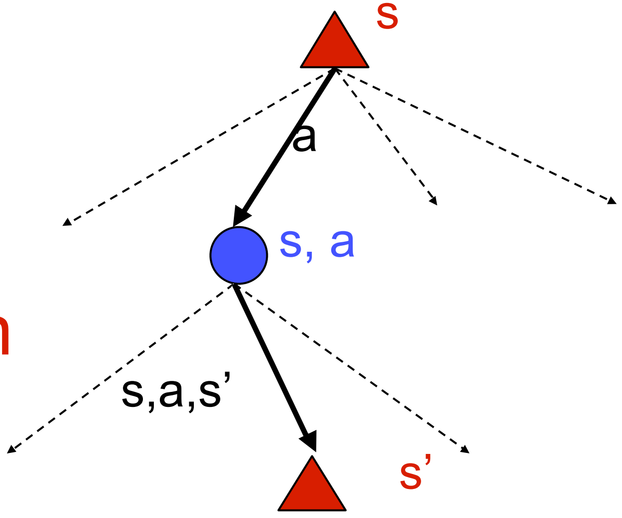
  $V^*(s)$ = expected utility starting in s and acting optimally

- Define the value of a q-state (s,a):

  $Q^*(s,a)$ = expected utility starting in s, taking action a and thereafter acting optimally

- Define the optimal policy:

  $\pi^*(s)$ = optimal action from state s

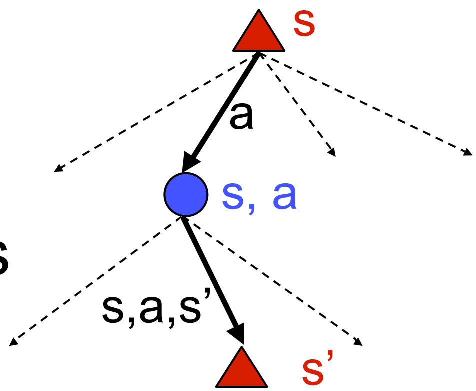



0.85
0.89
0.93
1.00
0.81
0.68
-1.00
0.77
0.73
0.70
0.47
VALUES AFTER 100 ITERATIONS


0.83
0.82
0.85
0.80
0.86
0.83
0.89
0.86
0.90
0.85
0.93
0.71
1.00
0.81
0.78
0.78
0.75
0.68
0.68
-0.67
0.49
-1.00
0.77
0.75
0.72
0.74
0.70
0.73
0.67
0.70
0.64
0.70
0.49
0.65
-0.71
0.47
0.30
0.47
Q-VALUES AFTER 100 ITERATIONS

The Bellman Equations

- Definition of "optimal utility" leads to a simple one-step lookahead relationship amongst optimal utility values:
  - Expected utility under optimal action
  - Average sum of (discounted) rewards
  - This is just what expectimax does



- Formally:

$$V^*(s) = \max_a Q^*(s,a)$$

$$Q^*(s,a) = \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V^*(s')\right]$$

$$V^*(s) = \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V^*(s')\right]$$

Solving MDPs

- Find V*(s) for all the states in S
  - |S| non-linear equations with |S| unknown

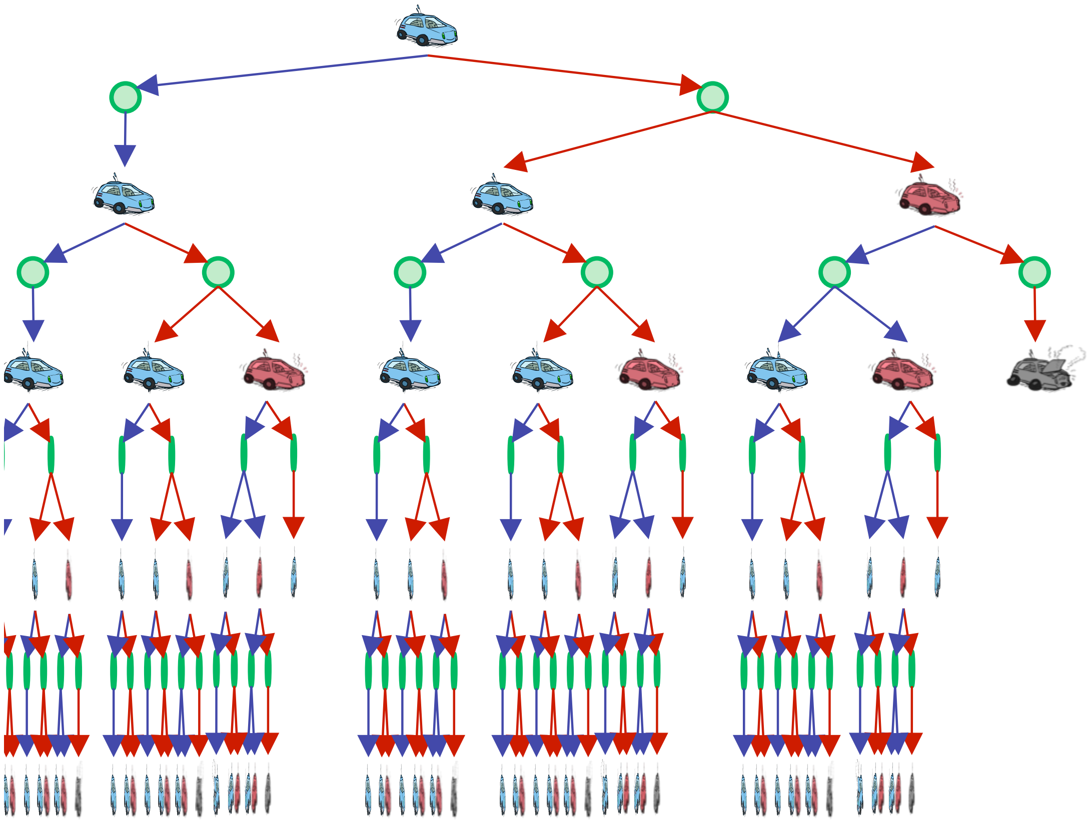
$$V^*(s) = \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V^*(s') \right]$$

- Our proposal:
  - Dynamic programming
  - Define V*i(s) as the optimal value of s if game ends in i steps
  - V*0(s)=0 for all the states

$$V_{i+1}(s) = \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V_i(s') \right]$$

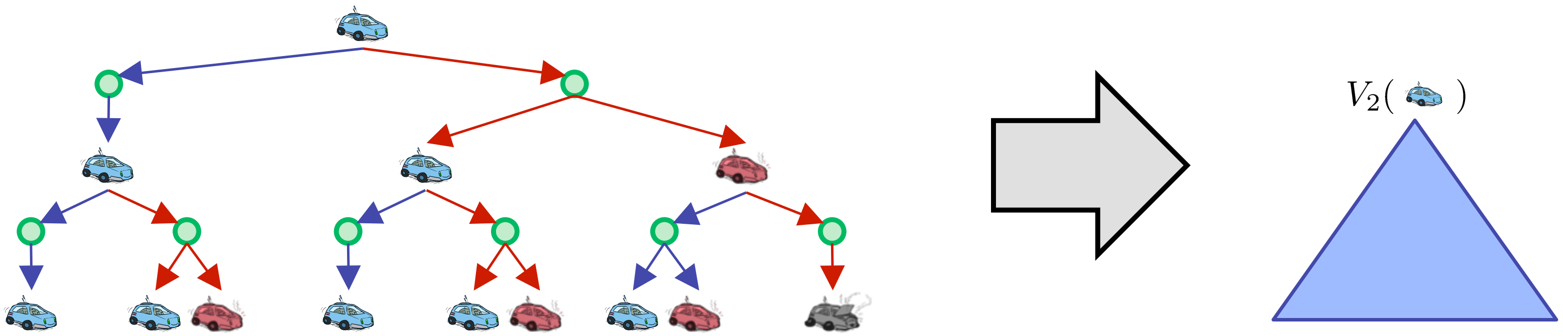
Racing Car Search Tree

- We're doing way too much work with expectimax!
- Problem: States are repeated
  - Idea: Only compute needed quantities once
- Problem: Tree goes on forever
  - Idea: Do a depth-limited computation, but with increasing depths until change is small
  - Note: deep parts of the tree eventually don't matter if $\gamma < 1$



Time Limited Values

- Key idea: time-limited values
- Define $V_k(s)$ to be the optimal value of s if the game ends in k more time steps
  - Equivalently, it's what a depth-k expectimax would give from s



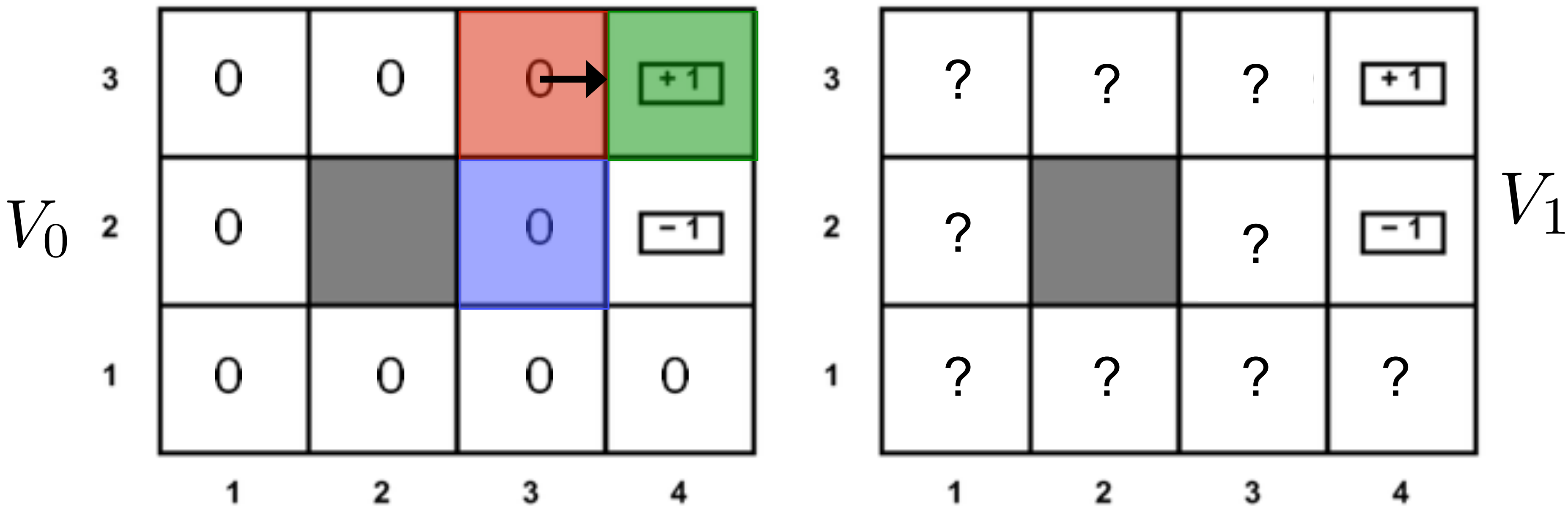
Example: γ=0.9, living reward=0, noise=0.2


0.00
0.00
0.00
1.00
0.00
0.00
-1.00
0.00
0.00
0.00
0.00
VALUES AFTER 1 ITERATIONS


0.00
0.00
0.72
1.00
0.00
0.00
-1.00
0.00
0.00
0.00
0.00
VALUES AFTER 2 ITERATIONS

Example: γ=0.9, living reward=0, noise=0.2

Example: Bellman Updates



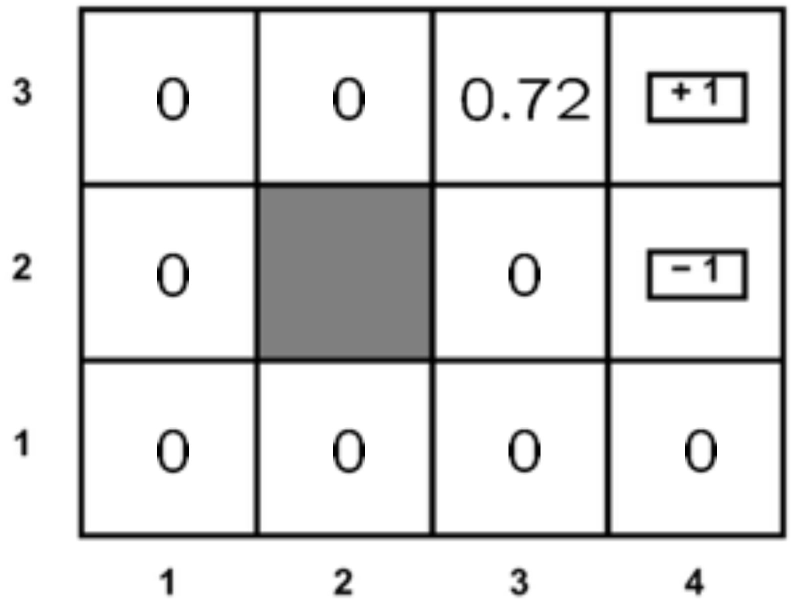
$$V_{i+1}(s) = \max_a \sum_{s'} T(s,a,s')\left[R(s,a,s') + \gamma V_i(s')\right] = \max_a Q_{i+1}(s,a)$$

$$Q_1(\langle 3,3\rangle, \text{right}) = \sum_{s'} T(\langle 3,3\rangle, \text{right}, s')\left[R(\langle 3,3\rangle, \text{right}, s') + \gamma V_i(s')\right]$$

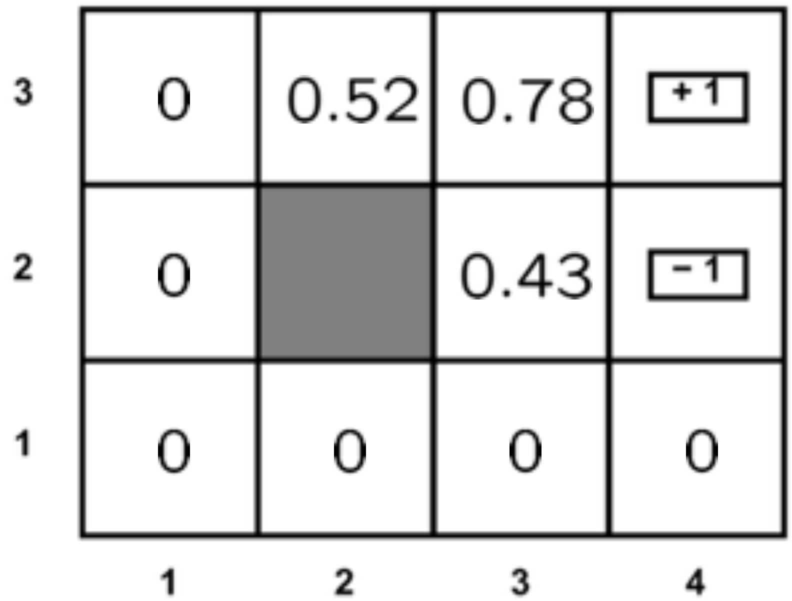
$$= 0.8 * [0.0 + 0.9 * 1.0] + 0.1 * [0.0 + 0.9 * 0.0] + 0.1 * [0.0 + 0.9 * 0.0]$$

Example: Value Iteration

V_1



V_2



- Information propagates outward from terminal states and eventually all states have correct value estimates


0.00
0.52
0.78
1.00
0.00
0.43
-1.00
0.00
0.00
0.00
0.00
VALUES AFTER 3 ITERATIONS

VALUES AFTER 4 ITERATIONS


0.51
0.72
0.84
1.00
0.27
0.55
-1.00
0.00
0.22
0.37
0.13
VALUES AFTER 5 ITERATIONS


0.59
0.73
0.85
1.00
0.41
0.57
-1.00
0.21
0.31
0.43
0.19
VALUES AFTER 6 ITERATIONS


0.62
0.74
0.85
1.00
0.50
0.57
-1.00
0.34
0.36
0.45
0.24
VALUES AFTER 7 ITERATIONS

Recap: Value Iteration

- Idea:
  - Start with $V_0^*(s) = 0$, which we know is right (why?)
  - Given V_i^*, calculate the values for all states for depth i+1:

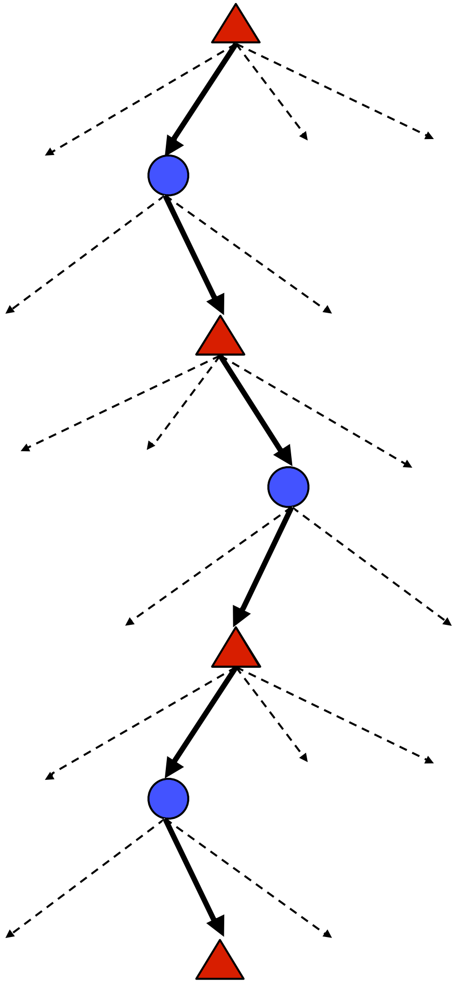
$$V_{i+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V_i(s')\right]$$

  - This is called a value update or Bellman update
  - Repeat until convergence

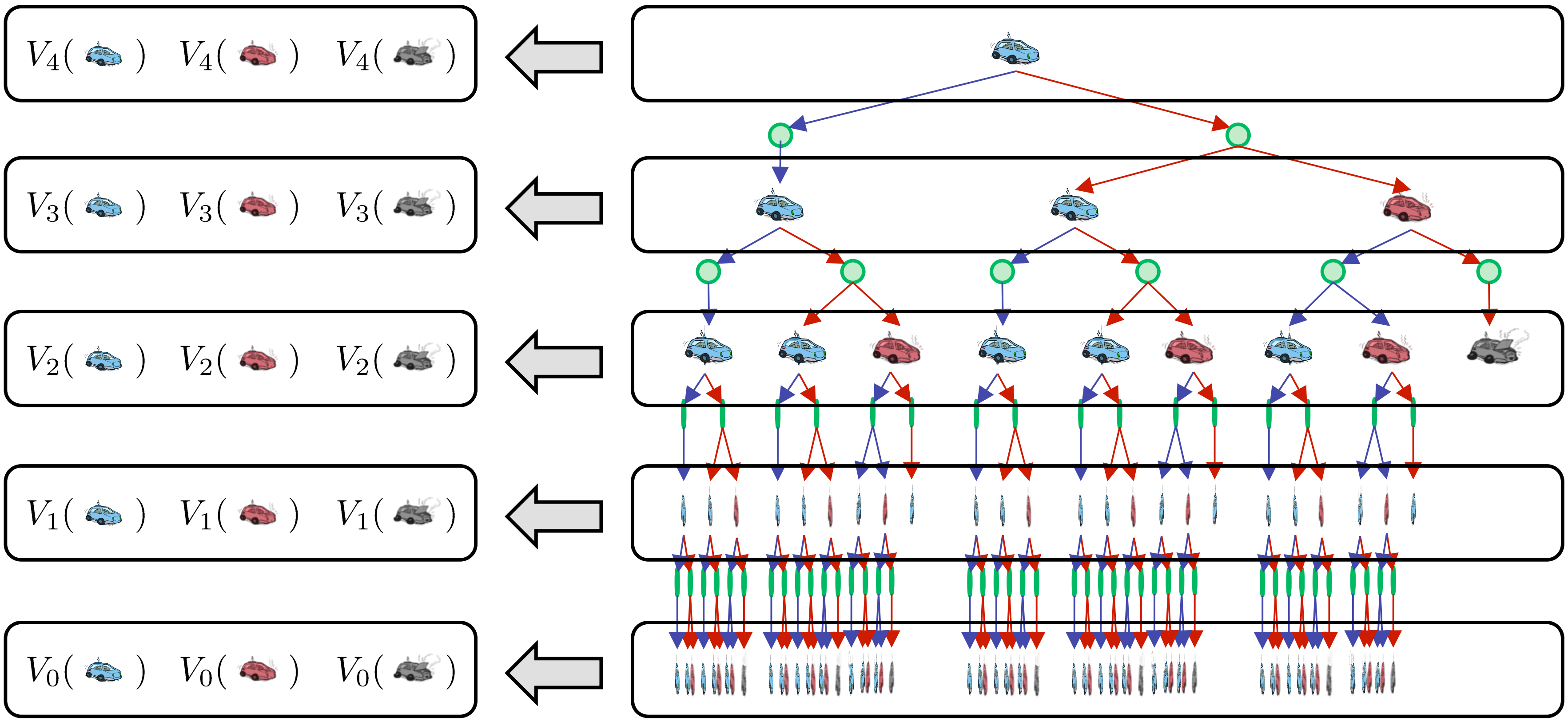
- Theorem: will converge to unique optimal values
  - Basic idea: approximations get refined towards optimal values
  - Policy may converge long before values do

Why Not Search Trees?

- Why not solve with expectimax?
- Problems:
  - This tree is usually infinite (why?)
  - Same states appear over and over (why?)
  - We would search once per state (why?)
- Idea: Value iteration
  - Compute optimal values for all states all at once using successive approximations
  - Will be a bottom-up dynamic program similar in cost to memoization
  - Do all planning offline, no replanning needed!

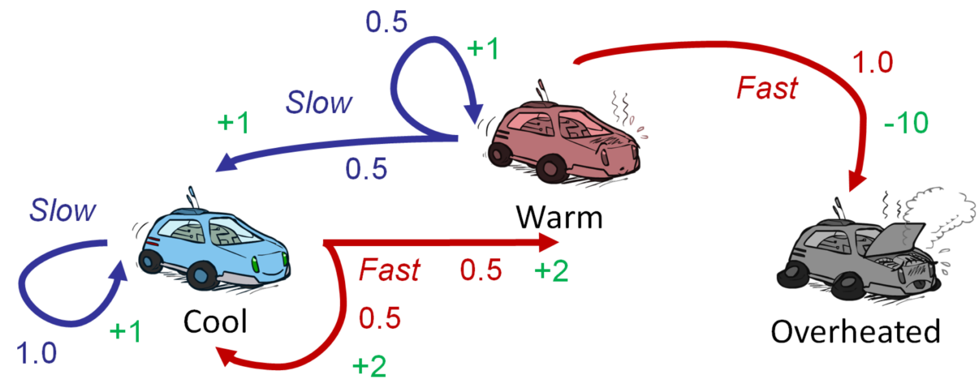


Computing time limited values



Example of Value iteration

V_2	3.5	2.5	0
V_1	2	1	0
V_0	0	0	0

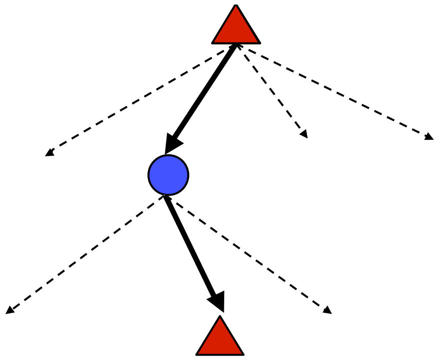


Assume no discount!

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V_k(s')\right]$$

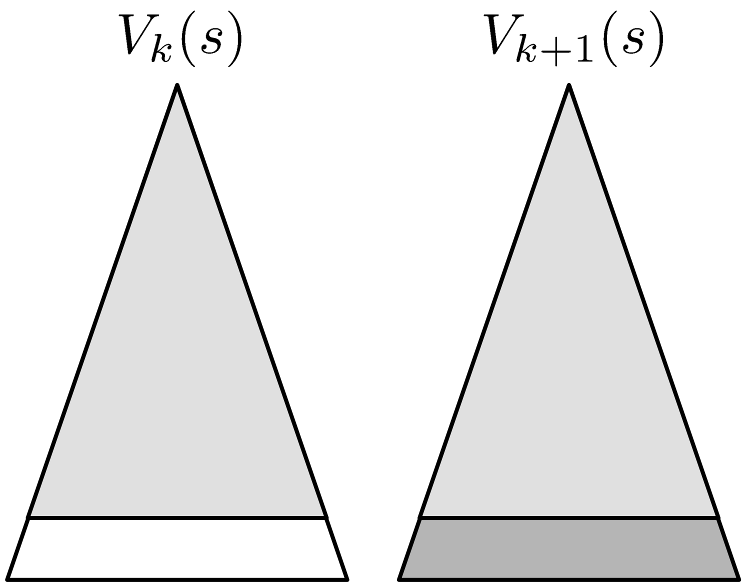
Recap: Value Estimates

- Calculate estimates $V_k^*(s)$
  - The optimal value considering only next k time steps (k rewards)
  - As $k \rightarrow \infty$, it approaches the optimal value
- Why:
  - If discounting, distant rewards become negligible
  - If terminal states reachable from everywhere, fraction of episodes not ending becomes negligible
  - Otherwise, can get infinite expected utility and then this approach actually won't work



Convergence

- How do we know the V_k vectors are going to converge?
- Case 1: If the tree has maximum depth M, then V_M holds the actual untruncated values
- Case 2: If the discount is less than 1
  - Sketch: For any state V_k and V_{k+1} can be viewed as depth k +1 expectimax results in nearly identical search trees
  - The difference is that on the bottom layer, V_{k+1} has actual rewards while V_k has zeros
  - That last layer is at best all R_{MAX}
  - It is at worst R_{MIN}
  - But everything is discounted by γ^k that far out
  - So V_k and V_{k+1} are at most γ^k max|R| different
  - So as k increases, the values converge



Value Iteration Complexity

- Problem size:
  - |A| actions and |S| states
- Each Iteration
  - Computation: $O(|A| \cdot |S|^2)$
  - Space: $O(|S|)$
- Num of iterations
  - Can be exponential in the discount factor γ

Practice: Computing Actions

- Which action should we chose from state s:
  - Given optimal values Q?

$$\arg\max_a Q^*(s,a)$$

  - Given optimal values V?

$$\arg\max_a \sum_{s'} T(s,a,s')[R(s,a,s') + \gamma V^*(s')]$$

  - Lesson: actions are easier to select from Q's!

Aside: Q-Value Iteration

- Value iteration: find successive approx optimal values
  - Start with $V_0^*(s) = 0$
  - Given V_i^*, calculate the values for all states for depth i+1:

$$V_{i+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V_i(s') \right]$$

- But Q-values are more useful!
  - Start with $Q_0^*(s,a) = 0$
  - Given Q_i^*, calculate the q-values for all q-states for depth i+1:

$$Q_{i+1}(s,a) \leftarrow \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma \max_{a'} Q_i(s',a') \right]$$

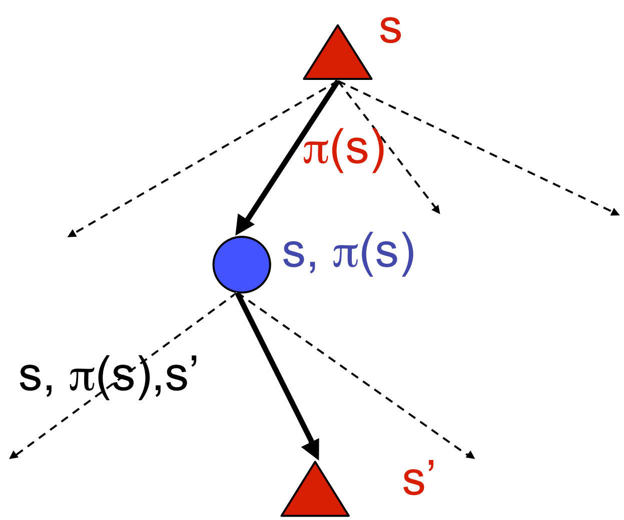
Example: Value Iteration

Utilities for Fixed Policies

- Another basic operation: compute the utility of a state s under a fix (general non-optimal) policy
- Define the utility of a state s, under a fixed policy π:

  $V^{\pi}(s)$ = expected total discounted rewards (return) starting in s and following π

- Recursive relation (one-step look-ahead / Bellman equation):



$$V^{\pi}(s) = \sum_{s'} T(s, \pi(s), s')[R(s, \pi(s), s') + \gamma V^{\pi}(s')]$$

Policy Evaluation

- How do we calculate the V's for a fixed policy?
- Idea one: modify Bellman updates

$$V_0^\pi(s) = 0$$

$$V_{i+1}^\pi(s) \leftarrow \sum_{s'} T(s, \pi(s), s')[R(s, \pi(s), s') + \gamma V_i^\pi(s')]$$

- Idea two: it's just a linear system, solve with Matlab (or whatever)

Policy Iteration

- Problem with value iteration:
  - Considering all actions each iteration is slow: takes |A| times longer than policy evaluation
  - But policy doesn’t change each iteration, time wasted

- Alternative to value iteration:
  - Step 1: Policy evaluation: calculate utilities for a fixed policy (not optimal utilities!) until convergence (fast)
  - Step 2: Policy improvement: update policy using one-step lookahead with resulting converged (but not optimal!) utilities (slow but infrequent)
  - Repeat steps until policy converges

Policy Iteration

- Policy evaluation: with fixed current policy π, find values with simplified Bellman updates
  - Iterate until values converge

$$V_{i+1}^{\pi_k}(s) \leftarrow \sum_{s'} T(s, \pi_k(s), s') \left[R(s, \pi_k(s), s') + \gamma V_i^{\pi_k}(s') \right]$$

  - Note: could also solve value equations with other techniques

- Policy improvement: with fixed utilities, find the best action according to one-step look-ahead

$$\pi_{k+1}(s) = \arg\max_a \sum_{s'} T(s, a, s') \left[R(s, a, s') + \gamma V^{\pi_k}(s') \right]$$

Policy Iteration Complexity

- Problem size:
  - |A| actions and |S| states

- Each Iteration
  - Computation: $O(|S|^3 + |A| \cdot |S|^2)$
  - Space: $O(|S|)$

- Num of iterations
  - Unknown, but can be faster in practice
  - Convergence is guaranteed

Comparison

- In value iteration:
  - Every pass (or “backup”) updates both utilities (explicitly, based on current utilities) and policy (possibly implicitly, based on current policy)
- In policy iteration:
  - Several passes to update utilities with frozen policy
  - Occasional passes to update policies
- Hybrid approaches (asynchronous policy iteration):
  - Any sequences of partial updates to either policy entries or utilities will converge if every state is visited infinitely often

Reinforcement Learning

- Basic idea:
  - Receive feedback in the form of rewards
  - Agent's utility is defined by the reward function
  - Must learn to act so as to maximize expected rewards