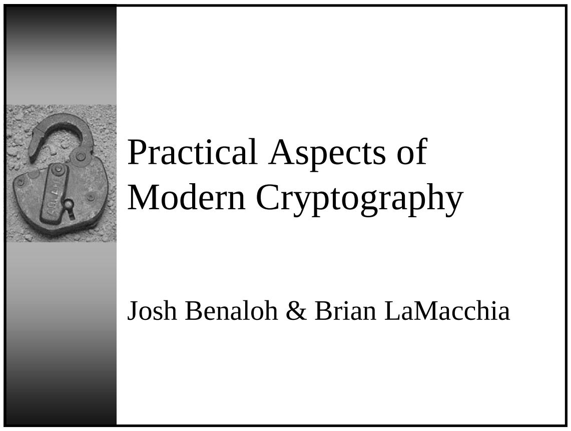
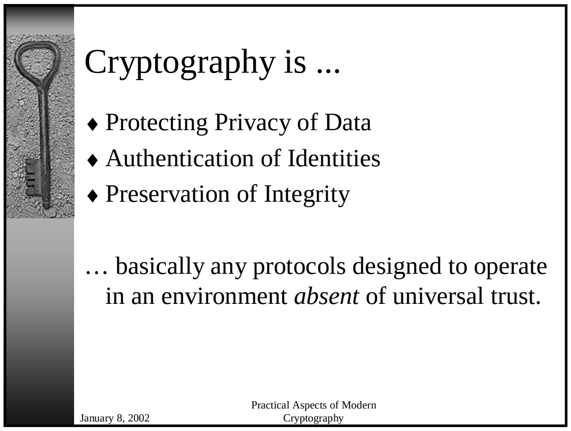


Practical Aspects of Modern Cryptography

Josh Benaloh & Brian LaMacchia



Cryptography is ...

- ♦ Protecting Privacy of Data
- ♦ Authentication of Identities
- ♦ Preservation of Integrity

… basically any protocols designed to operate in an environment *absent* of universal trust.

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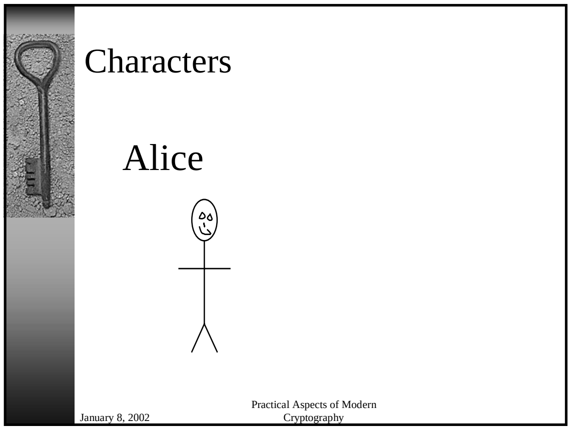

Characters

Alice

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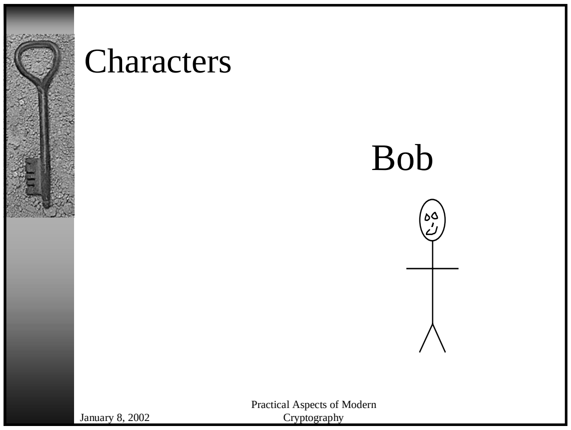

Characters

Bob

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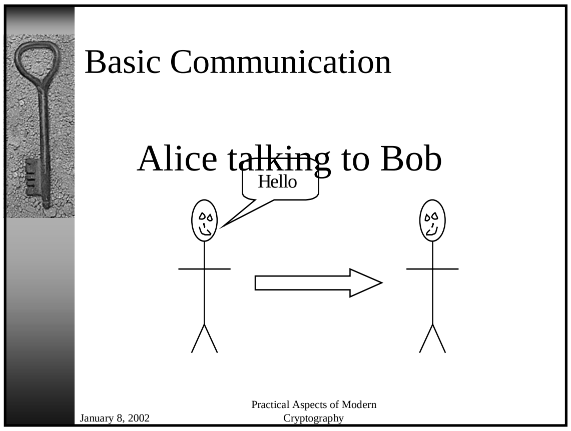

Basic Communication

Alice talking to Bob

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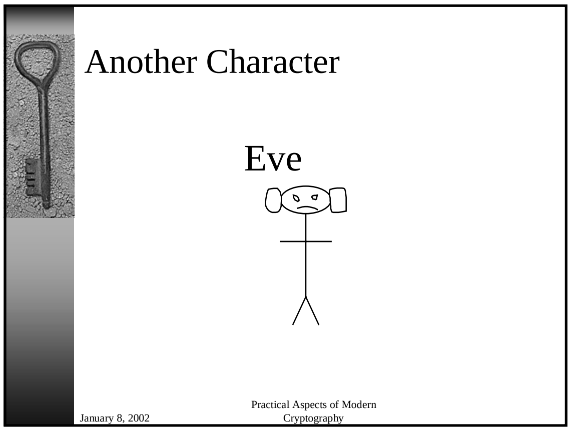

Another Character

Eve

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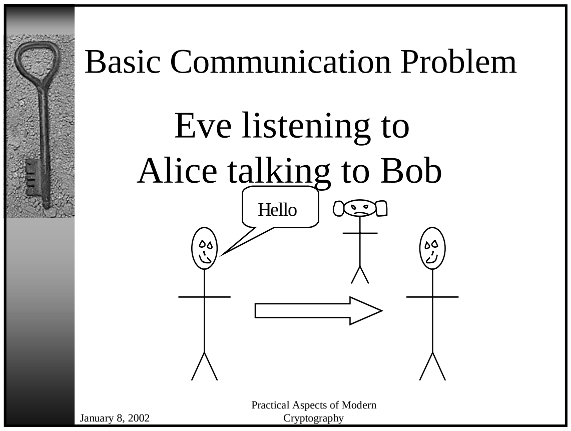
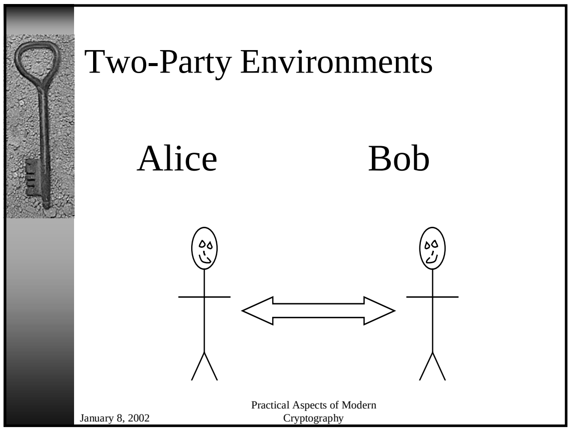


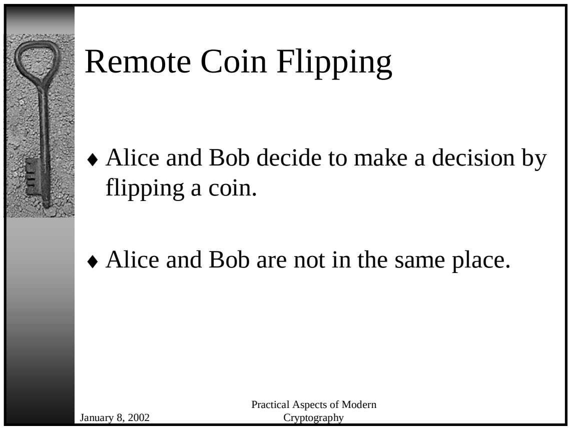
Basic Communication Problem

Eve listening to
Alice talking to Bob



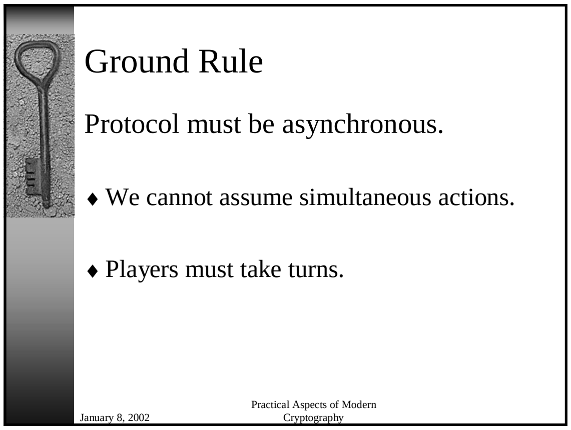
Two-Party Environments

Alice Bob



Remote Coin Flipping

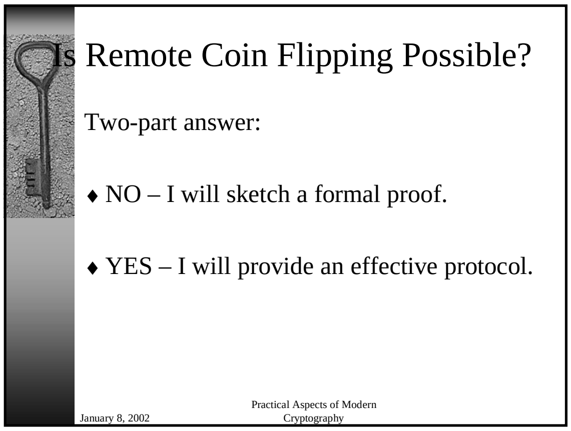
- ♦ Alice and Bob decide to make a decision by flipping a coin.
- ♦ Alice and Bob are not in the same place.



Ground Rule

Protocol must be asynchronous.

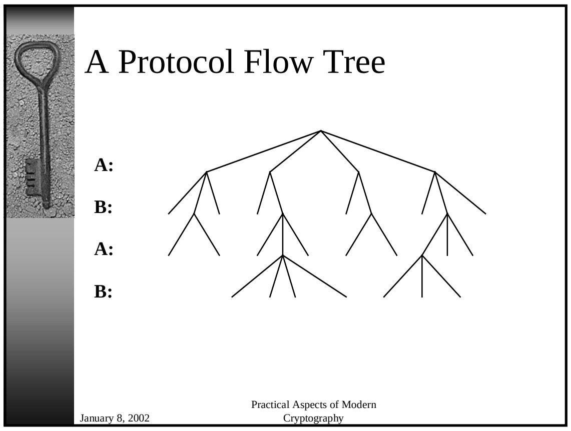
- ♦ We cannot assume simultaneous actions.
- ♦ Players must take turns.



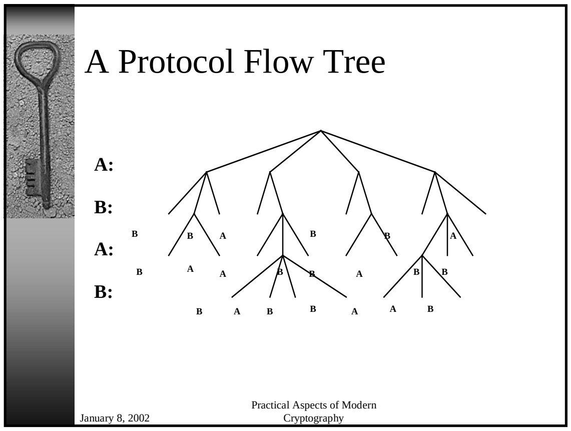
Is Remote Coin Flipping Possible?

Two-part answer:

- ♦ NO – I will sketch a formal proof.
- ♦ YES – I will provide an effective protocol.



A Protocol Flow Tree



A Protocol Flow Tree

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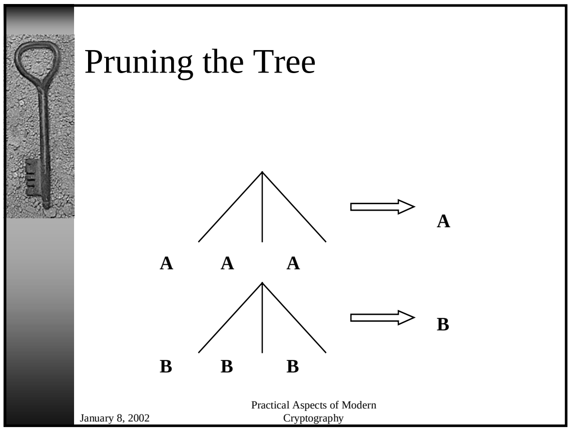

Pruning the Tree

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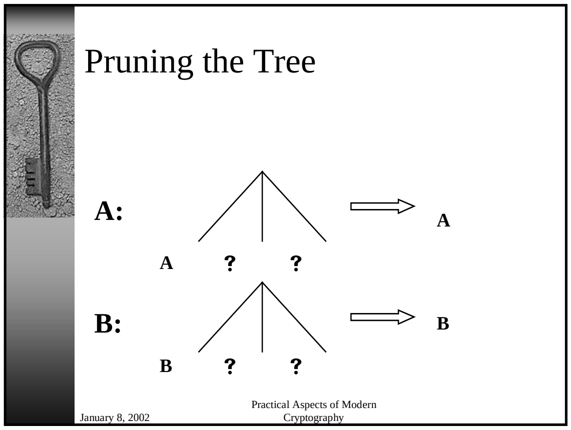

Pruning the Tree

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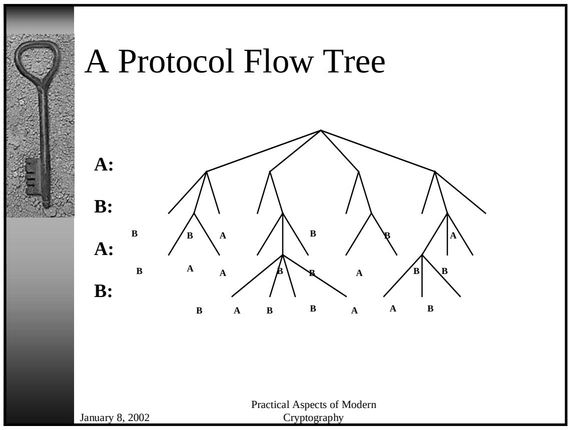

A Protocol Flow Tree

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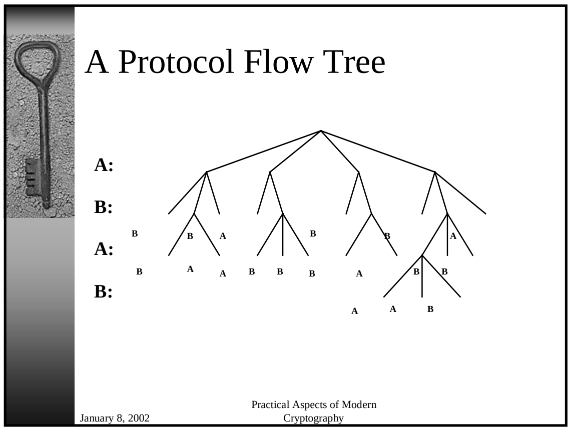

A Protocol Flow Tree

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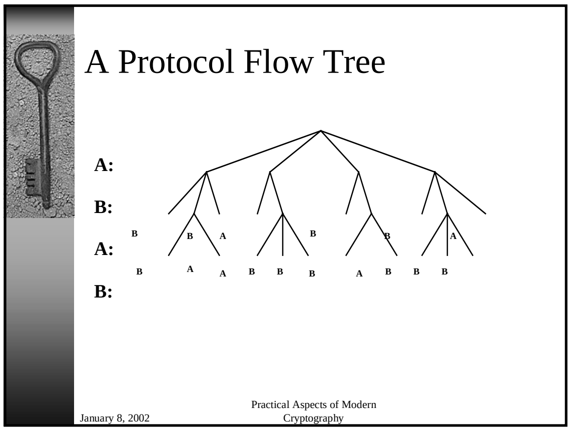
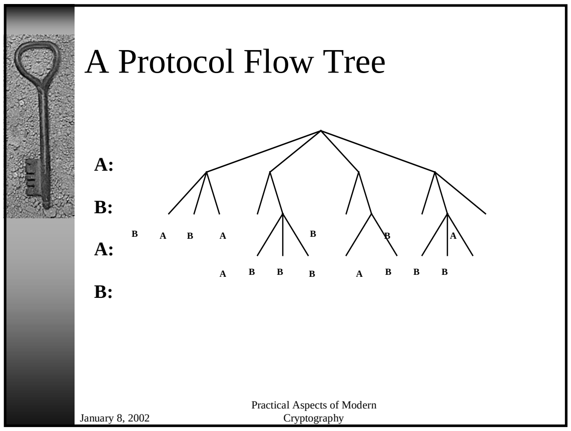


A Protocol Flow Tree

January 8, 2002 Practical Aspects of Modern Cryptography



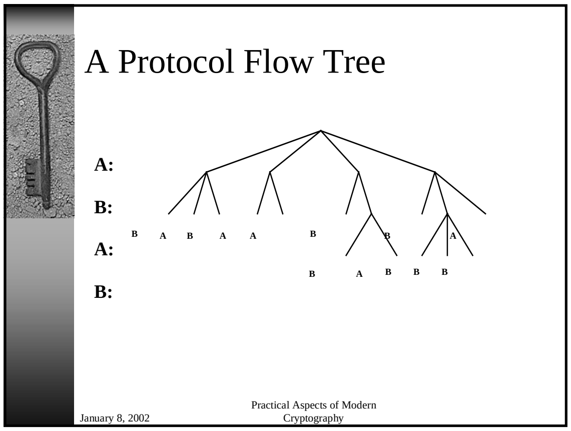
A Protocol Flow Tree

A:

B:

A:

B:



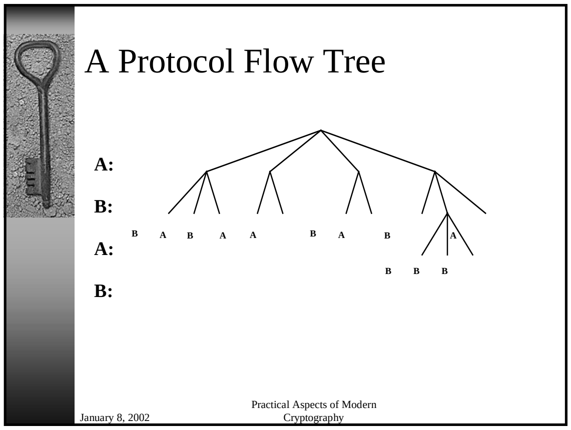
A Protocol Flow Tree

A:

B:

A:

B:



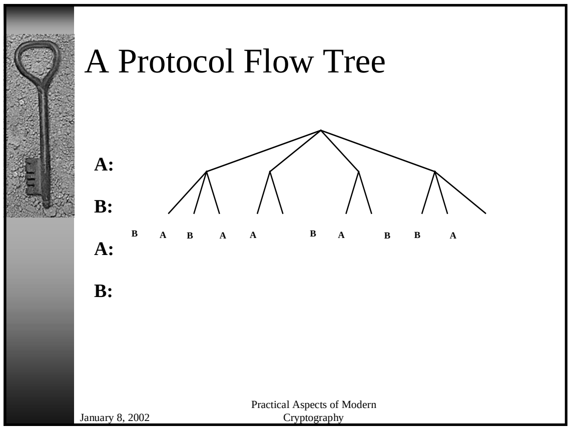
A Protocol Flow Tree

A:

B:

A:

B:



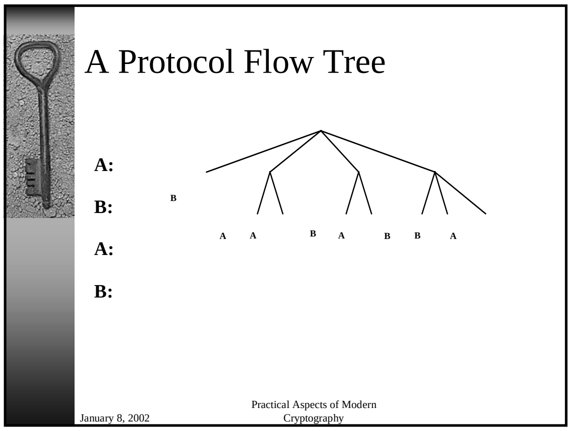
A Protocol Flow Tree

A:

B:

A:

B:



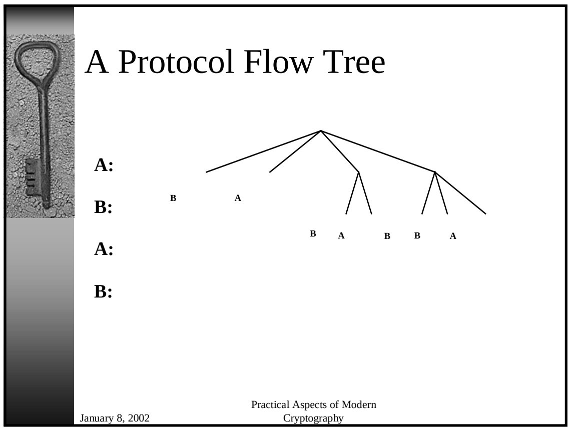
A Protocol Flow Tree

A:

B:

A:

B:



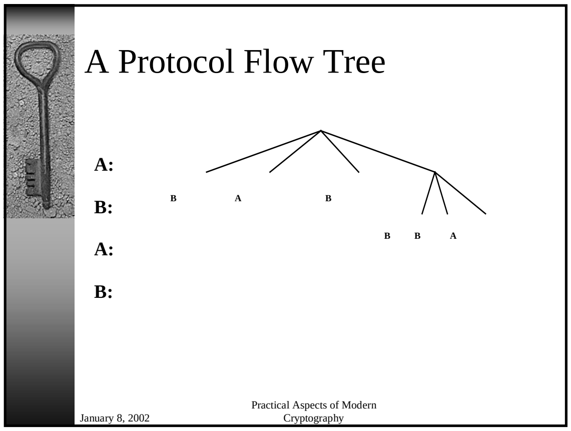
A Protocol Flow Tree

A:

B:

A:

B:



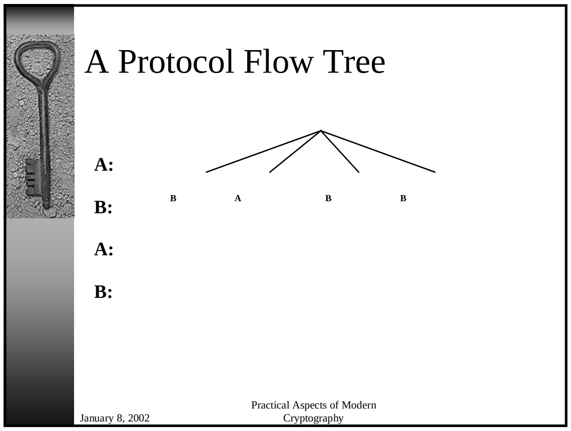
A Protocol Flow Tree

A:

B:

A:

B:



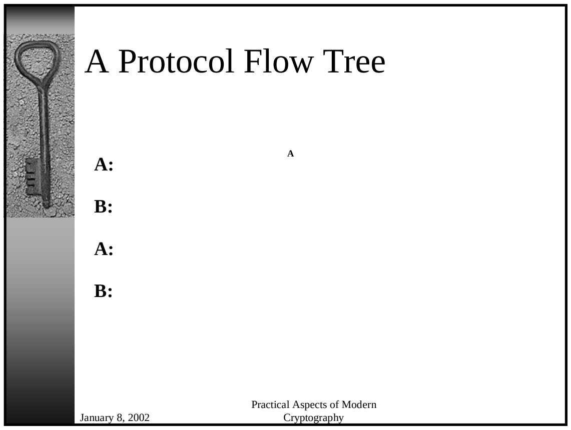
A Protocol Flow Tree

A:

B:

A:

B:



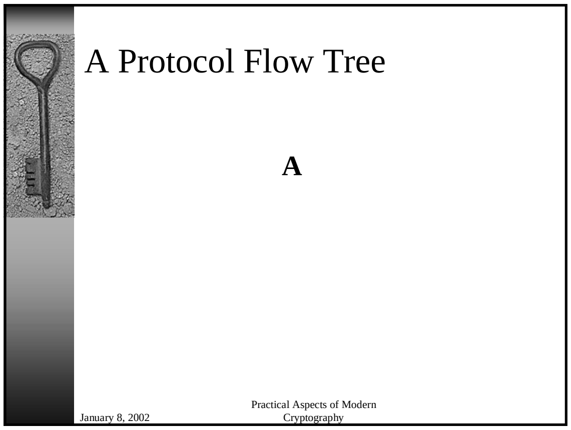
A Protocol Flow Tree

A:

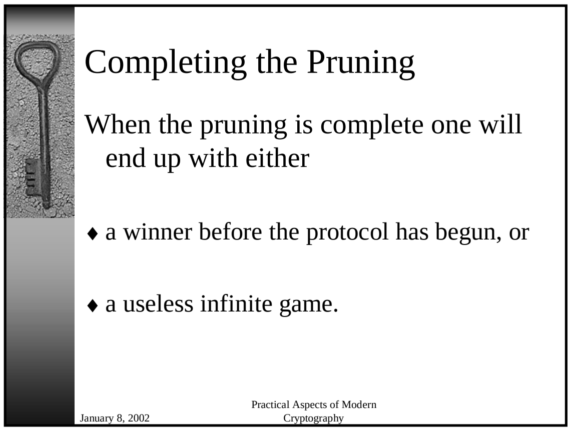
B:

A:

B:



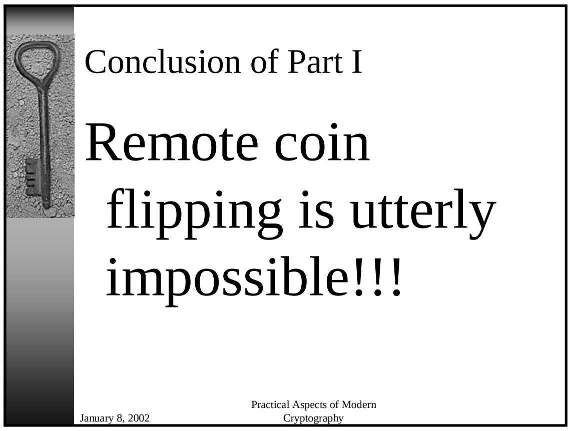
A Protocol Flow Tree



Completing the Pruning

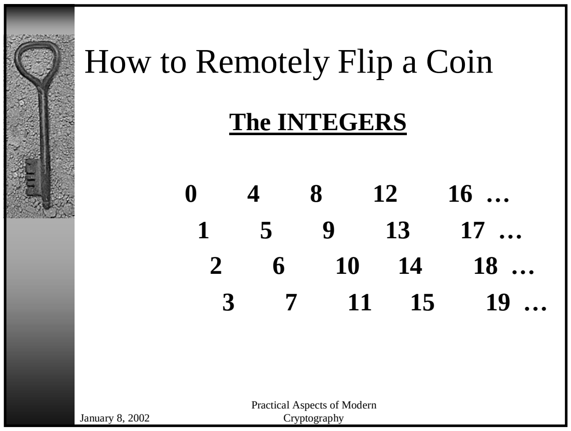
When the pruning is complete one will end up with either

- a winner before the protocol has begun, or
- a useless infinite game.



Conclusion of Part I

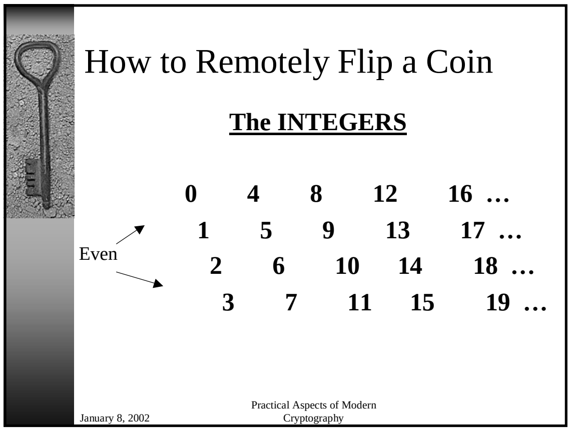
Remote coin flipping is utterly impossible!!!



How to Remotely Flip a Coin

The INTEGERS

0 4 8 12 16 …
1 5 9 13 17 …
2 6 10 14 18 …
3 7 11 15 19 …

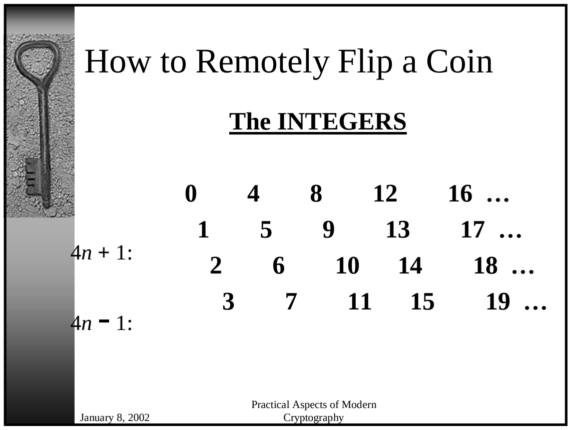


How to Remotely Flip a Coin

The INTEGERS

0 4 8 12 16 …
1 5 9 13 17 …
2 6 10 14 18 …
3 7 11 15 19 …

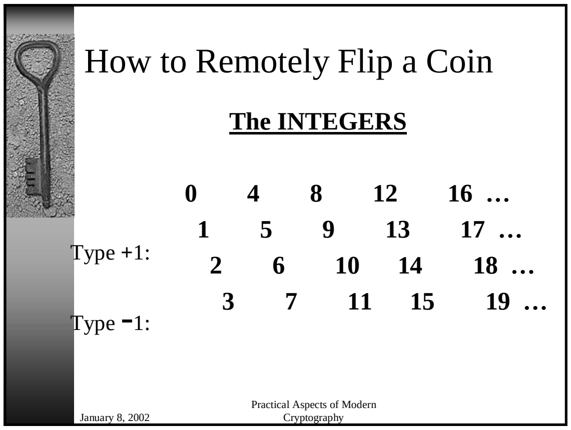
Even



How to Remotely Flip a Coin

The INTEGERS

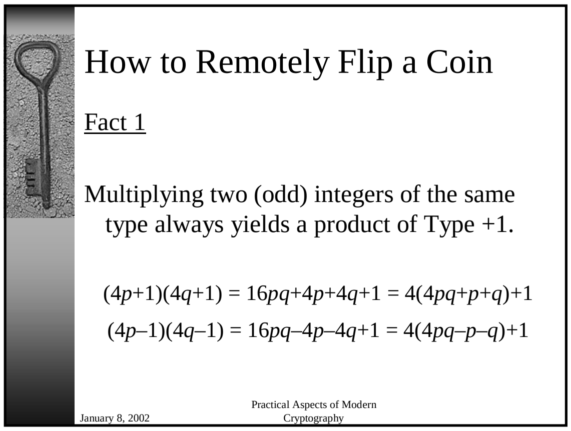
0 4 8 12 16 …
$4n + 1$: **1 5 9 13 17 …**
2 6 10 14 18 …
$4n - 1$: **3 7 11 15 19 …**



How to Remotely Flip a Coin

The INTEGERS

0 4 8 12 16 …
Type +1: **1 5 9 13 17 …**
2 6 10 14 18 …
Type −1: **3 7 11 15 19 …**



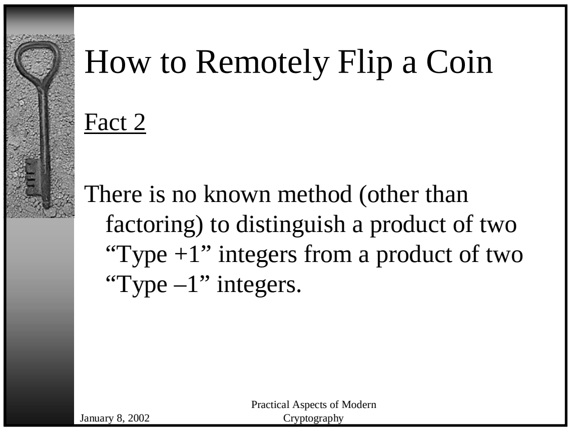
How to Remotely Flip a Coin

Fact 1

Multiplying two (odd) integers of the same type always yields a product of Type +1.

$$(4p+1)(4q+1) = 16pq+4p+4q+1 = 4(4pq+p+q)+1$$

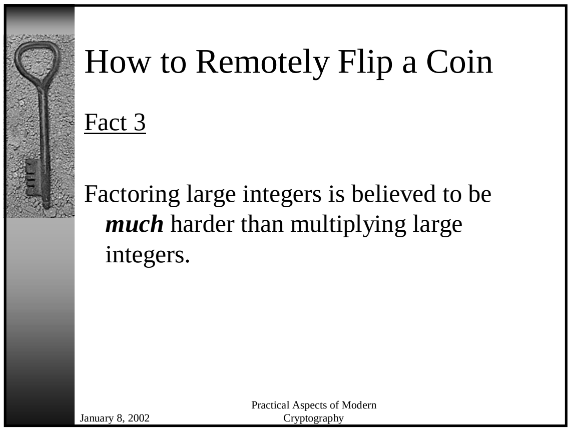
$$(4p–1)(4q–1) = 16pq–4p–4q+1 = 4(4pq–p–q)+1$$



How to Remotely Flip a Coin

Fact 2

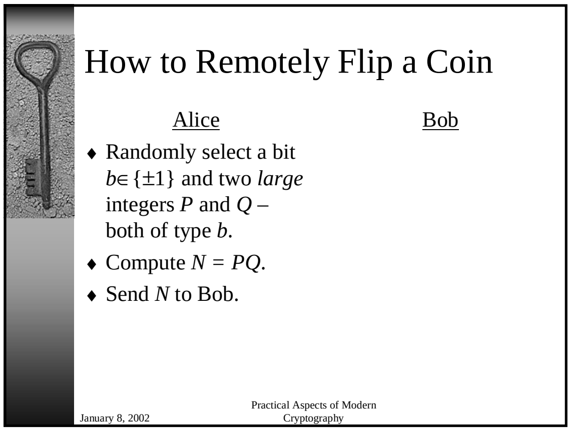
There is no known method (other than factoring) to distinguish a product of two “Type +1” integers from a product of two “Type –1” integers.



How to Remotely Flip a Coin

Fact 3

Factoring large integers is believed to be ***much*** harder than multiplying large integers.

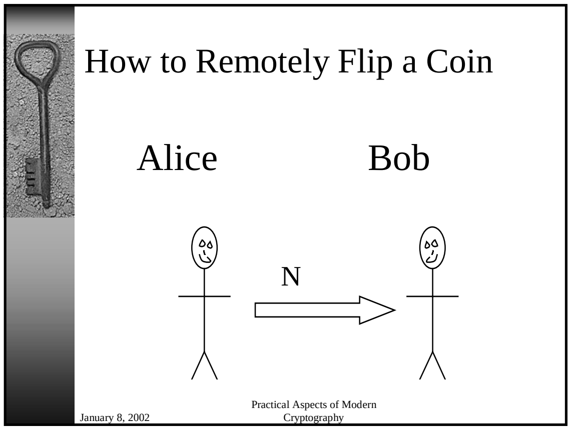


How to Remotely Flip a Coin

Alice

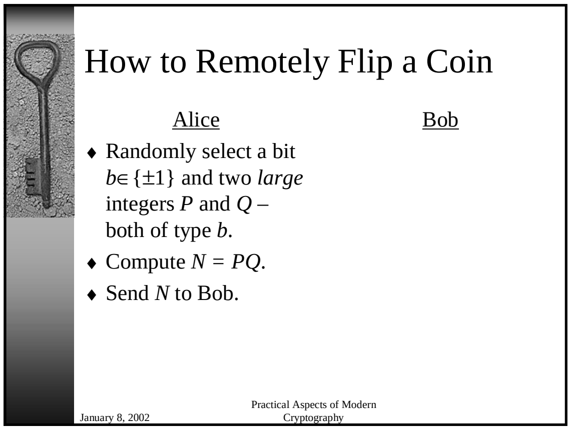
- Randomly select a bit $b \in \{\pm 1\}$ and two *large* integers P and Q – both of type b.
- Compute $N = PQ$.
- Send N to Bob.

Bob



How to Remotely Flip a Coin

Alice Bob

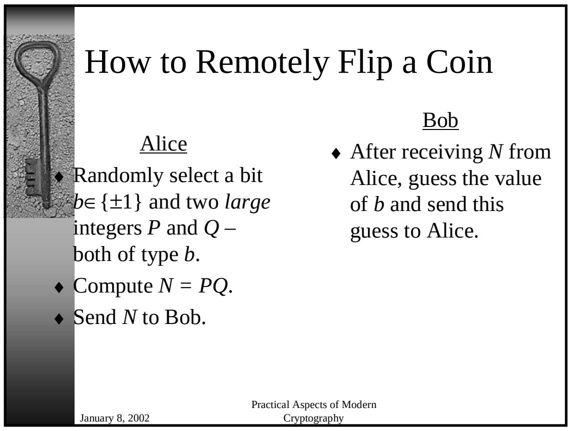


How to Remotely Flip a Coin

Alice

- Randomly select a bit $b \in \{\pm 1\}$ and two *large* integers P and Q – both of type b.
- Compute $N = PQ$.
- Send N to Bob.

Bob



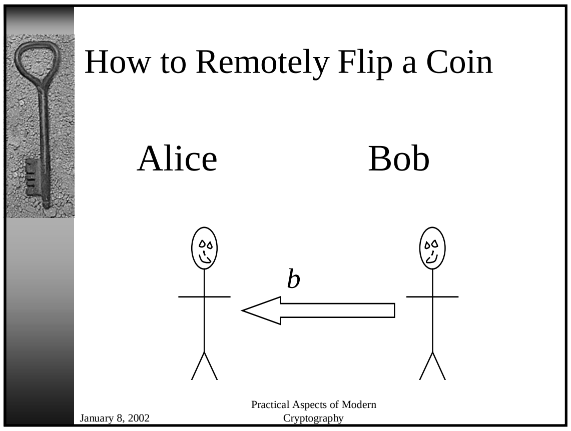
How to Remotely Flip a Coin

Alice

- Randomly select a bit $b \in \{\pm 1\}$ and two *large* integers P and Q – both of type b.
- Compute $N = PQ$.
- Send N to Bob.

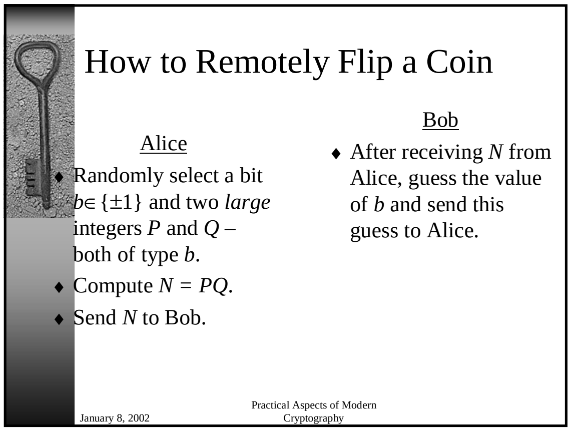
Bob

- After receiving N from Alice, guess the value of b and send this guess to Alice.



How to Remotely Flip a Coin

Alice Bob



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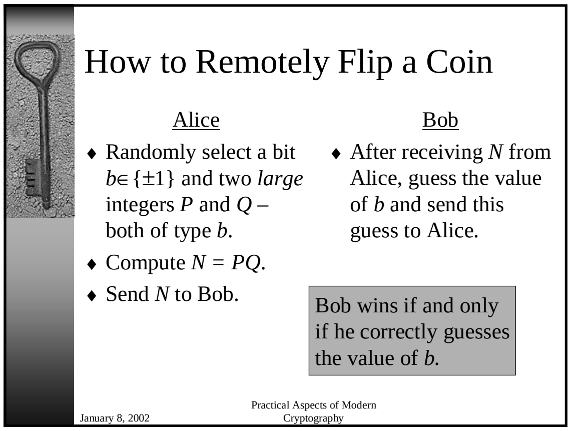


How to Remotely Flip a Coin

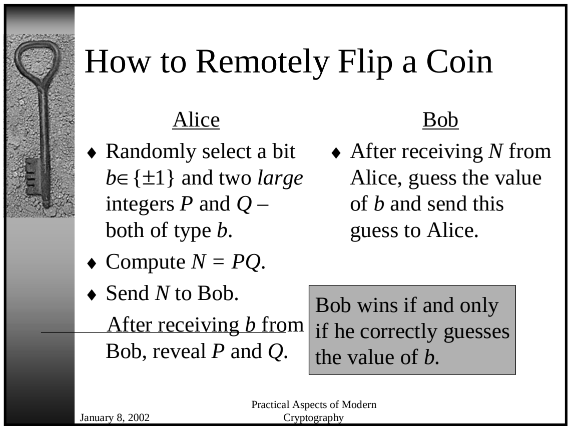
Alice

- Randomly select a bit $b \in \{\pm 1\}$ and two *large* integers P and Q – both of type b.
- Compute $N = PQ$.
- Send N to Bob.

Bob

- After receiving N from Alice, guess the value of b and send this guess to Alice.

Bob wins if and only if he correctly guesses the value of b.



How to Remotely Flip a Coin

Alice

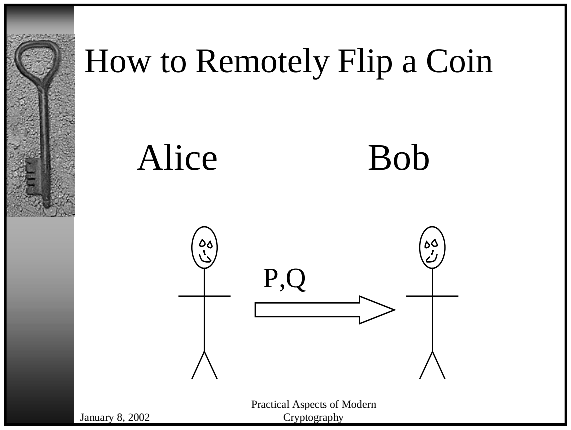
- Randomly select a bit $b \in \{\pm 1\}$ and two *large* integers P and Q – both of type b.
- Compute $N = PQ$.
- Send N to Bob.

After receiving b from Bob, reveal P and Q.

Bob

- After receiving N from Alice, guess the value of b and send this guess to Alice.

Bob wins if and only if he correctly guesses the value of b.

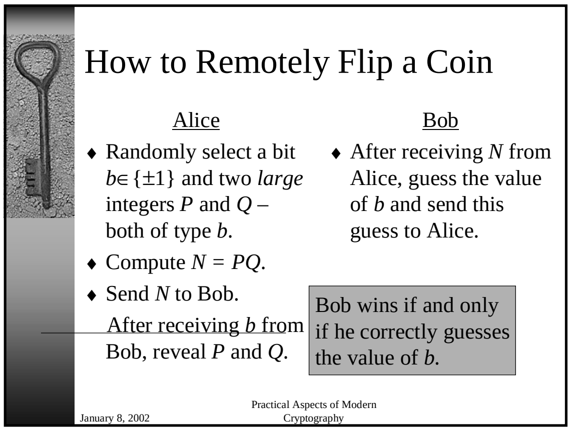


How to Remotely Flip a Coin

Alice

Bob

P,Q



How to Remotely Flip a Coin

Alice

- Randomly select a bit $b \in \{\pm 1\}$ and two *large* integers P and Q – both of type b.
- Compute $N = PQ$.
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After receiving b from Bob, reveal P and Q.

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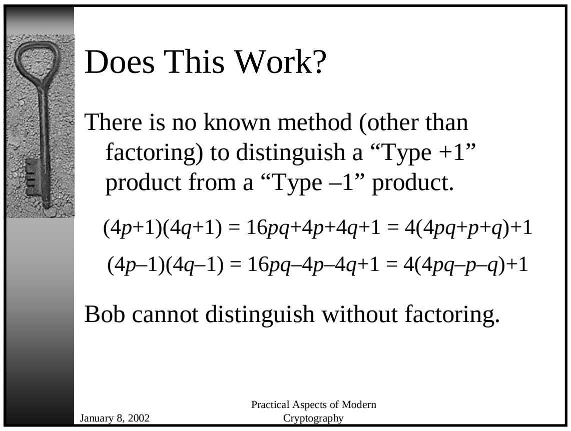
Bob wins if and only if he correctly guesses the value of b.



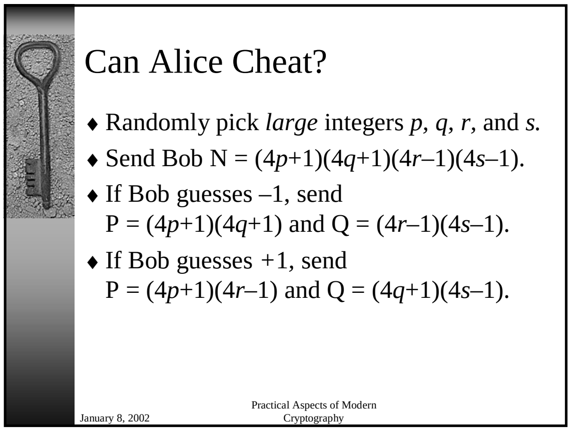
Does This Work?

There is no known method (other than factoring) to distinguish a “Type +1” product from a “Type –1” product.

$$(4p+1)(4q+1) = 16pq+4p+4q+1 = 4(4pq+p+q)+1$$

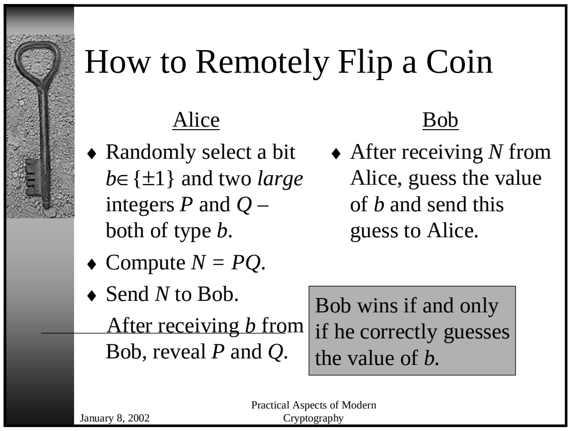
$$(4p–1)(4q–1) = 16pq–4p–4q+1 = 4(4pq–p–q)+1$$

Bob cannot distinguish without factoring.



Can Alice Cheat?

- ♦ Randomly pick *large* integers $p, q, r,$ and s.
- ♦ Send Bob N = $(4p+1)(4q+1)(4r–1)(4s–1)$.
- ♦ If Bob guesses –1, send P = $(4p+1)(4q+1)$ and Q = $(4r–1)(4s–1)$.
- ♦ If Bob guesses +1, send P = $(4p+1)(4r–1)$ and Q = $(4q+1)(4s–1)$.



How to Remotely Flip a Coin

Alice

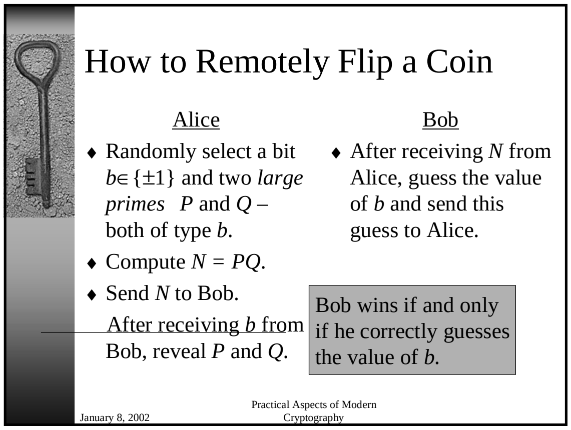
- ♦ Randomly select a bit $b \in \{\pm 1\}$ and two *large* integers P and Q – both of type b.
- ♦ Compute $N = PQ$.
- ♦ Send N to Bob.

After receiving b from Bob, reveal P and Q.

Bob

- ♦ After receiving N from Alice, guess the value of b and send this guess to Alice.

Bob wins if and only if he correctly guesses the value of b.



How to Remotely Flip a Coin

Alice

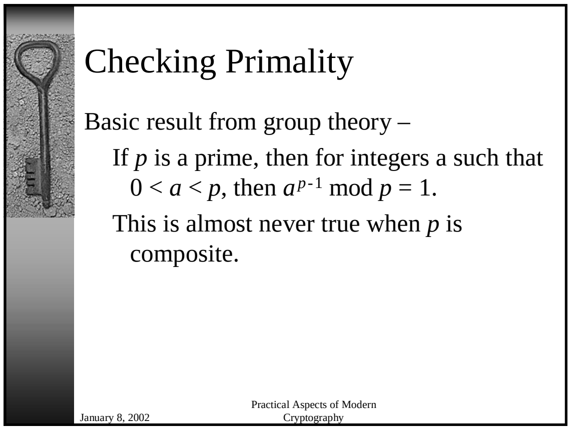
- ♦ Randomly select a bit $b \in \{\pm 1\}$ and two *large primes* P and Q – both of type b.
- ♦ Compute $N = PQ$.
- ♦ Send N to Bob.

After receiving b from Bob, reveal P and Q.

Bob

- ♦ After receiving N from Alice, guess the value of b and send this guess to Alice.

Bob wins if and only if he correctly guesses the value of b.

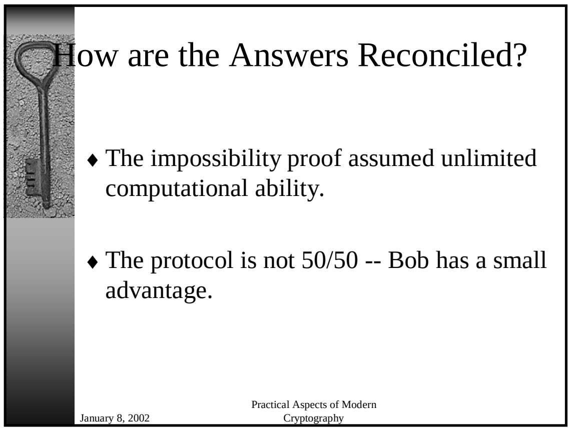


Checking Primality

Basic result from group theory –

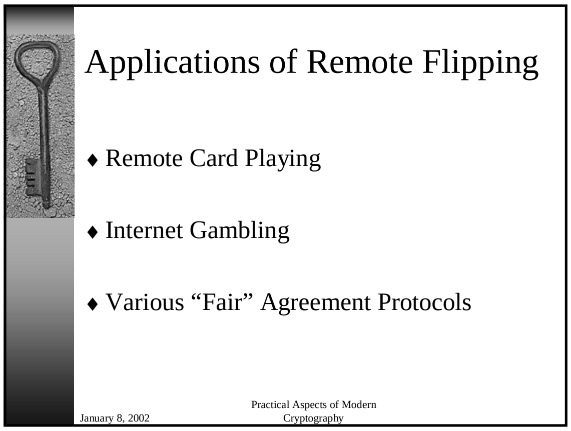
If p is a prime, then for integers a such that $0 < a < p$, then $a^{p-1} \bmod p = 1$.

This is almost never true when p is composite.



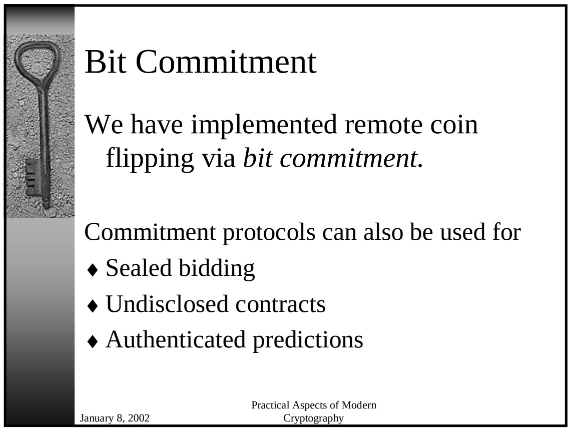
How are the Answers Reconciled?

- ♦ The impossibility proof assumed unlimited computational ability.
- ♦ The protocol is not 50/50 -- Bob has a small advantage.



Applications of Remote Flipping

- ♦ Remote Card Playing
- ♦ Internet Gambling
- ♦ Various “Fair” Agreement Protocols

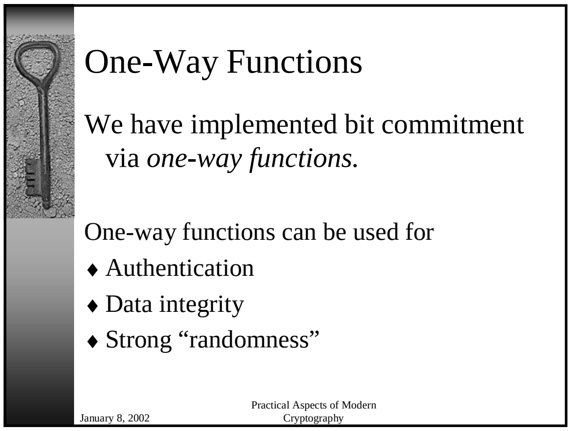


Bit Commitment

We have implemented remote coin flipping via *bit commitment.*

Commitment protocols can also be used for

- ♦ Sealed bidding
- ♦ Undisclosed contracts
- ♦ Authenticated predictions

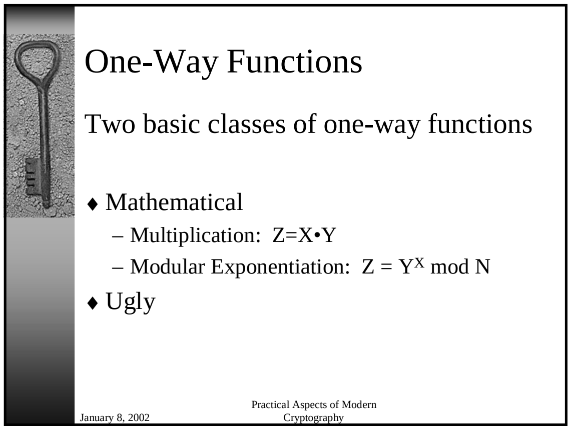


One-Way Functions

We have implemented bit commitment via *one-way functions.*

One-way functions can be used for

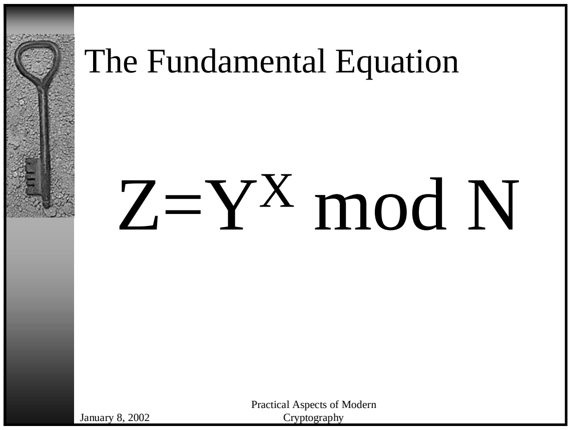
- ♦ Authentication
- ♦ Data integrity
- ♦ Strong “randomness”



One-Way Functions

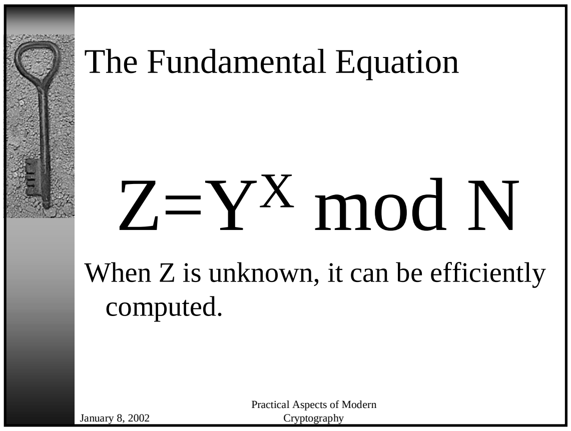
Two basic classes of one-way functions

- ♦ Mathematical
  - – Multiplication: $Z=X\bullet Y$
  - – Modular Exponentiation: $Z = Y^X \bmod N$
- ♦ Ugly



The Fundamental Equation

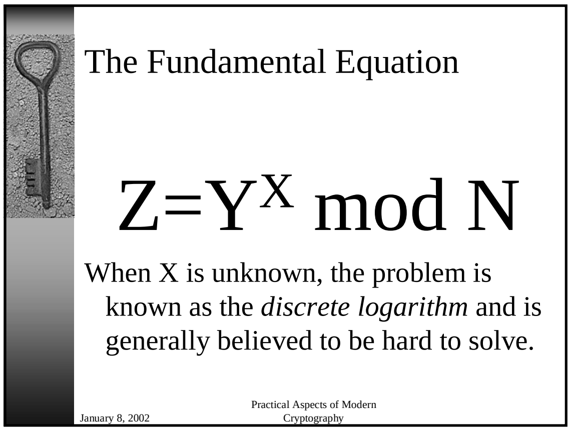
$$Z=Y^X \bmod N$$



The Fundamental Equation

$$Z=Y^X \bmod N$$

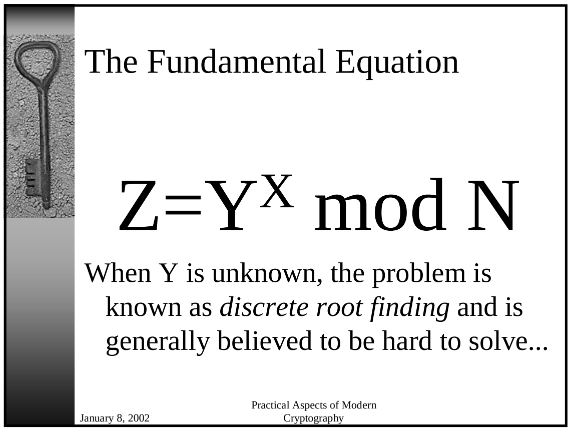
When Z is unknown, it can be efficiently computed.



The Fundamental Equation

$$Z=Y^X \bmod N$$

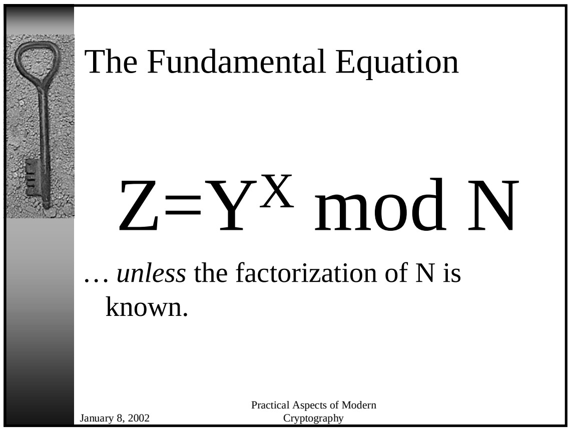
When X is unknown, the problem is known as the *discrete logarithm* and is generally believed to be hard to solve.



The Fundamental Equation

$$Z=Y^X \bmod N$$

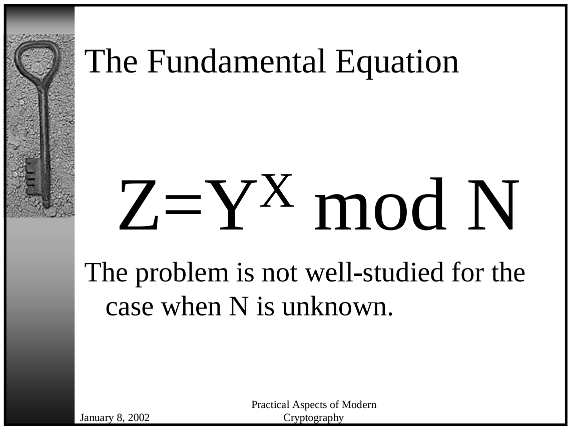
When Y is unknown, the problem is known as *discrete root finding* and is generally believed to be hard to solve...



The Fundamental Equation

$$Z=Y^X \bmod N$$

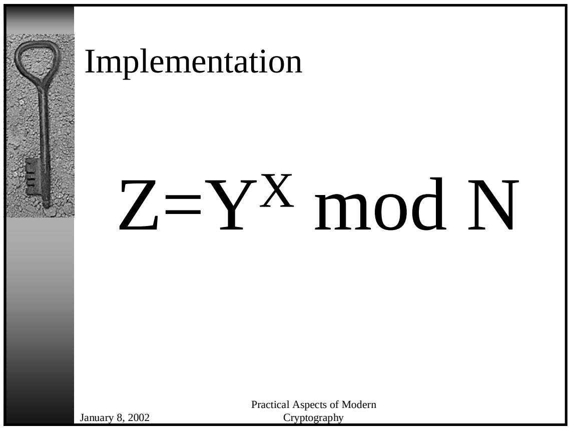
… *unless* the factorization of N is known.



The Fundamental Equation

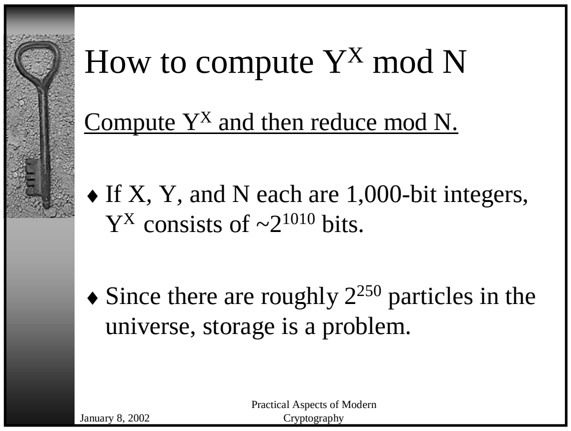
$$Z=Y^X \bmod N$$

The problem is not well-studied for the case when N is unknown.



Implementation

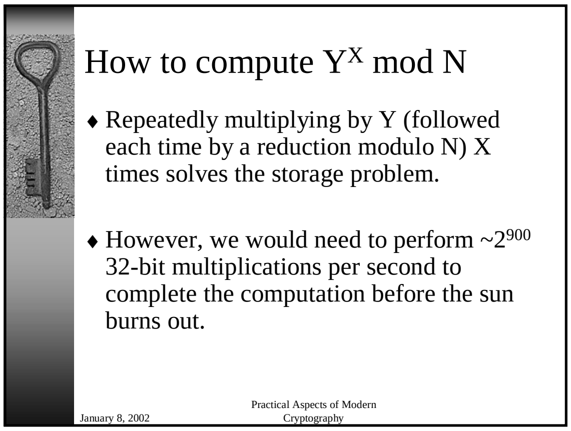
$$Z=Y^X \bmod N$$



How to compute Y^X mod N

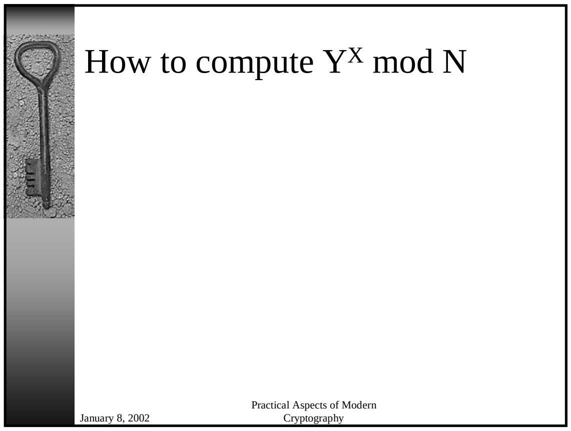
Compute Y^X and then reduce mod N.

- If X, Y, and N each are 1,000-bit integers, Y^X consists of ~2^{1010} bits.
- Since there are roughly 2^{250} particles in the universe, storage is a problem.



How to compute Y^X mod N

- Repeatedly multiplying by Y (followed each time by a reduction modulo N) X times solves the storage problem.
- However, we would need to perform ~2^{900} 32-bit multiplications per second to complete the computation before the sun burns out.



How to compute Y^X mod N

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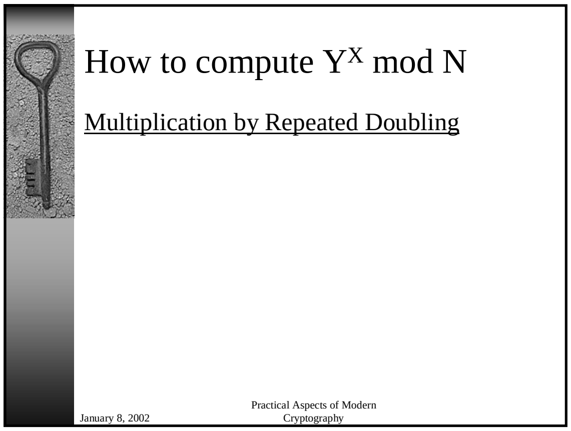

How to compute Y^X mod N

Multiplication by Repeated Doubling

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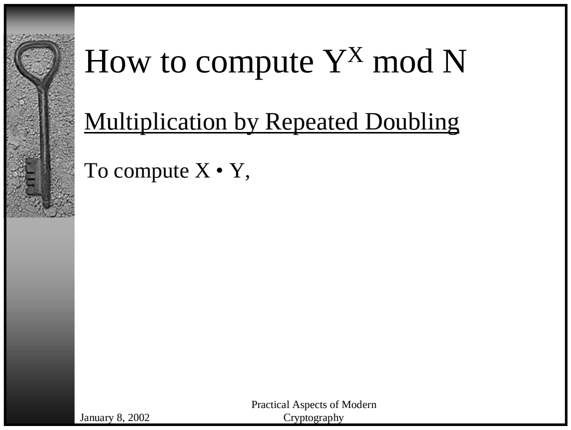

How to compute Y^X mod N

Multiplication by Repeated Doubling

To compute X • Y,

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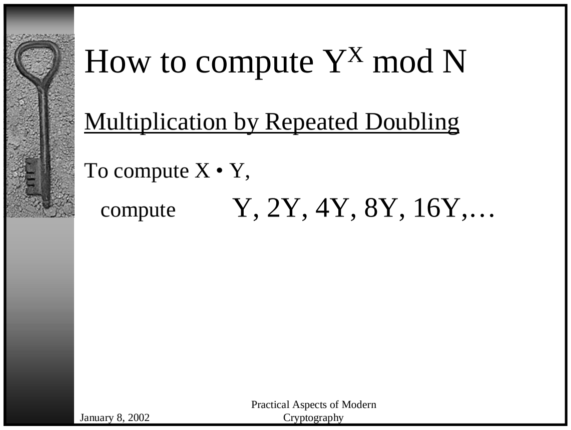

How to compute Y^X mod N

Multiplication by Repeated Doubling

To compute X • Y,

compute Y, 2Y, 4Y, 8Y, 16Y,…

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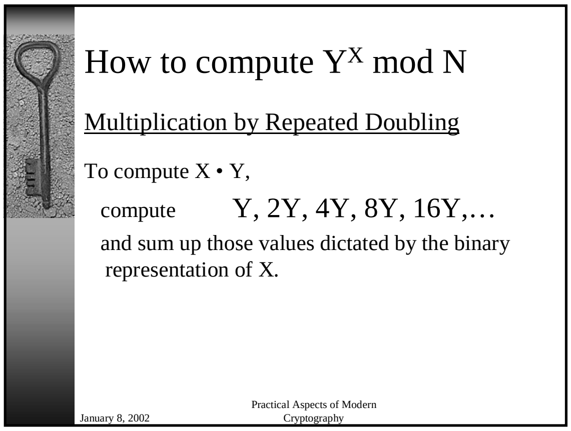

How to compute Y^X mod N

Multiplication by Repeated Doubling

To compute X • Y,

compute Y, 2Y, 4Y, 8Y, 16Y,…

and sum up those values dictated by the binary representation of X.

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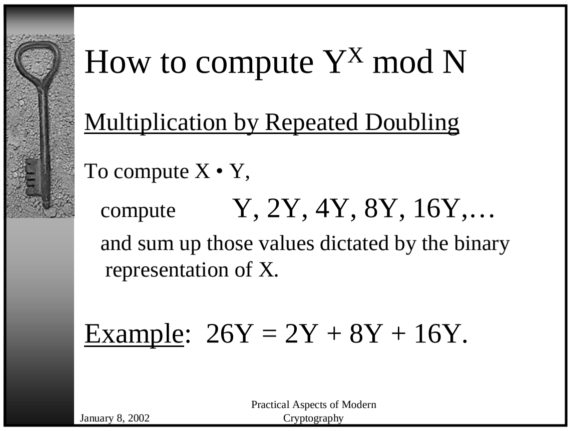

How to compute Y^X mod N

Multiplication by Repeated Doubling

To compute X • Y,

compute Y, 2Y, 4Y, 8Y, 16Y,…

and sum up those values dictated by the binary representation of X.

Example: 26Y = 2Y + 8Y + 16Y.

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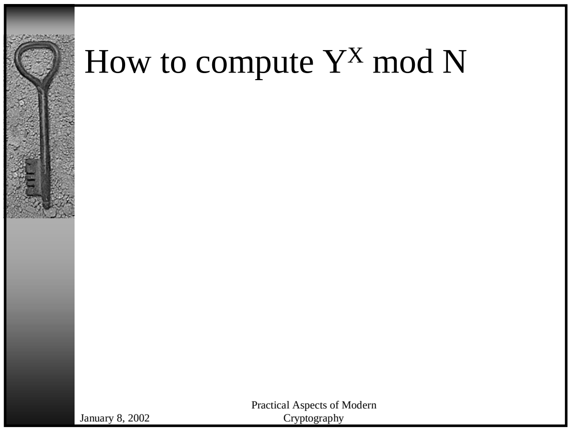
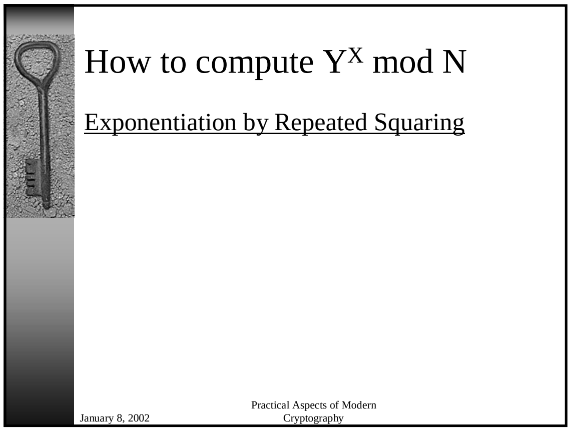
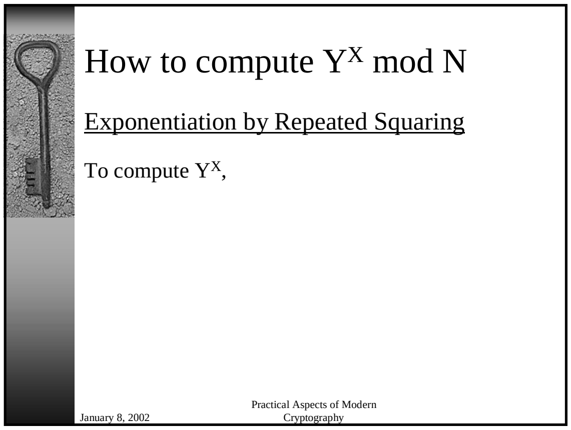


How to compute Y^X mod N



How to compute Y^X mod N

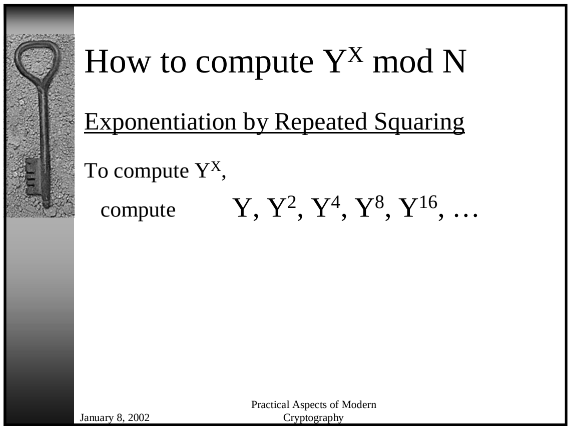
Exponentiation by Repeated Squaring



How to compute Y^X mod N

Exponentiation by Repeated Squaring

To compute Y^X,

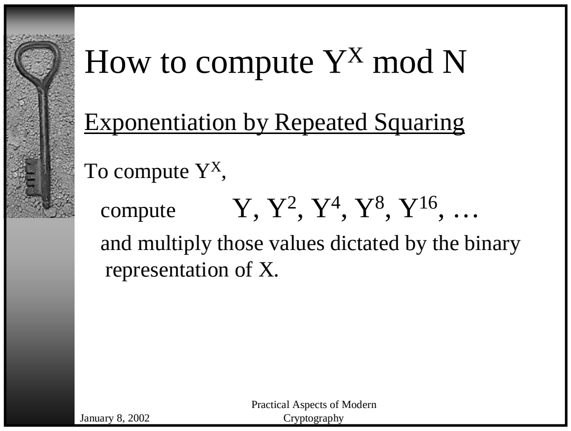


How to compute Y^X mod N

Exponentiation by Repeated Squaring

To compute Y^X,

compute $Y, Y^2, Y^4, Y^8, Y^{16}, \ldots$



How to compute Y^X mod N

Exponentiation by Repeated Squaring

To compute Y^X,

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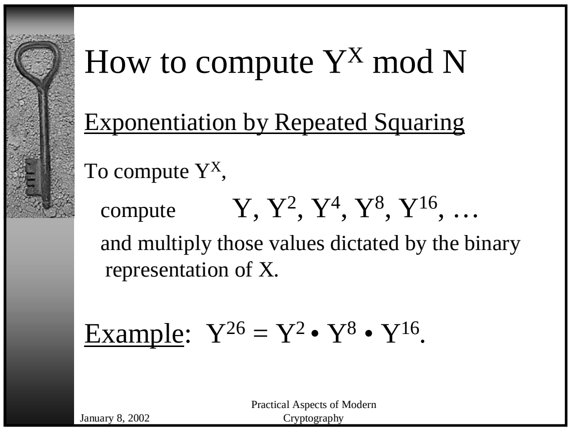
and multiply those values dictated by the binary representation of X.



How to compute Y^X mod N

Exponentiation by Repeated Squaring

To compute Y^X,

compute $Y, Y^2, Y^4, Y^8, Y^{16}, \ldots$

and multiply those values dictated by the binary representation of X.

Example: $Y^{26} = Y^2 \bullet Y^8 \bullet Y^{16}$.

How to compute Y^X mod N

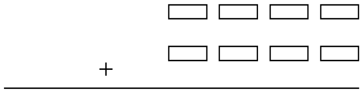
We can now perform a 1,000-bit modular exponentiation using ~1,500 1,000-bit modular multiplications.

- ♦ 1,000 squarings: $y, y^2, y^4, \ldots, y^{2^{1000}}$
- ♦ ~500 “ordinary” multiplications

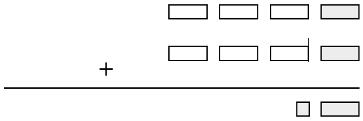
Large-Integer Operations

- ♦ Addition and Subtraction
- ♦ Multiplication
- ♦ Division and Remainder (Mod N)
- ♦ Exponentiation

Large-Integer Addition



Large-Integer Addition



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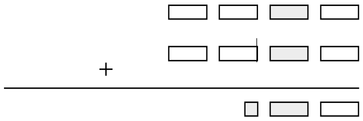

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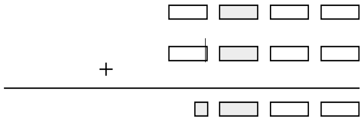

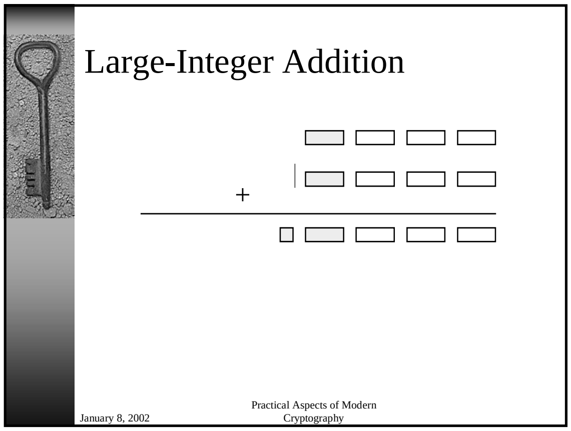

Large-Integer Addition

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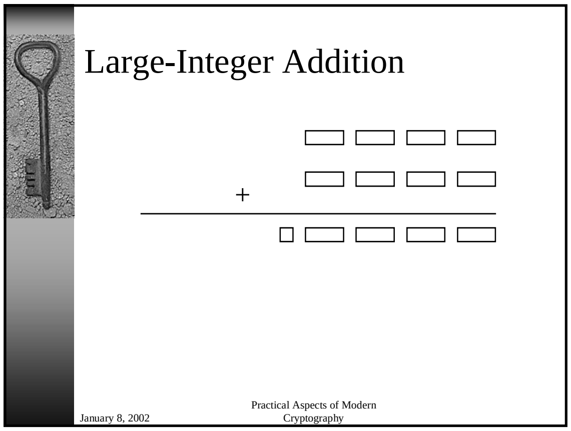

Large-Integer Addition

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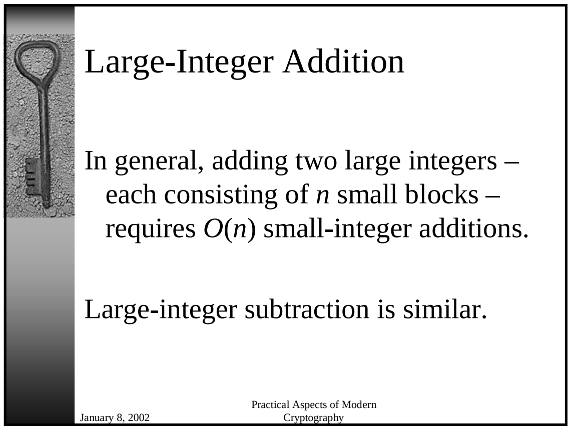

Large-Integer Addition

In general, adding two large integers – each consisting of *n* small blocks – requires *O*(*n*) small-integer additions.

Large-integer subtraction is similar.

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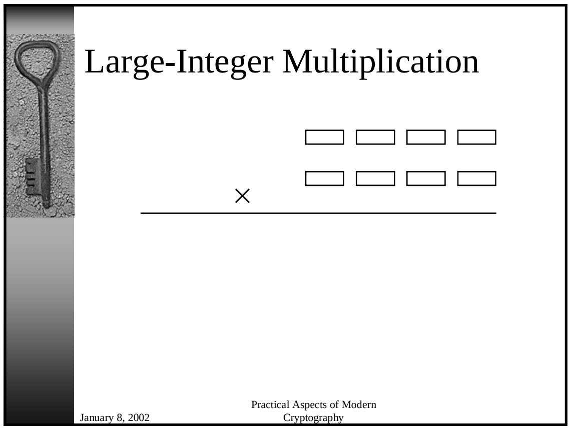

Large-Integer Multiplication

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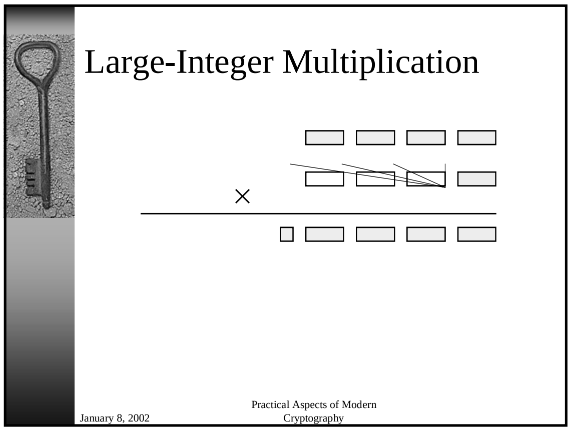

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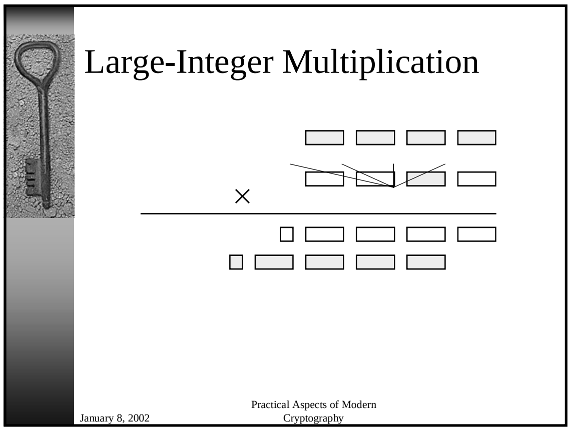

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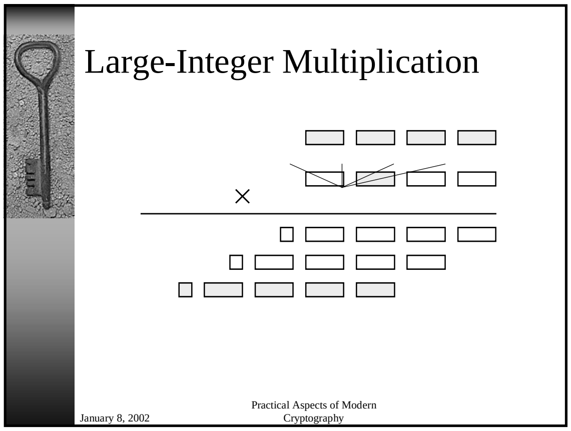

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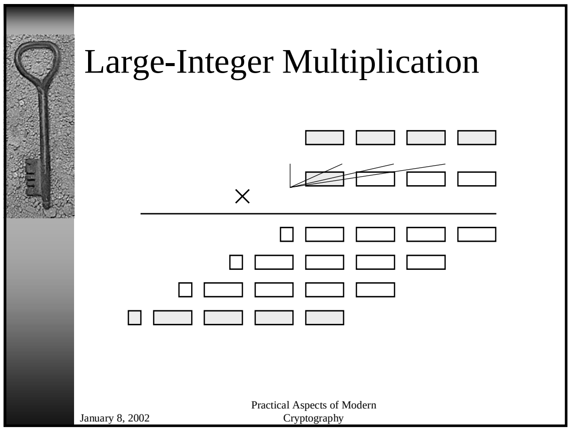

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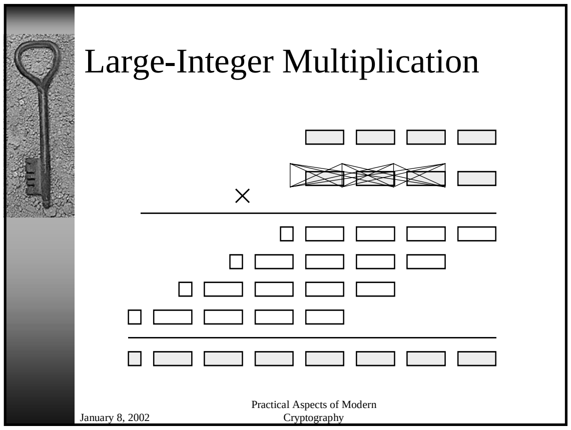

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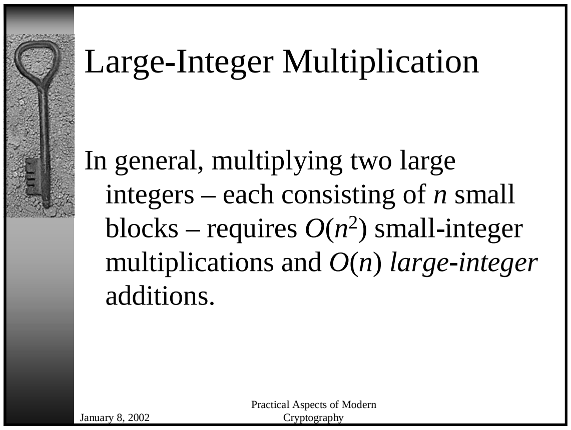
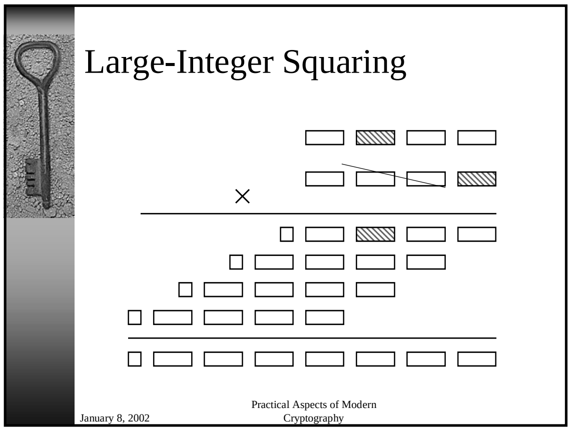
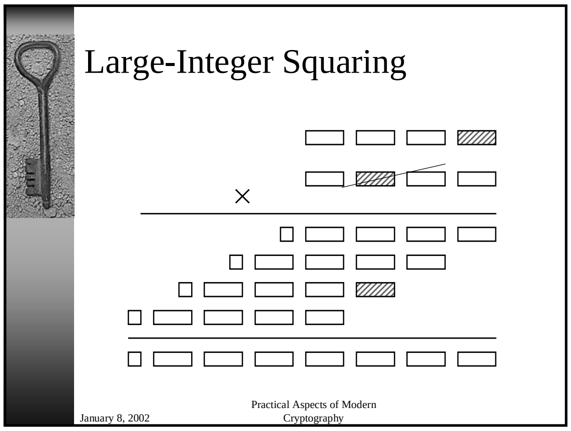


Large-Integer Multiplication

In general, multiplying two large integers – each consisting of *n* small blocks – requires $O(n^2)$ small-integer multiplications and $O(n)$ *large-integer* additions.



Large-Integer Squaring



Large-Integer Squaring

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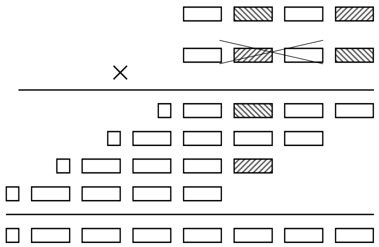

Large-Integer Squaring

Careful bookkeeping can save nearly half of the small-integer multiplications (and nearly half of the time).

Recall computing Y^X mod N

- About 2/3 of the multiplications required to compute Y^X are actually squarings.
- Overall, efficient squaring can save about 1/3 of the small multiplications required for modular exponentiation.

Karatsuba Multiplication

$(Ax+B)(Cx+D) = ACx^2 + (AD+BC)x + BD$

Karatsuba Multiplication

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4 multiplications, 1 addition

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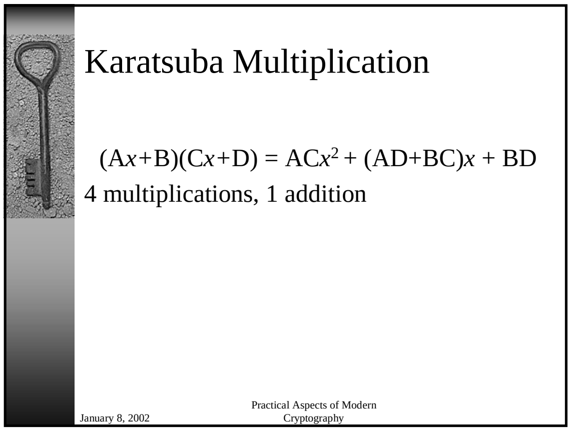
4 multiplications, 1 addition



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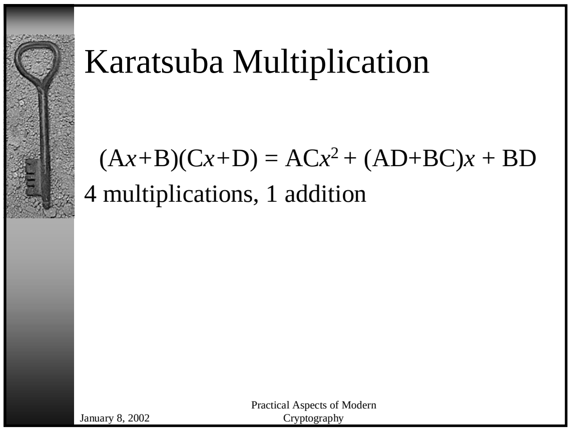

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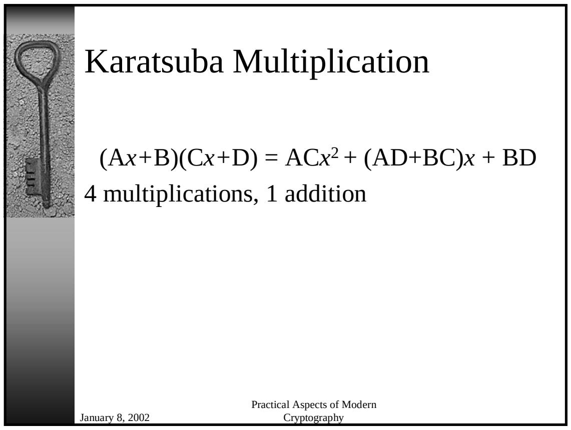

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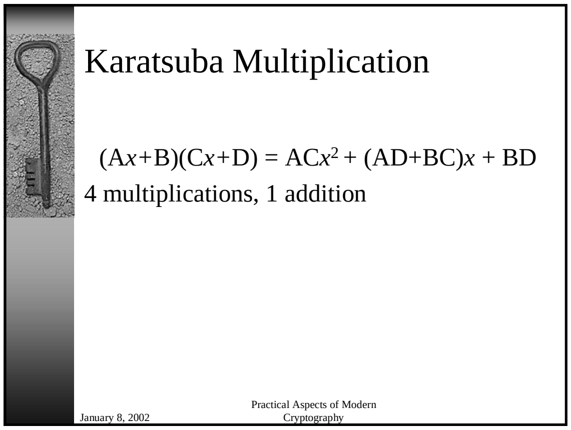

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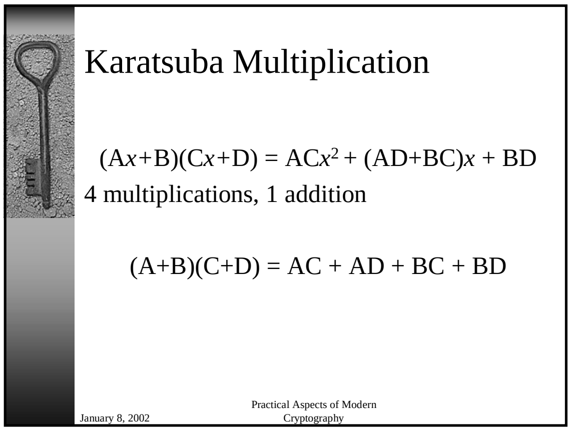
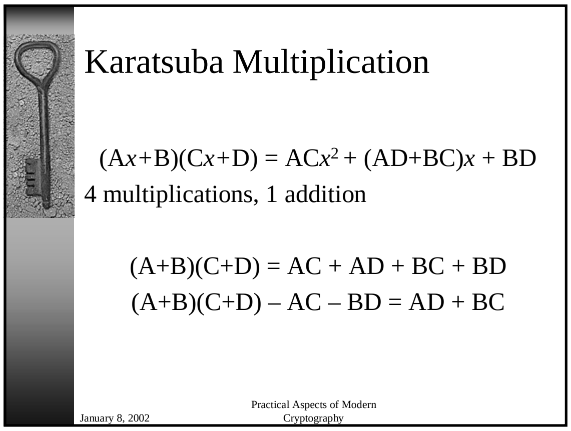


Karatsuba Multiplication

$(Ax+B)(Cx+D) = ACx^2 + (AD+BC)x + BD$

4 multiplications, 1 addition

$$(A+B)(C+D) = AC + AD + BC + BD$$



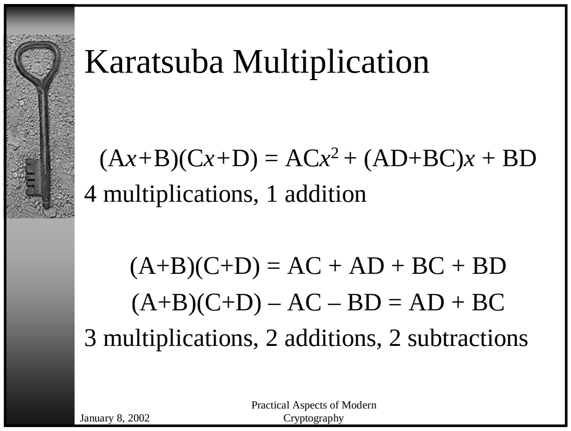
Karatsuba Multiplication

$(Ax+B)(Cx+D) = ACx^2 + (AD+BC)x + BD$

4 multiplications, 1 addition

$$(A+B)(C+D) = AC + AD + BC + BD$$

$$(A+B)(C+D) – AC – BD = AD + BC$$



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4 multiplications, 1 addition

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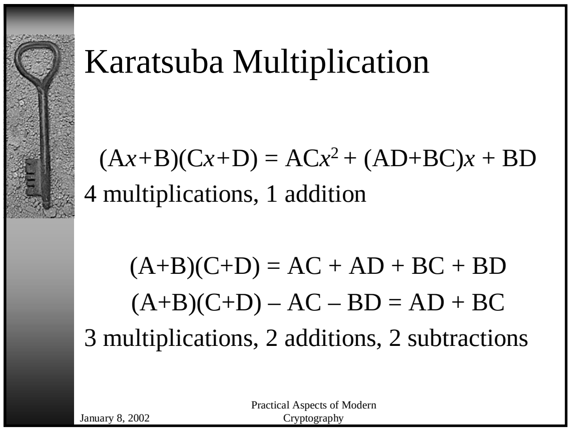
3 multiplications, 2 additions, 2 subtractions



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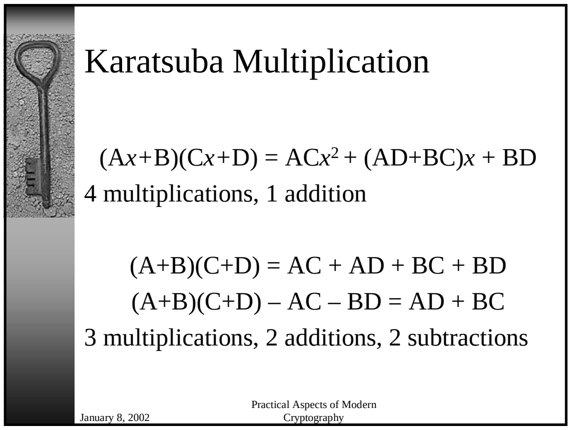

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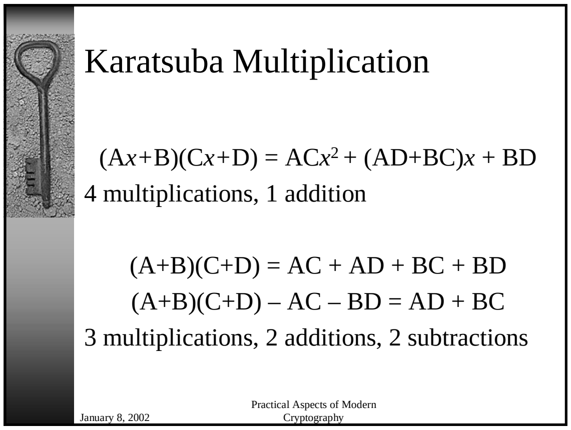

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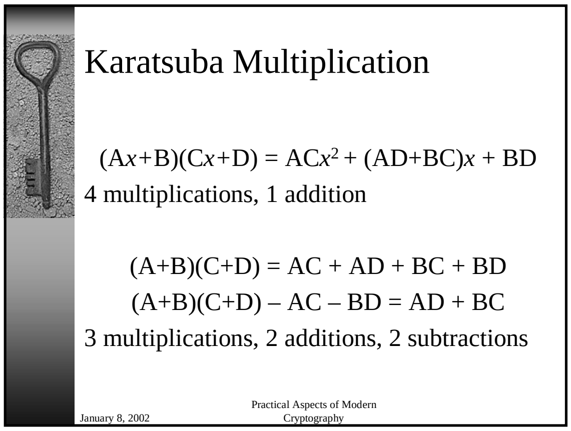

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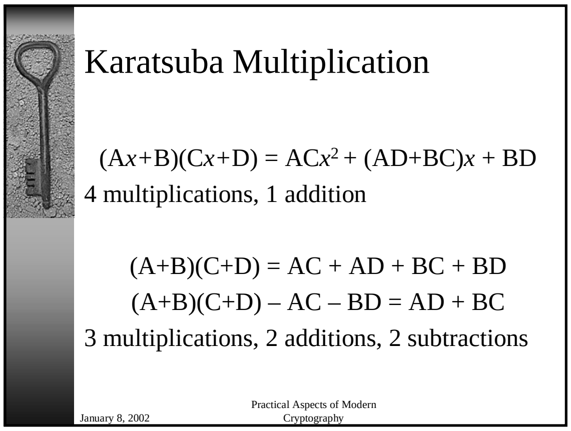

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$(A+B)(C+D) – AC – BD = AD + BC$

3 multiplications, 2 additions, 2 subtractions

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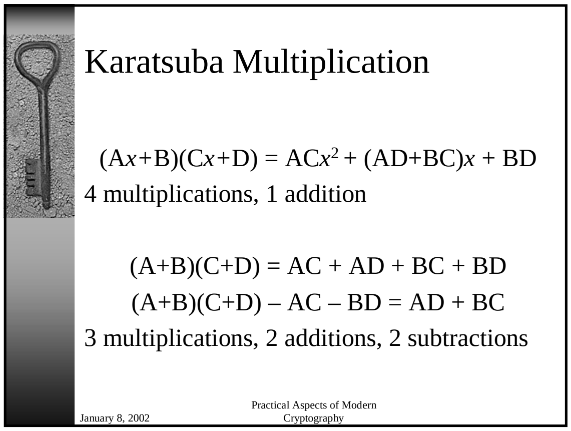

Karatsuba Multiplication

$(Ax+B)(Cx+D) = ACx^2 + (AD+BC)x + BD$

4 multiplications, 1 addition

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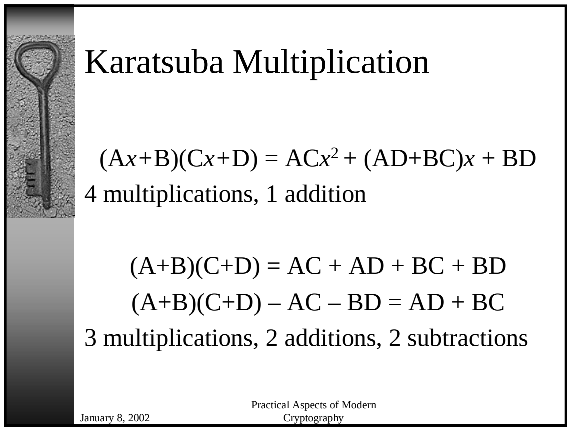

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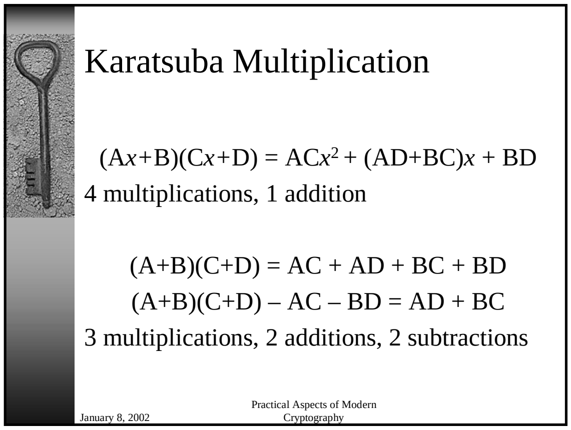

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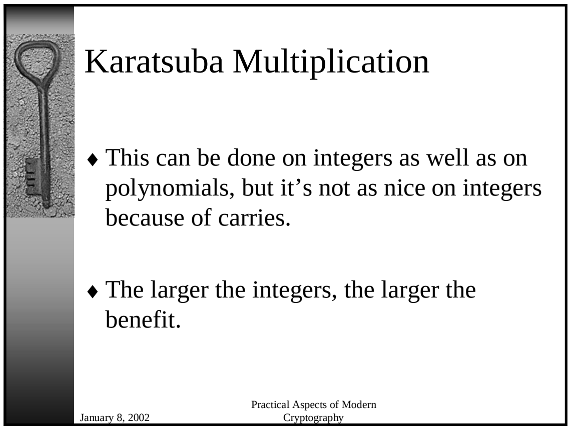
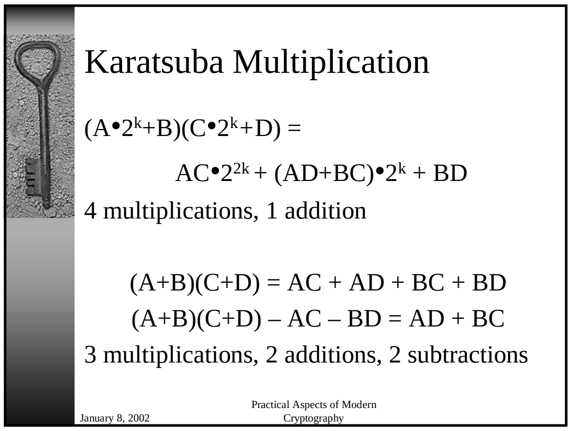


Karatsuba Multiplication

- ♦ This can be done on integers as well as on polynomials, but it’s not as nice on integers because of carries.
- ♦ The larger the integers, the larger the benefit.



Karatsuba Multiplication

$(A\bullet 2^k+B)(C\bullet 2^k+D) =$

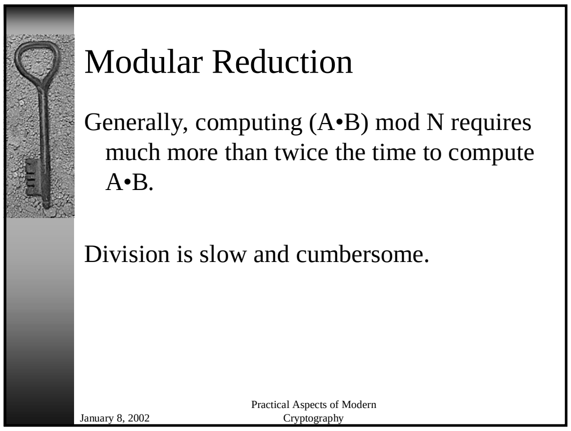
$AC\bullet 2^{2k} + (AD+BC)\bullet 2^k + BD$

4 multiplications, 1 addition

$(A+B)(C+D) = AC + AD + BC + BD$

$(A+B)(C+D) – AC – BD = AD + BC$

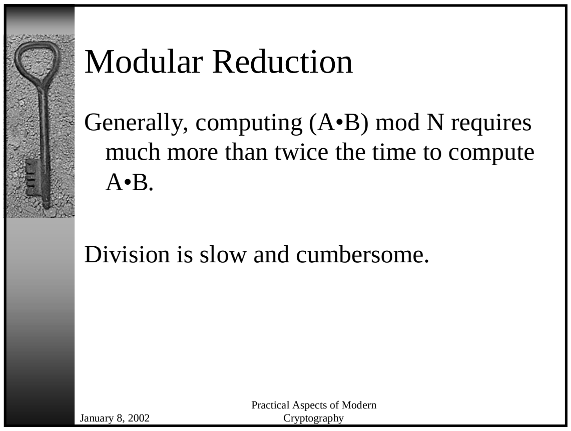
3 multiplications, 2 additions, 2 subtractions



Modular Reduction

Generally, computing (A•B) mod N requires much more than twice the time to compute A•B.

Division is slow and cumbersome.



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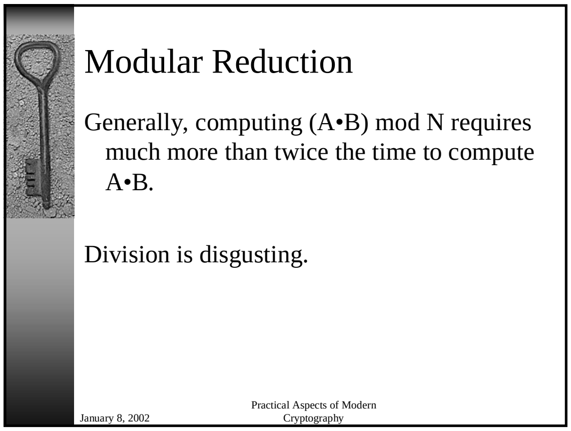

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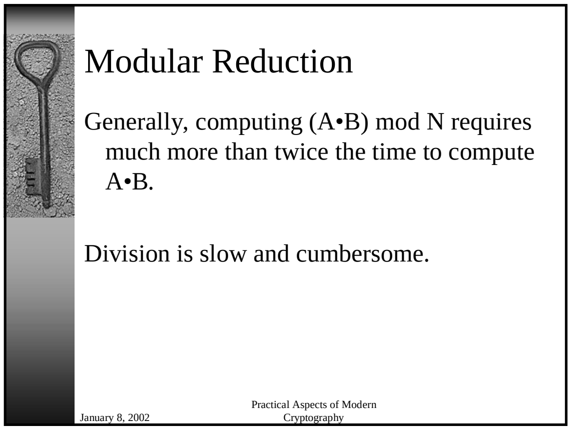

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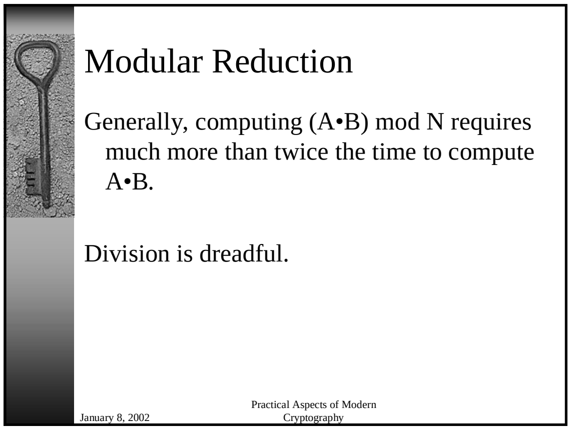

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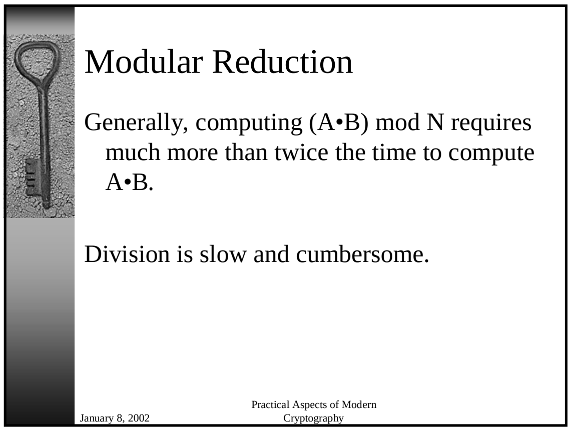

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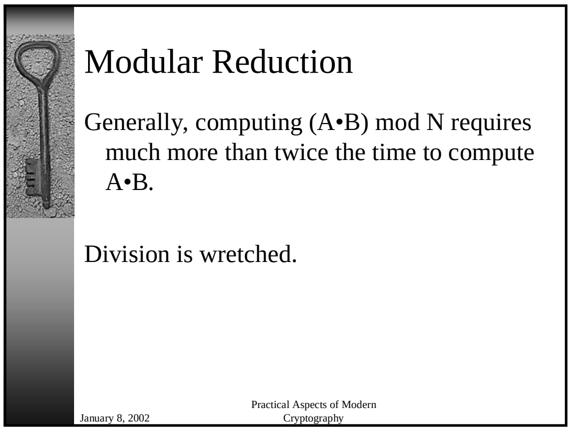

Modular Reduction

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Division is wretched.

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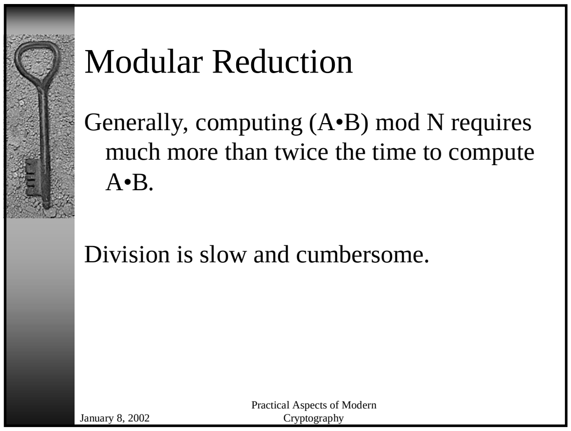

Modular Reduction

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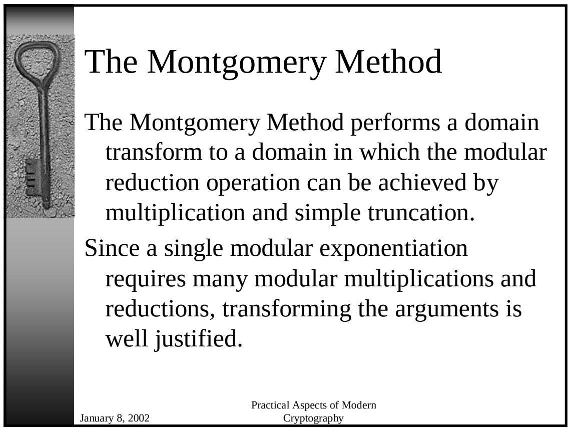
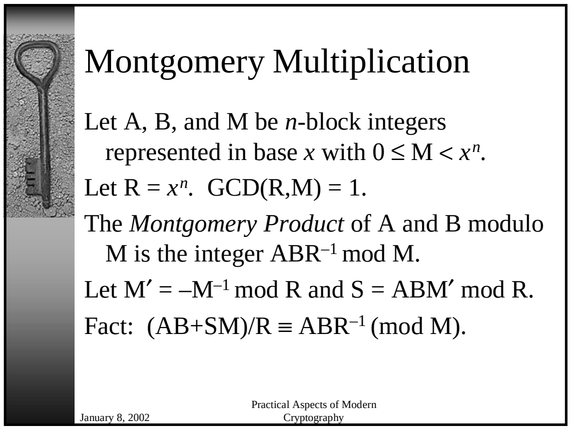


The Montgomery Method

The Montgomery Method performs a domain transform to a domain in which the modular reduction operation can be achieved by multiplication and simple truncation.

Since a single modular exponentiation requires many modular multiplications and reductions, transforming the arguments is well justified.



Montgomery Multiplication

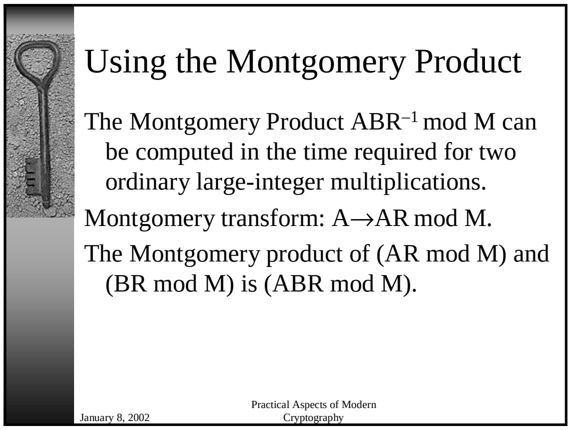
Let A, B, and M be n-block integers represented in base x with $0 \le M < x^n$.

Let $R = x^n$. GCD(R,M) = 1.

The *Montgomery Product* of A and B modulo M is the integer ABR^{-1} mod M.

Let $M' = -M^{-1}$ mod R and $S = ABM'$ mod R.

Fact: $(AB+SM)/R \equiv ABR^{-1}$ (mod M).

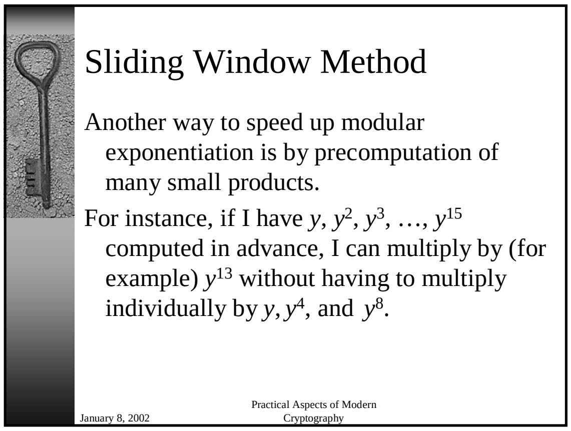


Using the Montgomery Product

The Montgomery Product ABR^{-1} mod M can be computed in the time required for two ordinary large-integer multiplications.

Montgomery transform: $A \rightarrow AR$ mod M.

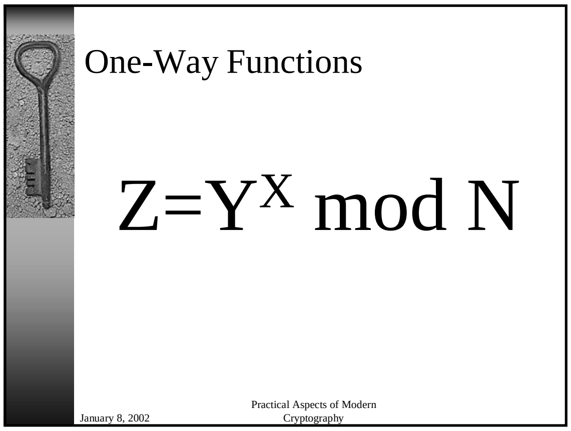
The Montgomery product of (AR mod M) and (BR mod M) is (ABR mod M).



Sliding Window Method

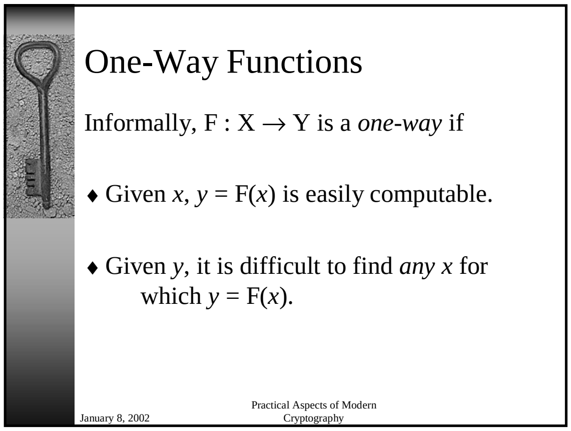
Another way to speed up modular exponentiation is by precomputation of many small products.

For instance, if I have y, y^2, y^3, …, y^{15} computed in advance, I can multiply by (for example) y^{13} without having to multiply individually by y, y^4, and y^8.



One-Way Functions

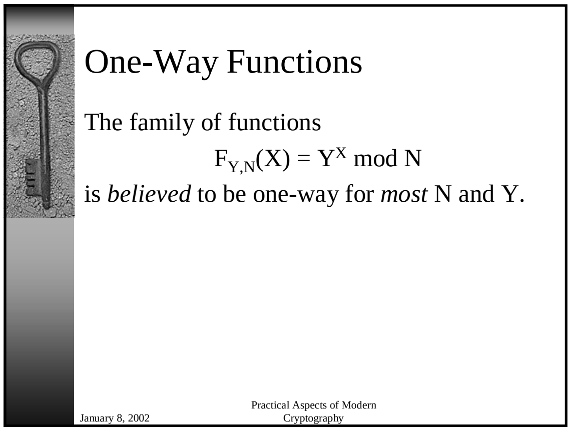
$$Z=Y^X \bmod N$$



One-Way Functions

Informally, $F : X \rightarrow Y$ is a *one-way* if

- Given x, $y = F(x)$ is easily computable.
- Given y, it is difficult to find *any* x for which $y = F(x)$.

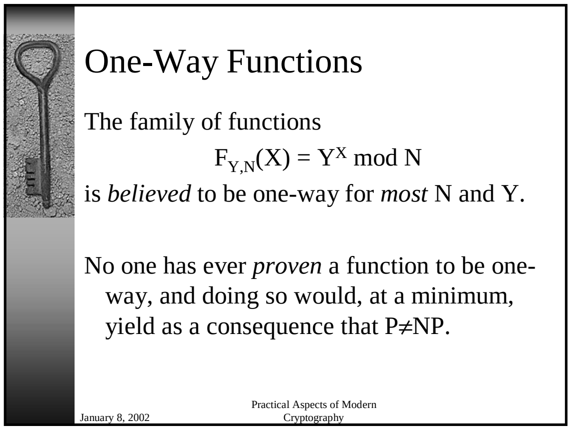


One-Way Functions

The family of functions

$$F_{Y,N}(X) = Y^X \bmod N$$

is *believed* to be one-way for *most* N and Y.



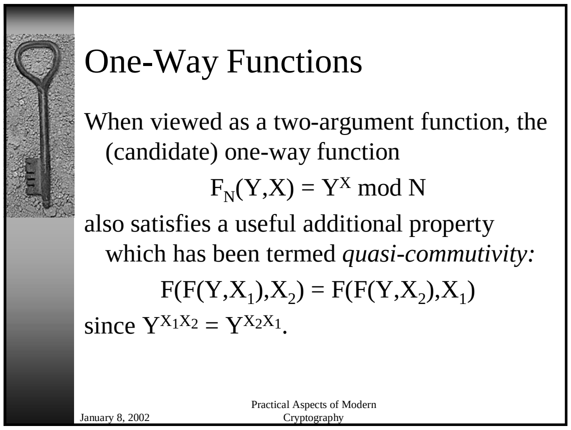
One-Way Functions

The family of functions

$$F_{Y,N}(X) = Y^X \bmod N$$

is *believed* to be one-way for *most* N and Y.

No one has ever *proven* a function to be one-way, and doing so would, at a minimum, yield as a consequence that P≠NP.



One-Way Functions

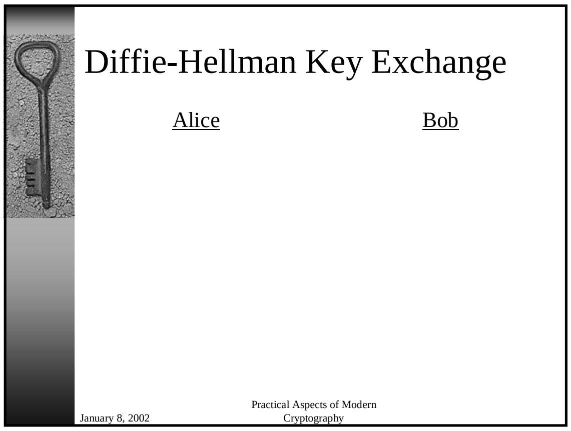
When viewed as a two-argument function, the (candidate) one-way function

$$F_N(Y,X) = Y^X \bmod N$$

also satisfies a useful additional property which has been termed *quasi-commutivity:*

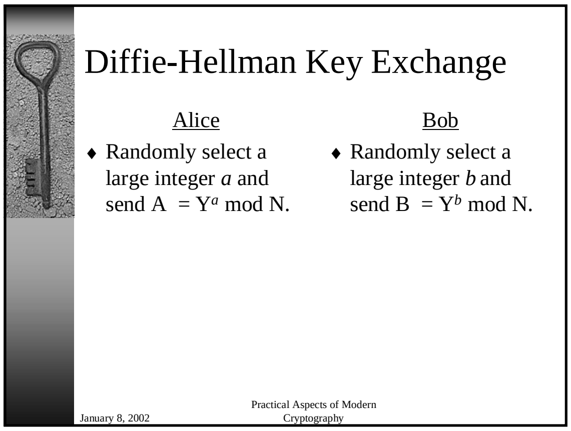
$$F(F(Y,X_1),X_2) = F(F(Y,X_2),X_1)$$

since $Y^{X_1X_2} = Y^{X_2X_1}$.



Diffie-Hellman Key Exchange

Alice	Bob

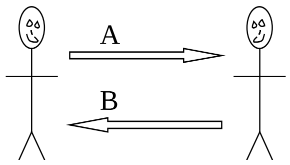


Diffie-Hellman Key Exchange

Alice	Bob
♦ Randomly select a large integer a and send A $= Y^a \bmod N$.	♦ Randomly select a large integer b and send B $= Y^b \bmod N$.

Diffie-Hellman Key Exchange

Alice Bob



Diffie-Hellman Key Exchange

Alice

- Randomly select a large integer *a* and send A = Y^a mod N.

Bob

- Randomly select a large integer *b* and send B = Y^b mod N.

Diffie-Hellman Key Exchange

Alice

- Randomly select a large integer *a* and send A = Y^a mod N.
- Compute the key K = B^a mod N.

Bob

- Randomly select a large integer *b* and send B = Y^b mod N.
- Compute the key K = A^b mod N.

Diffie-Hellman Key Exchange

Alice

- Randomly select a large integer *a* and send A = Y^a mod N.
- Compute the key K = B^a mod N.

Bob

- Randomly select a large integer *b* and send B = Y^b mod N.
- Compute the key K = A^b mod N.

$$B^a = Y^{ba} = Y^{ab} = A^b$$

Diffie-Hellman Key Exchange

What does Eve see?

$$Y, Y^a, Y^b$$

… but the exchanged key is Y^{ab}.

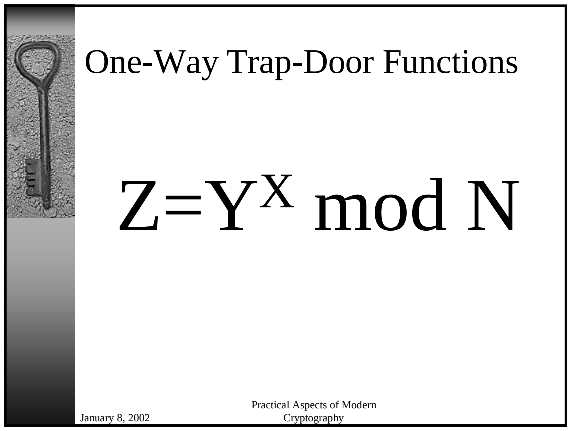
Belief: Given Y, Y^a, Y^b it is difficult to compute Y^{ab}.

Contrast with discrete logarithm assumption: Given Y, Y^a it is difficult to compute *a* .

More on *Quasi-Commutivity*

Quasi-commutivity has additional applications.

- decentralized digital signatures
- membership testing
- digital time-stamping



One-Way Trap-Door Functions

$$Z=Y^X \bmod N$$

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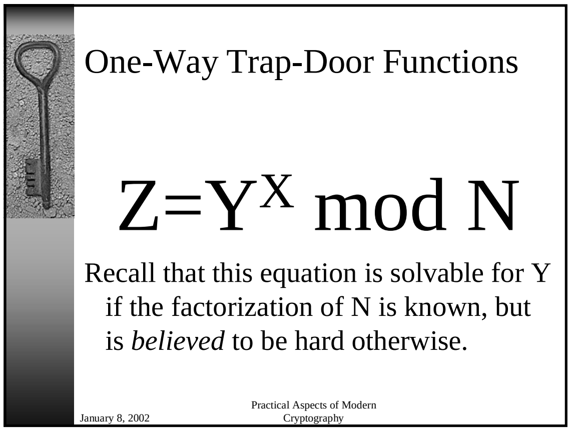

One-Way Trap-Door Functions

$$Z=Y^X \bmod N$$

Recall that this equation is solvable for Y if the factorization of N is known, but is *believed* to be hard otherwise.

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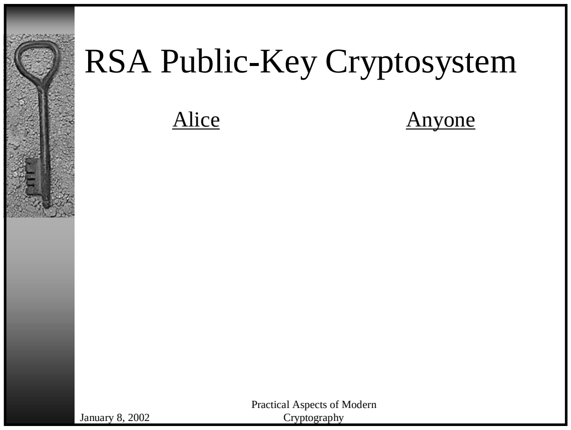

RSA Public-Key Cryptosystem

Alice

Anyone

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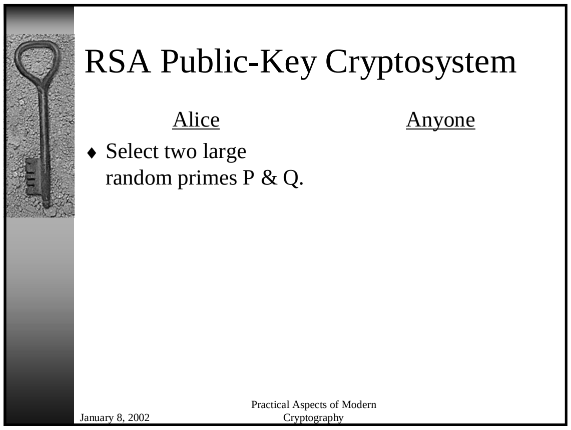

RSA Public-Key Cryptosystem

Alice

- Select two large random primes P & Q.

Anyone

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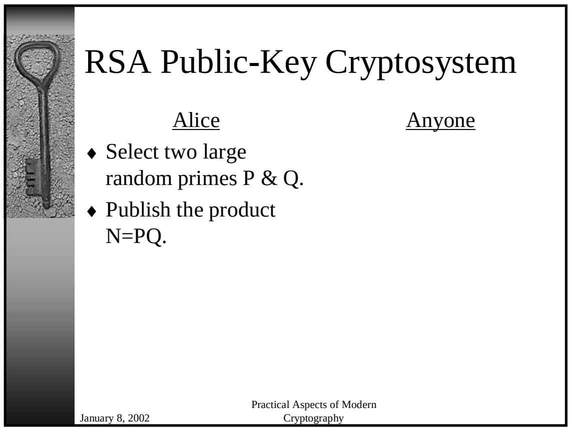

RSA Public-Key Cryptosystem

Alice

- Select two large random primes P & Q.
- Publish the product N=PQ.

Anyone

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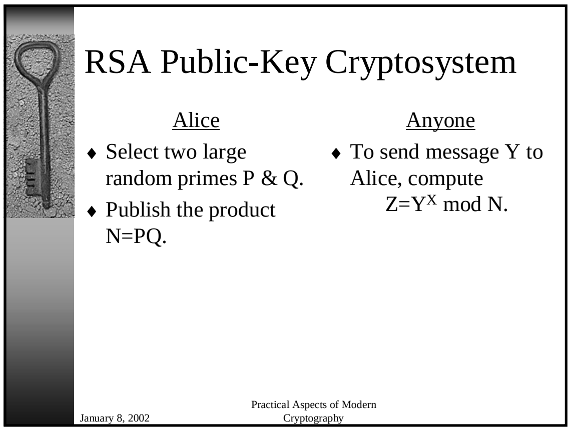

RSA Public-Key Cryptosystem

Alice

- Select two large random primes P & Q.
- Publish the product N=PQ.

Anyone

- To send message Y to Alice, compute
  $Z=Y^X$ mod N.

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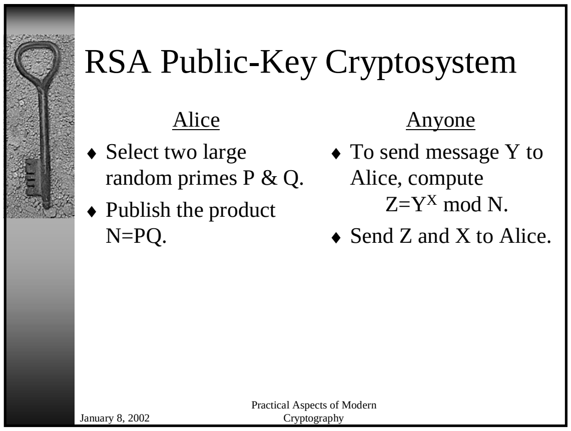


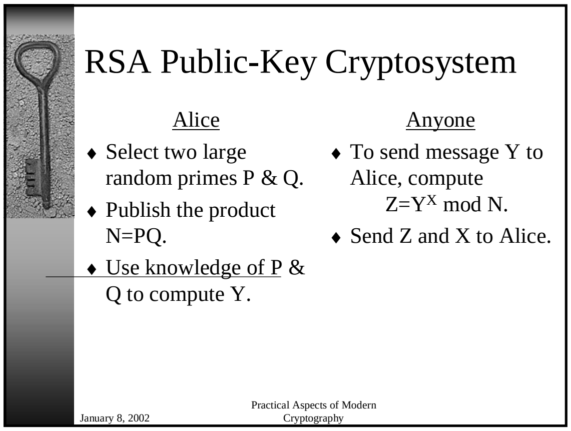
RSA Public-Key Cryptosystem

Alice

- ♦ Select two large random primes P & Q.
- ♦ Publish the product N=PQ.

Anyone

- ♦ To send message Y to Alice, compute $Z=Y^X \bmod N$.
- ♦ Send Z and X to Alice.



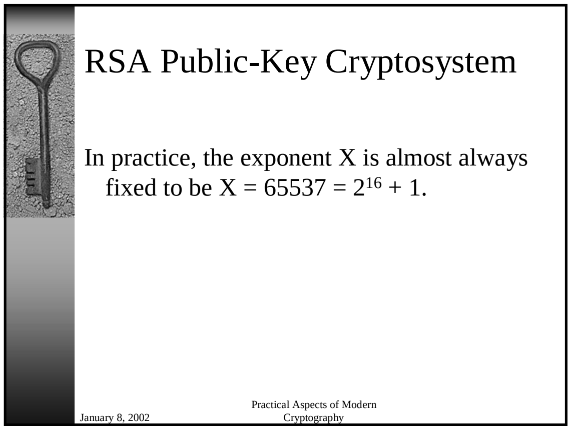
RSA Public-Key Cryptosystem

Alice

- ♦ Select two large random primes P & Q.
- ♦ Publish the product N=PQ.
- ♦ Use knowledge of P & Q to compute Y.

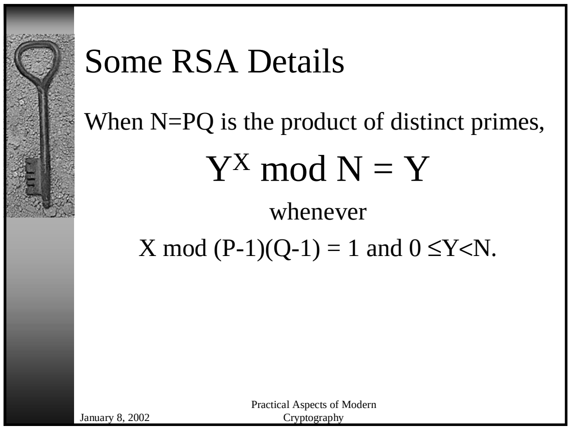
Anyone

- ♦ To send message Y to Alice, compute $Z=Y^X \bmod N$.
- ♦ Send Z and X to Alice.



RSA Public-Key Cryptosystem

In practice, the exponent X is almost always fixed to be $X = 65537 = 2^{16} + 1$.



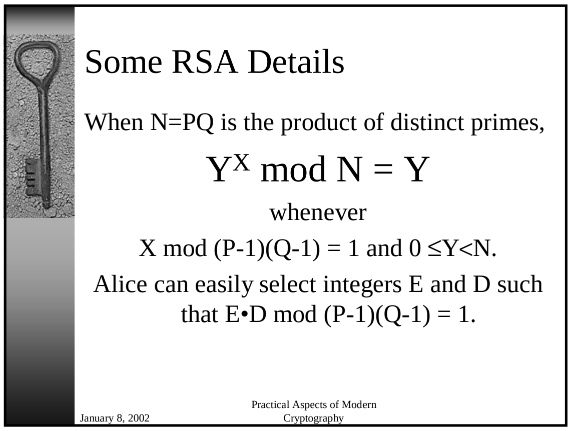
Some RSA Details

When N=PQ is the product of distinct primes,

$$Y^X \bmod N = Y$$

whenever

$X \bmod (P-1)(Q-1) = 1$ and $0 \le Y<N$.



Some RSA Details

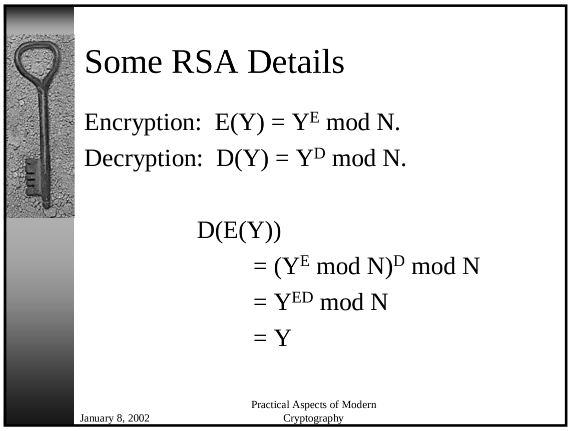
When N=PQ is the product of distinct primes,

$$Y^X \bmod N = Y$$

whenever

$X \bmod (P-1)(Q-1) = 1$ and $0 \le Y<N$.

Alice can easily select integers E and D such that $E•D \bmod (P-1)(Q-1) = 1$.

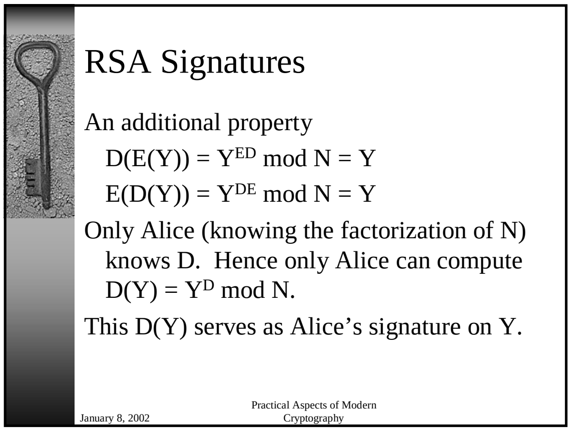


Some RSA Details

Encryption: $E(Y) = Y^E \bmod N$.

Decryption: $D(Y) = Y^D \bmod N$.

$$\begin{aligned} &D(E(Y)) \\ &\quad = (Y^E \bmod N)^D \bmod N \\ &\quad = Y^{ED} \bmod N \\ &\quad = Y \end{aligned}$$



RSA Signatures

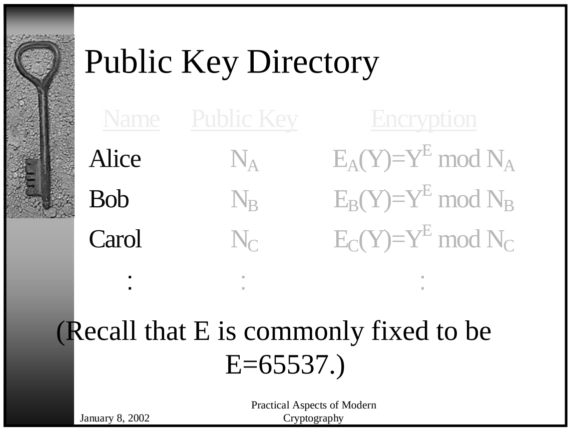
An additional property

$D(E(Y)) = Y^{ED} \bmod N = Y$

$E(D(Y)) = Y^{DE} \bmod N = Y$

Only Alice (knowing the factorization of N) knows D. Hence only Alice can compute $D(Y) = Y^D \bmod N$.

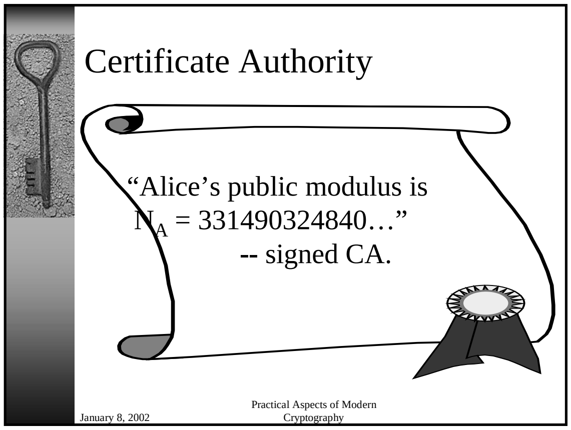
This D(Y) serves as Alice's signature on Y.



Public Key Directory

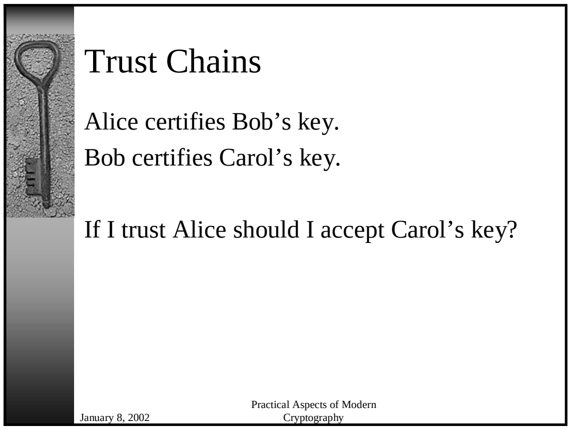
Name	Public Key	Encryption
Alice	N_A	$E_A(Y)=Y^E \bmod N_A$
Bob	N_B	$E_B(Y)=Y^E \bmod N_B$
Carol	N_C	$E_C(Y)=Y^E \bmod N_C$
:	:	:

(Recall that E is commonly fixed to be E=65537.)



Certificate Authority

"Alice's public modulus is N_A = 331490324840…"
-- signed CA.

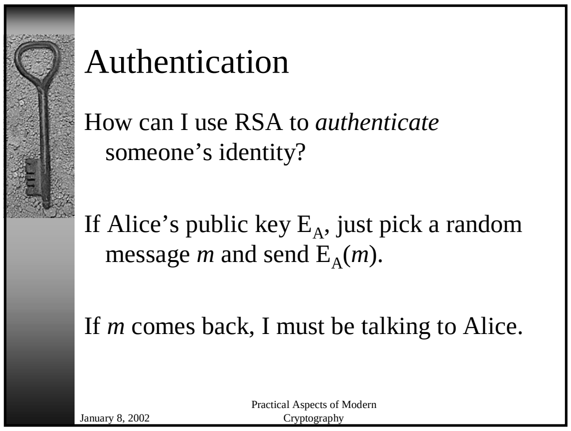


Trust Chains

Alice certifies Bob's key.

Bob certifies Carol's key.

If I trust Alice should I accept Carol's key?

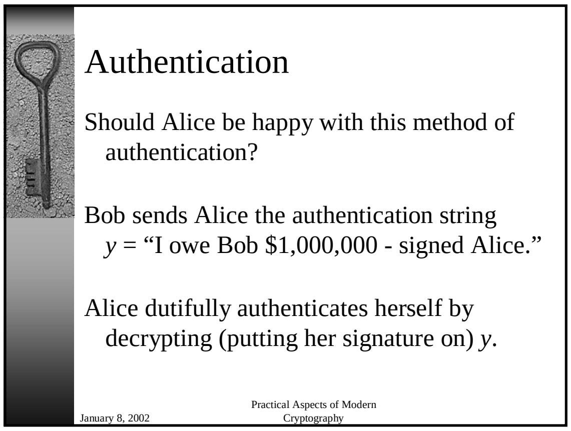


Authentication

How can I use RSA to *authenticate* someone's identity?

If Alice's public key E_A, just pick a random message *m* and send $E_A(m)$.

If *m* comes back, I must be talking to Alice.

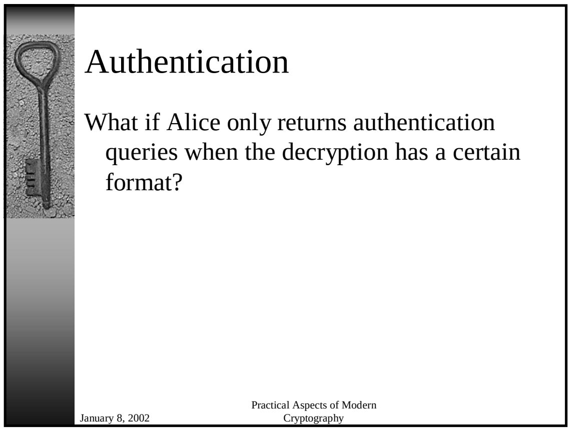


Authentication

Should Alice be happy with this method of authentication?

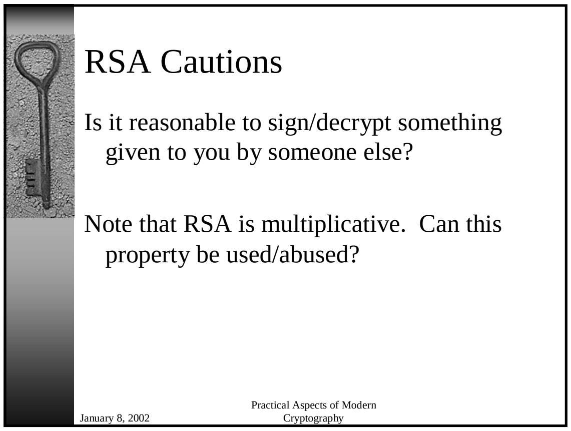
Bob sends Alice the authentication string *y* = "I owe Bob $1,000,000 - signed Alice."

Alice dutifully authenticates herself by decrypting (putting her signature on) *y*.



Authentication

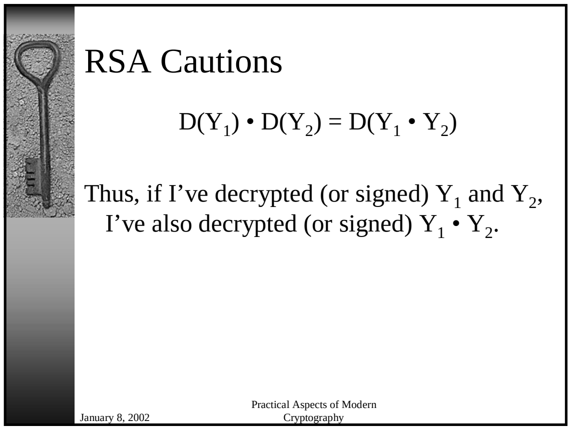
What if Alice only returns authentication queries when the decryption has a certain format?



RSA Cautions

Is it reasonable to sign/decrypt something given to you by someone else?

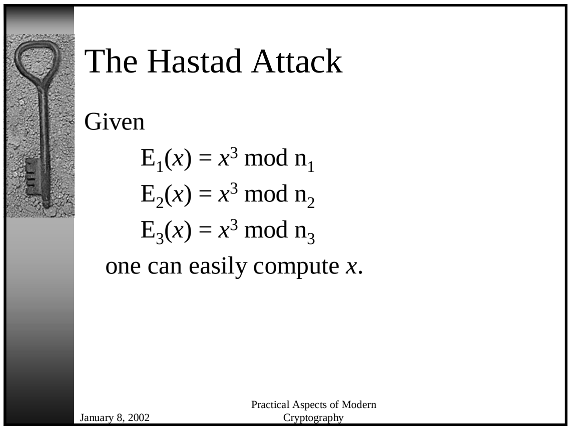
Note that RSA is multiplicative. Can this property be used/abused?



RSA Cautions

$$D(Y_1) \bullet D(Y_2) = D(Y_1 \bullet Y_2)$$

Thus, if I’ve decrypted (or signed) Y_1 and Y_2, I’ve also decrypted (or signed) $Y_1 \bullet Y_2$.



The Hastad Attack

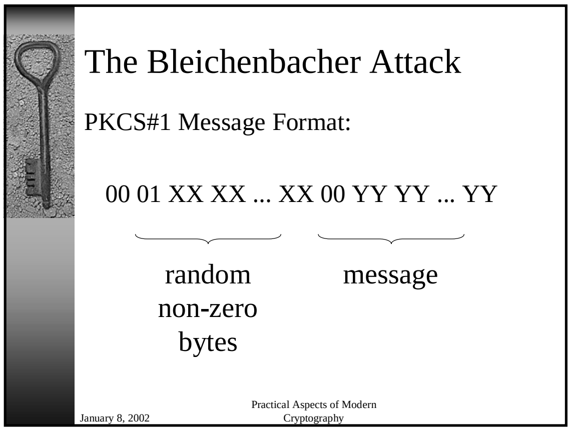
Given

$E_1(x) = x^3 \bmod n_1$

$E_2(x) = x^3 \bmod n_2$

$E_3(x) = x^3 \bmod n_3$

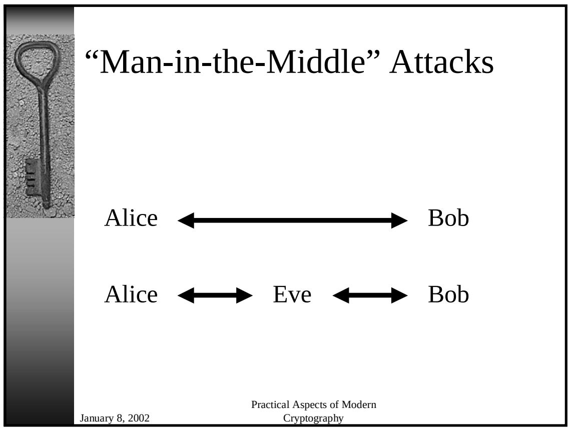
one can easily compute x.



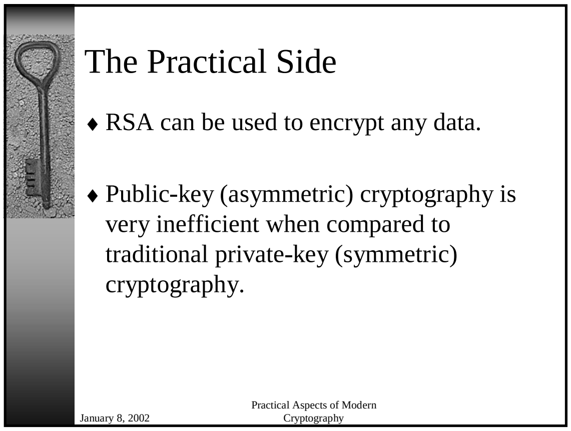
The Bleichenbacher Attack

PKCS#1 Message Format:

$$00\ 01\ \underbrace{XX\ XX \ldots XX}_{\text{random non-zero bytes}}\ 00\ \underbrace{YY\ YY \ldots YY}_{\text{message}}$$

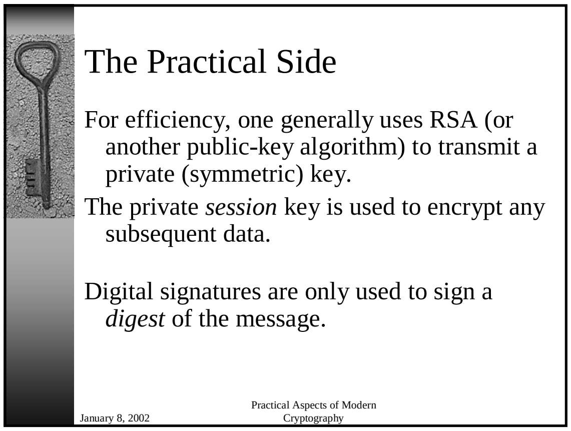


“Man-in-the-Middle” Attacks



The Practical Side

- ♦ RSA can be used to encrypt any data.
- ♦ Public-key (asymmetric) cryptography is very inefficient when compared to traditional private-key (symmetric) cryptography.



The Practical Side

For efficiency, one generally uses RSA (or another public-key algorithm) to transmit a private (symmetric) key.

The private *session* key is used to encrypt any subsequent data.

Digital signatures are only used to sign a *digest* of the message.