Practical Aspects of Modern Cryptography

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Public-Key History

- 1976 *New Directions in Cryptograhy*
  Whit Diffie and Marty Hellman
  - One-Way functions
  - Diffie-Hellman Key Exchange
- 1978 RSA paper
  Ron Rivest, Adi Shamir, and Len Adleman
  - RSA Encryption System
  - RSA Digital Signature Mechanism

The Fundamental Equation

$$Z=Y^X \bmod N$$

Diffie-Hellman

$$Z=Y^X \bmod N$$

When X is unknown, the problem is known as the *discrete logarithm* and is generally believed to be hard to solve.

Diffie-Hellman Key Exchange

Alice	Bob
Randomly select a large integer a and send $A = Y^a \bmod N$.	Randomly select a large integer b and send $B = Y^b \bmod N$.
Compute the key $K = B^a \bmod N$.	Compute the key $K = A^b \bmod N$.

$$B^a = Y^{ba} = Y^{ab} = A^b$$

Diffie-Hellman Key Exchange

What does Eve see?

Y, Y^a, Y^b

… but the exchanged key is Y^{ab}.

Belief: Given Y, Y^a, Y^b it is difficult to compute Y^{ab}.

Contrast with discrete logarithm assumption: Given Y, Y^x it is difficult to compute x.

One-Way Trap-Door Functions

$$Z=Y^X \bmod N$$

Recall that this equation is solvable for Y if the factorization of N is known, but is *believed* to be hard otherwise.

RSA Public-Key Cryptosystem

Alice

- Select two large random primes P & Q.
- Publish the product N=PQ.
- Use knowledge of P & Q to compute Y.

Anyone

- To send message Y to Alice, compute $Z=Y^X \bmod N$.
- Send Z and X to Alice.

Some RSA Details

When N=PQ is the product of distinct primes,

$$Y^X \bmod N = Y$$

whenever

$X \bmod (P-1)(Q-1) = 1$ and $0 \leq Y<N$.

Some RSA Details

When N=PQ is the product of distinct primes,

$$Y^X \bmod N = Y$$

whenever

$X \bmod (P-1)(Q-1) = 1$ and $0 \leq Y<N$.

Alice can easily select integers E and D such that $E \bullet D \bmod (P-1)(Q-1) = 1$.

Some RSA Details

Encryption: $E(Y) = Y^E \bmod N$.

Decryption: $D(Y) = Y^D \bmod N$.

$$\begin{aligned} &D(E(Y)) \\ &\quad = (Y^E \bmod N)^D \bmod N \\ &\quad = Y^{ED} \bmod N \\ &\quad = Y \end{aligned}$$

RSA Signatures

An additional property

$D(E(Y)) = Y^{ED} \bmod N = Y$

$E(D(Y)) = Y^{DE} \bmod N = Y$

Only Alice (knowing the factorization of N) knows D. Hence only Alice can compute $D(Y) = Y^D \bmod N$.

This D(Y) serves as Alice's signature on Y.

Remaining RSA Basics

- Why is Y^X mod PQ = Y whenever
  X mod (P-1)(Q-1) = 1, 0 ≤Y<PQ,
  and P and Q are distinct primes?

- How can Alice can select integers E and D such that E•D mod (P-1)(Q-1) = 1?

Modular Arithmetic

- To compute (A+B) mod N,
  compute (A+B) and take the result mod N.
- To compute (A-B) mod N,
  compute (A-B) and take the result mod N.
- To compute (A×B) mod N,
  compute (A×B) and take the result mod N.
- To compute (A÷B) mod N, …

Modular Division

What is the value of (1÷2) mod 7?
We need a solution to 2*x* mod 7 = 1.
Try *x* = 4.

What is the value of (7÷5) mod 11?
We need a solution to 5*x* mod 11 = 7.
Try *x* = 8.

Modular Division

Is modular division always well-defined?
(1÷3) mod 6 = ?
3*x* mod 6 = 1 has no solution!

<u>Fact</u>
(A÷B) mod N always has a solution when gcd(B,N) = 1.

Greatest Common Divisors

gcd(A , B) = gcd(B , A - B)
gcd(21,12) = gcd(12,9) = gcd(9,3)
= gcd(6,3) = gcd(3,3) = gcd(0,3) = 3
gcd(A , B) = gcd(B , A mod B)
gcd(21,12) = gcd(12,9) = gcd(9,3)
= gcd(0,3) = 3

Extended Euclidean Algorithm

Given integers A and B, find integers X and Y such that AX + BY = gcd(A,B).

When gcd(A,B) = 1, solve AX mod B = 1, by finding X and Y such that
AX + BY = gcd(A,B) = 1.

Compute (C÷A) mod B as C×(1÷A) mod B.

Extended Euclidean Algorithm

Given A,B > 0, set x_1=1, x_2=0, y_1=0, y_2=1, a_1=A, b_1=B, i=1.

Repeat while b_i>0: {$i = i + 1$;
$q = a_{i-1}$ div b_{i-1}; $b_i = a_{i-1} - qb_{i-1}$; $a_i = b_{i-1}$;
$x_{i+1} = x_{i-1} - qx_i$; $y_{i+1} = y_{i-1} - qy_i$}.

$Ax_i + By_i = a_i = \gcd(A,B)$.

Remaining RSA Basics

- Why is Y^X mod PQ = Y whenever
  X mod (P-1)(Q-1) = 1, 0 ≤Y<PQ,
  and P and Q are distinct primes?
- How can Alice can select integers E and D such that E•D mod (P-1)(Q-1) = 1?

Fermat's Little Theorem

If p is prime,
then $x^{p-1} \bmod p = 1$ for all $0 < x < p$.

Equivalently …

If p is prime,
then $x^p \bmod p = x \bmod p$ for all integers x.

Proof of Fermat's Little Theorem

The Binomial Theorem

$(x + y)^p = x^p + \binom{p}{1} x^{p-1}y + \ldots + \binom{p}{p-1} xy^{p-1} + y^p$

If p is prime, then $\binom{p}{i} \bmod p = 0$ for $0 < i < p$.

Thus, $(x + y)^p \bmod p = (x^p + y^p) \bmod p$.

Proof of Fermat's Little Theorem

By induction on x…

Basis

If $x = 0$, then $x^p \bmod p = 0 = x \bmod p$.
If $x = 1$, then $x^p \bmod p = 1 = x \bmod p$.

Proof of Fermat's Little Theorem

Inductive Step

Assume that $x^p \bmod p = x \bmod p$.
Then $(x + 1)^p \bmod p = (x^p + 1^p) \bmod p$
$= (x + 1) \bmod p$.
Hence, $x^p \bmod p = x \bmod p$ for integers $x \geq 0$.

Also true for negative x, since $(-x)^p = (-1)^p x^p$.

Proof of RSA

We have shown …

$Y^P \bmod P = Y$ whenever $0 \le Y < P$
and P is *prime*!

You will show …

$Y^{K(P-1)(Q-1)+1} \bmod PQ = Y$ when $0 \le Y < PQ$
P and Q are distinct primes and $K \ge 0$.

Authentication

How can I use RSA to *authenticate* someone's identity?

If Alice's public key E_A, just pick a random message *m* and send $E_A(m)$.

If *m* comes back, I must be talking to Alice.

Authentication

Should Alice be happy with this method of authentication?

Bob sends Alice the authentication string *y* = "I owe Bob $1,000,000 - signed Alice."

Alice dutifully authenticates herself by decrypting (putting her signature on) *y*.

Authentication

What if Alice only returns authentication queries when the decryption has a certain format?

RSA Cautions

Is it reasonable to sign/decrypt something given to you by someone else?

Note that RSA is multiplicative. Can this property be used/abused?

RSA Cautions

$$D(Y_1) \bullet D(Y_2) = D(Y_1 \bullet Y_2)$$

Thus, if I've decrypted (or signed) Y_1 and Y_2, I've also decrypted (or signed) $Y_1 \bullet Y_2$.

The Hastad Attack

Given

$E_1(x) = x^3 \bmod n_1$

$E_2(x) = x^3 \bmod n_2$

$E_3(x) = x^3 \bmod n_3$

one can easily compute x.

The Bleichenbacher Attack

PKCS#1 Message Format:

$$00\ 01\ \underbrace{XX\ XX\ ...\ XX}_{\text{random non-zero bytes}}\ 00\ \underbrace{YY\ YY\ ...\ YY}_{\text{message}}$$

“Man-in-the-Middle” Attacks

Alice ⟷ Bob

Alice ⟷ Eve ⟷ Bob

The Practical Side

- ♦ RSA can be used to encrypt any data.
- ♦ Public-key (asymmetric) cryptography is very inefficient when compared to traditional private-key (symmetric) cryptography.

The Practical Side

For efficiency, one generally uses RSA (or another public-key algorithm) to transmit a private (symmetric) key.

The private *session* key is used to encrypt and authenticate any subsequent data.

Digital signatures are only used to sign a *digest* of the message.

Symmetric Ciphers

Private-key (symmetric) ciphers are usually divided into two classes.

- ♦ Block ciphers
- ♦ Stream ciphers

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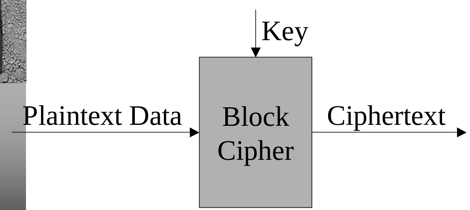
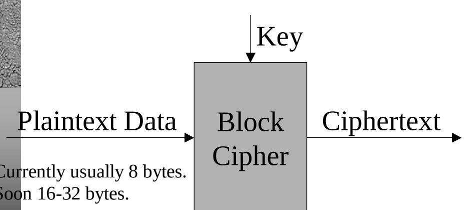
- Block ciphers
- Stream ciphers

Block Ciphers



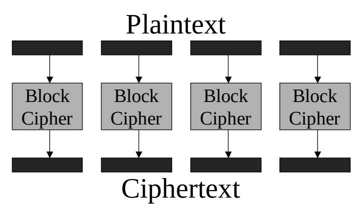
Block Ciphers



Currently usually 8 bytes.
Soon 16-32 bytes.

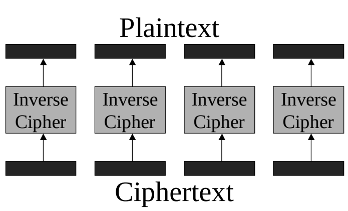
Block Cipher Modes

Electronic Code Book (ECB) Encryption:



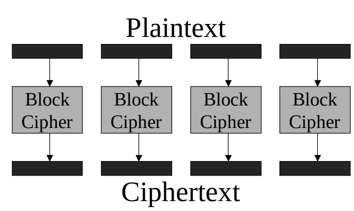
Block Cipher Modes

Electronic Code Book (ECB) Decryption:



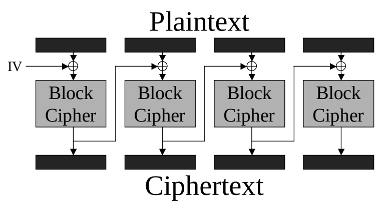
Block Cipher Modes

Electronic Code Book (ECB) Encryption:



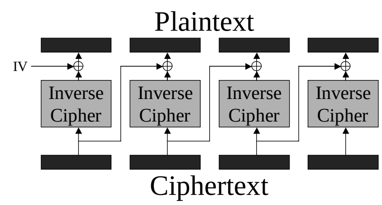
Block Cipher Modes

Cipher Block Chaining (CBC) Encryption:



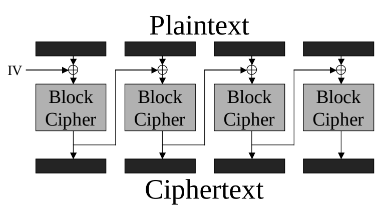
Block Cipher Modes

Cipher Block Chaining (CBC) Decryption:

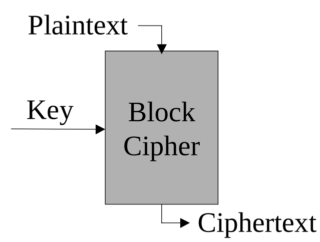


Block Cipher Modes

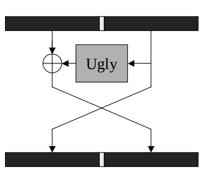
Cipher Block Chaining (CBC) Encryption:



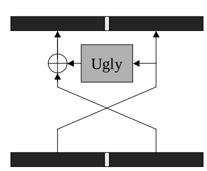
How to Build a Block Cipher



Feistel Ciphers



Feistel Ciphers



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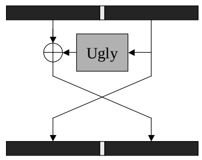

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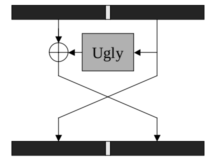
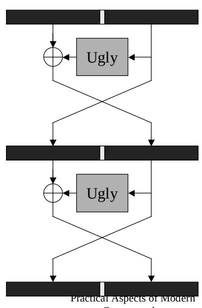


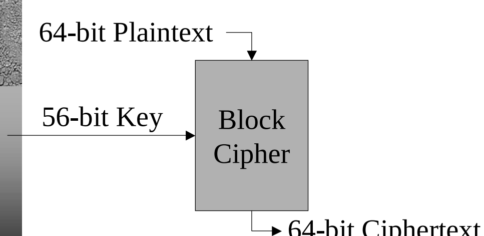
Feistel Ciphers



Feistel Ciphers

- Typically, most Feistel ciphers are iterated for about 16 rounds.
- Different “sub-keys” are used for each round.
- Even a weak round function can yield a strong Feistel cipher if iterated sufficiently.

Data Encryption Standard (DES)



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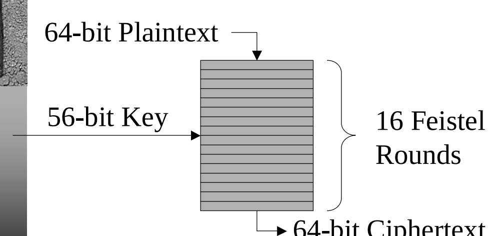
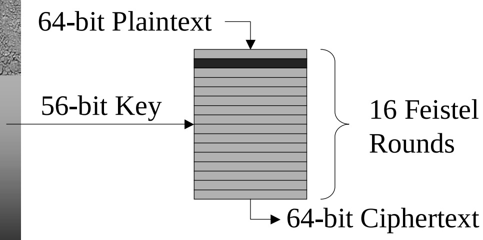
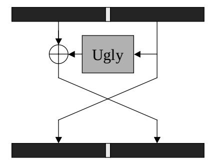


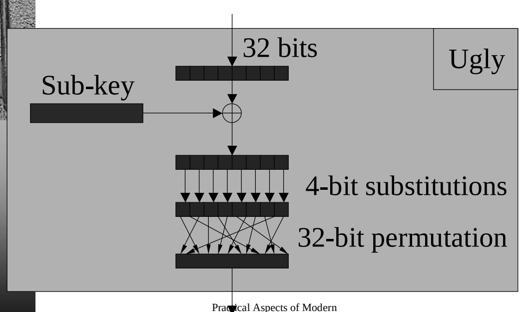
Data Encryption Standard (DES)



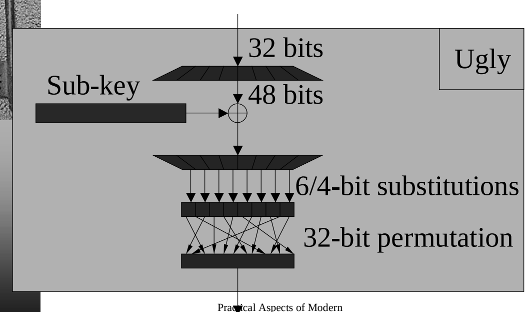
DES Round



Simplified DES Round Function



Actual DES Round Function



Symmetric Ciphers

Private-key (symmetric) ciphers are usually divided into two classes.

- Block ciphers
- Stream ciphers

Stream Ciphers

- Use the key as a seed to a pseudo-random number-generator.
- Take the stream of output bits from the PRNG and XOR it with the plaintext to form the ciphertext.

Stream Cipher Encryption

Plaintext: ■■■■■■■■■■■■■■■■
⊕⊕⊕⊕⊕⊕⊕⊕⊕⊕⊕⊕⊕⊕⊕⊕
PRNG(seed): □□□□□□□□□□□□□□□□
↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓
Ciphertext: ■■■■■■■■■■■■■■■■

Stream Cipher Decryption

Plaintext: ■■■■■■■■■■■■■■■■
↑↑↑↑↑↑↑↑↑↑↑↑↑↑↑↑
PRNG(seed): □□□□□□□□□□□□□□□□
⊕⊕⊕⊕⊕⊕⊕⊕⊕⊕⊕⊕⊕⊕⊕⊕
Ciphertext: ■■■■■■■■■■■■■■■■

A PRNG: Alleged RC4

<u>Initialization</u>

```
S[0..255] = 0,1,…,255
K[0..255] = Key,Key,Key,…
for i = 0 to 255
    j = (j + S[i] + K[i]) mod 256
    swap S[i] and S[j]
```


A PRNG: Alleged RC4

<u>Iteration</u>

```
i = (i + 1) mod 256
j = (j + S[i]) mod 256
swap S[i] and S[j]
t = (S[i] + S[j]) mod 256
Output S[t]
```


Stream Cipher Integrity

- It is easy for an adversary (even one who can’t decrypt the ciphertext) to alter the plaintext in a known way.

Bob to Bob’s Bank:
Please transfer $0,000,002.00 to the account of my good friend Alice.

Stream Cipher Integrity

- It is easy for an adversary (even one who can’t decrypt the ciphertext) to alter the plaintext in a known way.

Bob to Bob’s Bank:
Please transfer $1,000,002.00 to the account of my good friend Alice.

Stream Cipher Integrity

- It is easy for an adversary (even one who can't decrypt the ciphertext) to alter the plaintext in a known way.

Bob to Bob's Bank:
Please transfer $1,000,002.00 to the account of my good friend Alice.

- This can be protected against by the careful addition of appropriate redundancy.

One-Way Hash Functions

The idea of a *check sum* is great, but it is designed to prevent accidental changes in a message.

For cryptographic integrity, we need an integrity check that is resilient against a smart and determined adversary.

One-Way Hash Functions

Generally, a *one-way hash function* is a function $H : \{0,1\}^* \rightarrow \{0,1\}^k$ (typically k is 128 or 160) such that given an input value x, one cannot find a value $x' \neq x$ such $H(x) = H(x')$.

One-Way Hash Functions

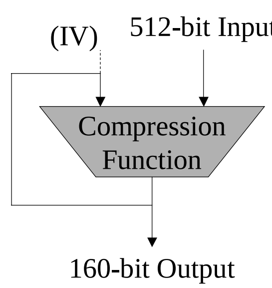
There are many measures for one-way hashes.

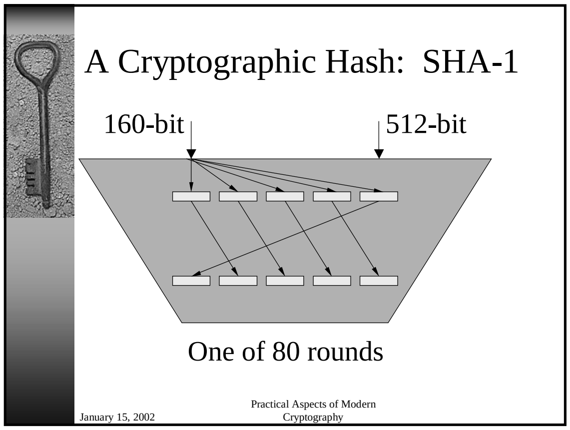
- Non-invertability: given y, it's difficult to find any x such that H(x) = y.
- Collision-intractability: one cannot find a pair of values $x' \neq x$ such that $H(x) = H(x')$.

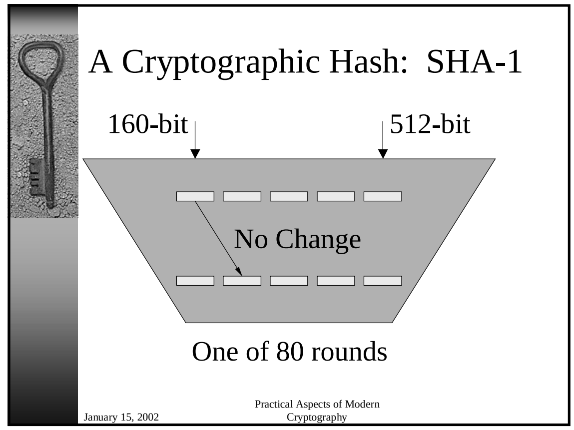
One-Way Hash Functions

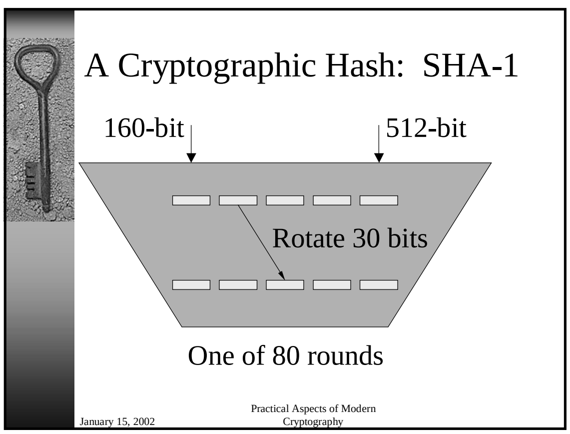
- When using a stream cipher, a hash of the message can be appended to ensure integrity. [Message Authentication Code]
- When forming a digital signature, the signature need only be applied to a hash of the message. [Message Digest]

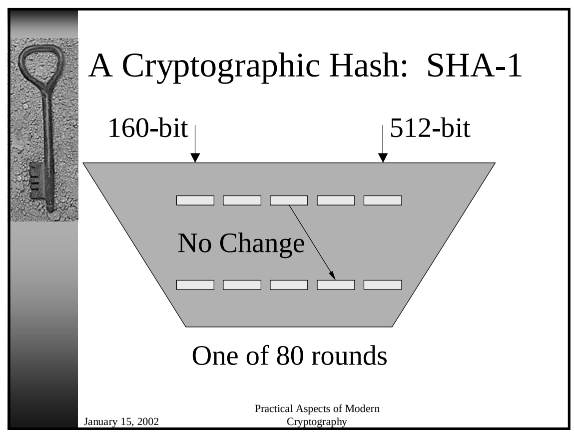
A Cryptographic Hash: SHA-1

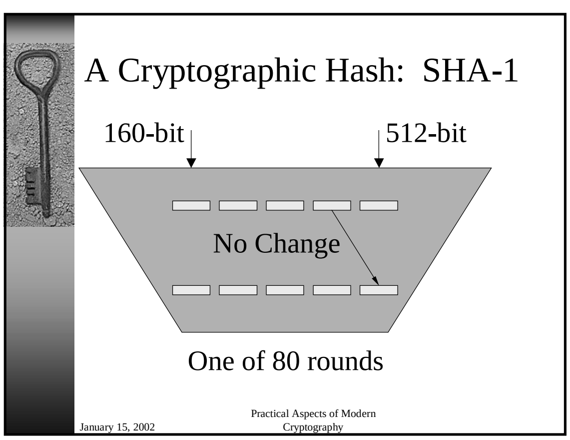


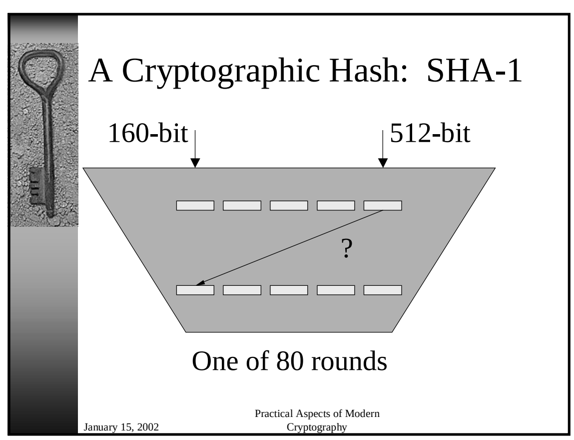

A Cryptographic Hash: SHA-1
160-bit
512-bit
One of 80 rounds
January 15, 2002
Practical Aspects of Modern Cryptography


A Cryptographic Hash: SHA-1
160-bit
512-bit
No Change
One of 80 rounds
January 15, 2002
Practical Aspects of Modern Cryptography


A Cryptographic Hash: SHA-1
160-bit
512-bit
Rotate 30 bits
One of 80 rounds
January 15, 2002
Practical Aspects of Modern Cryptography


A Cryptographic Hash: SHA-1
160-bit
512-bit
No Change
One of 80 rounds
January 15, 2002
Practical Aspects of Modern Cryptography


A Cryptographic Hash: SHA-1
160-bit
512-bit
No Change
One of 80 rounds
January 15, 2002
Practical Aspects of Modern Cryptography

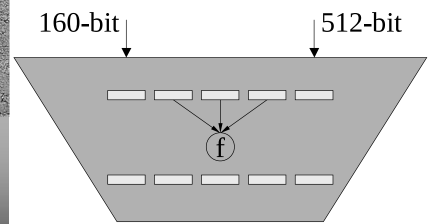

A Cryptographic Hash: SHA-1
160-bit
512-bit
?
One of 80 rounds
January 15, 2002
Practical Aspects of Modern Cryptography

A Cryptographic Hash: SHA-1

What's in the final 32-bit transform?

- Take the rightmost word.
- Add in the leftmost word rotated 5 bits.
- Add in a round-dependent function f of the middle three words.

A Cryptographic Hash: SHA-1



One of 80 rounds

A Cryptographic Hash: SHA-1

Depending on the round, the "non-linear" function f is one of the following.

$$f(X,Y,Z) = (X \wedge Y) \vee ((\neg X) \wedge Z)$$
$$f(X,Y,Z) = (X \wedge Y) \vee (X \wedge Z) \vee (Y \wedge Z)$$
$$f(X,Y,Z) = X \oplus Y \oplus Z$$

A Cryptographic Hash: SHA-1

What's in the final 32-bit transform?

- Take the rightmost word.
- Add in the leftmost word rotated 5 bits.
- Add in a round-dependent function f of the middle three words.

A Cryptographic Hash: SHA-1

What's in the final 32-bit transform?

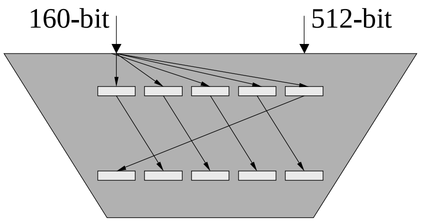
- Take the rightmost word.
- Add in the leftmost word rotated 5 bits.
- Add in a round-dependent function f of the middle three words.
- Add in a round-dependent constant.

A Cryptographic Hash: SHA-1

What's in the final 32-bit transform?

- Take the rightmost word.
- Add in the leftmost word rotated 5 bits.
- Add in a round-dependent function f of the middle three words.
- Add in a round-dependent constant.
- Add in a portion of the 512-bit message.

A Cryptographic Hash: SHA-1



One of 80 rounds

Cryptographic Tools

One-Way Trapdoor Functions
Public-Key Encryption Schemes
One-Way Functions
One-Way Hash Functions
Pseudo-Random Number-Generators
Secret-Key Encryption Schemes
Digital Signature Schemes