CSEP 590
Data Compression

Autumn 2007

Course Policies

Introduction to Data Compression

Entropy

Variable Length Codes

Instructors

- Instructor
  - Richard Ladner
  - ladner@cs.washington.edu
  - 206 543-9347
- TA
  - Rahul Vanam
  - rahulv@u.washington.edu

Helpful Knowledge

- Algorithm Design and Analysis
- Probability

Resources

- Text Book
  - Khalid Sayood, Introduction to Data Compression, Third Edition, Morgan Kaufmann Publishers, 2006.
- Course Web Page
  - http://www.cs.washington.edu/csep590a
- Papers and Sections from Books
- Discussion Board
  - For discussion

Engagement by Students

- Weekly Assignments
  - Understand compression methodology
  - Due in class on Fridays (except midterm Friday)
  - No late assignments accepted except with prior approval
- Programming Projects
  - Bi-level arithmetic coder and decoder.
  - Build code and experiment

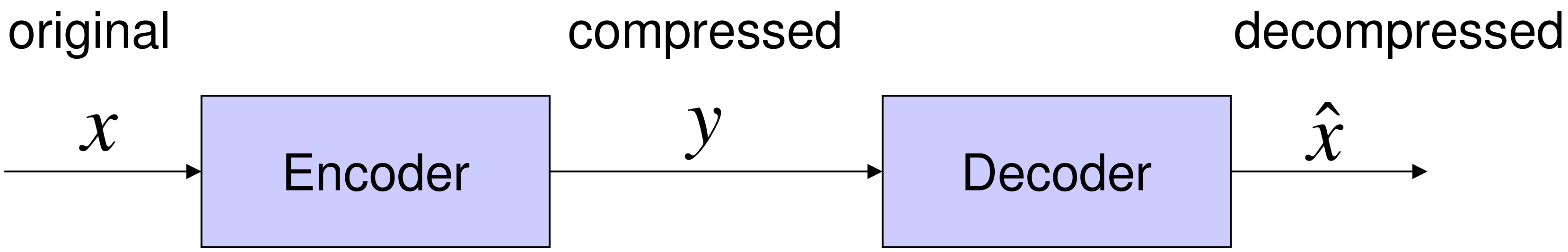
Final Exam and Grading

- 6:30-8:20 p.m. Thursday, Dec. 13, 2007
- Percentages
  - Weekly assignments (50%)
  - Project (20%)
  - Final exam (30%)

Logistics

- I will be gone the week of October 15th. We'll need to have a make up class.
- There is no class Thanksgiving week, November 19th.
- We have some guest speakers toward the end of the quarter.

Basic Data Compression Concepts



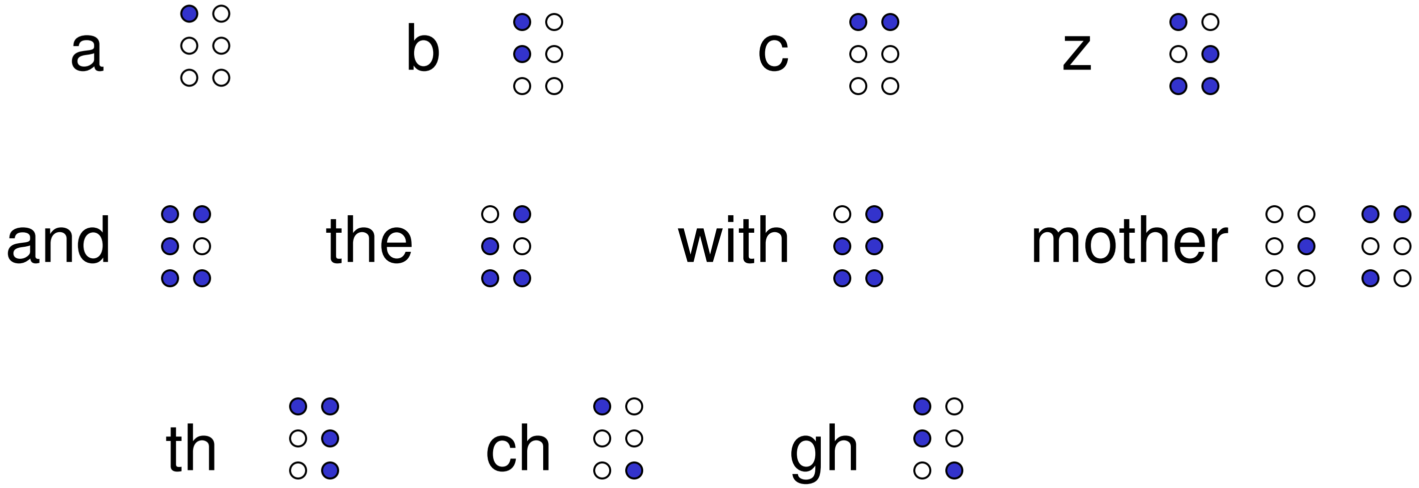
- Lossless compression $x = \hat{x}$
  - Also called entropy coding, reversible coding.
- Lossy compression $x \neq \hat{x}$
  - Also called irreversible coding.
- Compression ratio = $|x|/|y|$
  - $|x|$ is number of bits in x.

Why Compress

- Conserve storage space
- Reduce time for transmission
  - Faster to encode, send, then decode than to send the original
- Progressive transmission
  - Some compression techniques allow us to send the most important bits first so we can get a low resolution version of some data before getting the high fidelity version
- Reduce computation
  - Use less data to achieve an approximate answer

Braille

- System to read text by feeling raised dots on paper (or on electronic displays). Invented in 1820s by Louis Braille, a French blind man.



a b c z

and the with mother

th ch gh

Braille Example

Clear text:

Call me Ishmael. Some years ago -- never mind how long precisely -- having \\ little or no money in my purse, and nothing particular to interest me on shore, \\ I thought I would sail about a little and see the watery part of the world.   (238 characters)

Grade 2 Braille in ASCII.

,call me ,i\%mael4 ,``s ye$>$s ago -- n``e m9d h[l;g
precisely -- hav+ \\ ll or no m``oy 9 my purse1 \& no?+
``picul$>$ 6 9t]e/ me on \%ore1 \\ ,i $?$``$|$,i wd sail
ab a ll \& see ! wat]y ``p (! _w4 (203 characters)

Compression ratio = 238/203 = 1.17

Lossless Compression

- Data is not lost - the original is really needed.
  - text compression
  - compression of computer binary files
- Compression ratio typically no better than 4:1 for lossless compression on many kinds of files.
- Statistical Techniques
  - Huffman coding
  - Arithmetic coding
  - Golomb coding
- Dictionary techniques
  - LZW, LZ77
  - Sequitur
  - Burrows-Wheeler Method
- Standards - Morse code, Braille, Unix compress, gzip, zip, bzip, GIF, JBIG, Lossless JPEG

Lossy Compression

- Data is lost, but not too much.
  - audio
  - video
  - still images, medical images, photographs
- Compression ratios of 10:1 often yield quite high fidelity results.
- Major techniques include
  - Vector Quantization
  - Wavelets
  - Block transforms
  - Standards - JPEG, JPEG2000, MPEG 2, H.264

Why is Data Compression Possible

- Most data from nature has redundancy
  - There is more data than the actual information contained in the data.
  - Squeezing out the excess data amounts to compression.
  - However, unsqueezing is necessary to be able to figure out what the data means.
- Information theory is needed to understand the limits of compression and give clues on how to compress well.

What is Information

- Analog data
  - Also called continuous data
  - Represented by real numbers (or complex numbers)
- Digital data
  - Finite set of symbols $\{a_1, a_2, \ldots, a_m\}$
  - All data represented as sequences (strings) in the symbol set.
  - Example: {a,b,c,d,r} abracadabra
  - Digital data can be an approximation to analog data

Symbols

- Roman alphabet plus punctuation
- ASCII - 256 symbols
- Binary - {0,1}
  - 0 and 1 are called bits
  - All digital information can be represented efficiently in binary
  - {a,b,c,d} fixed length representation

symbol	a	b	c	d
binary	00	01	10	11

  - 2 bits per symbol

Exercise - How Many Bits Per Symbol?

- Suppose we have n symbols. How many bits (as a function of n) are needed in to represent a symbol in binary?
  - First try n a power of 2.

Discussion: Non-Powers of Two

- Can we do better than a fixed length representation for non-powers of two?

Information Theory

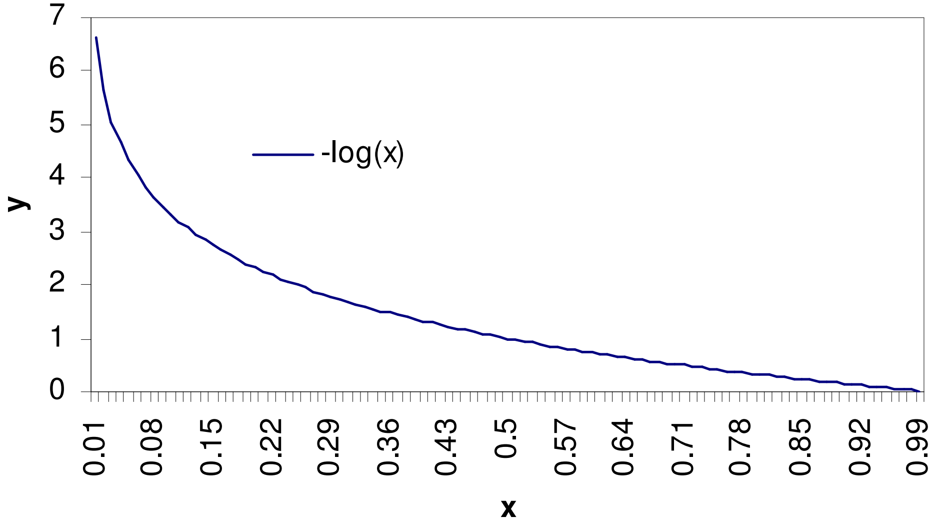
- Developed by Shannon in the 1940’s and 50’s
- Attempts to explain the limits of communication using probability theory.
- Example: Suppose English text is being sent
  - It is much more likely to receive an “e” than a “z”.
  - In some sense “z” has more information than “e”.

First-order Information

- Suppose we are given symbols $\{a_1, a_2, \dots , a_m\}$.
- $P(a_i)$ = probability of symbol a_i occurring in the absence of any other information.

$$P(a_1) + P(a_2) + \dots + P(a_m) = 1$$

- $\inf(a_i) = \log_2(1/P(a_i))$ bits is the information of a_i in bits.



Example

- {a, b, c} with P(a) = 1/8, P(b) = 1/4, P(c) = 5/8
  - inf(a) = $\log_2(8)$ = 3
  - inf(b) = $\log_2(4)$ = 2
  - inf(c) = $\log_2(8/5)$ = .678
- Receiving an “a” has more information than receiving a “b” or “c”.

First Order Entropy

- The first order entropy is defined for a probability distribution over symbols $\{a_1, a_2, \ldots, a_m\}$.

$$H = \sum_{i=1}^{m} P(a_i)\log_2\left(\frac{1}{P(a_i)}\right)$$

- H is the average number of bits required to code up a symbol, given all we know is the probability distribution of the symbols.
- H is the Shannon lower bound on the average number of bits to code a symbol in this “source model”.
- Stronger models of entropy include context.

Entropy Examples

- {a, b, c} with a 1/8, b 1/4, c 5/8.
  - H = 1/8 *3 + 1/4 *2 + 5/8* .678 = 1.3 bits/symbol

- {a, b, c} with a 1/3, b 1/3, c 1/3. (worst case)
  - H = 3* (1/3)*$\log_2(3)$ = 1.6 bits/symbol

- Note that a standard code takes 2 bits per symbol

symbol	a	b	c
binary code	00	01	10

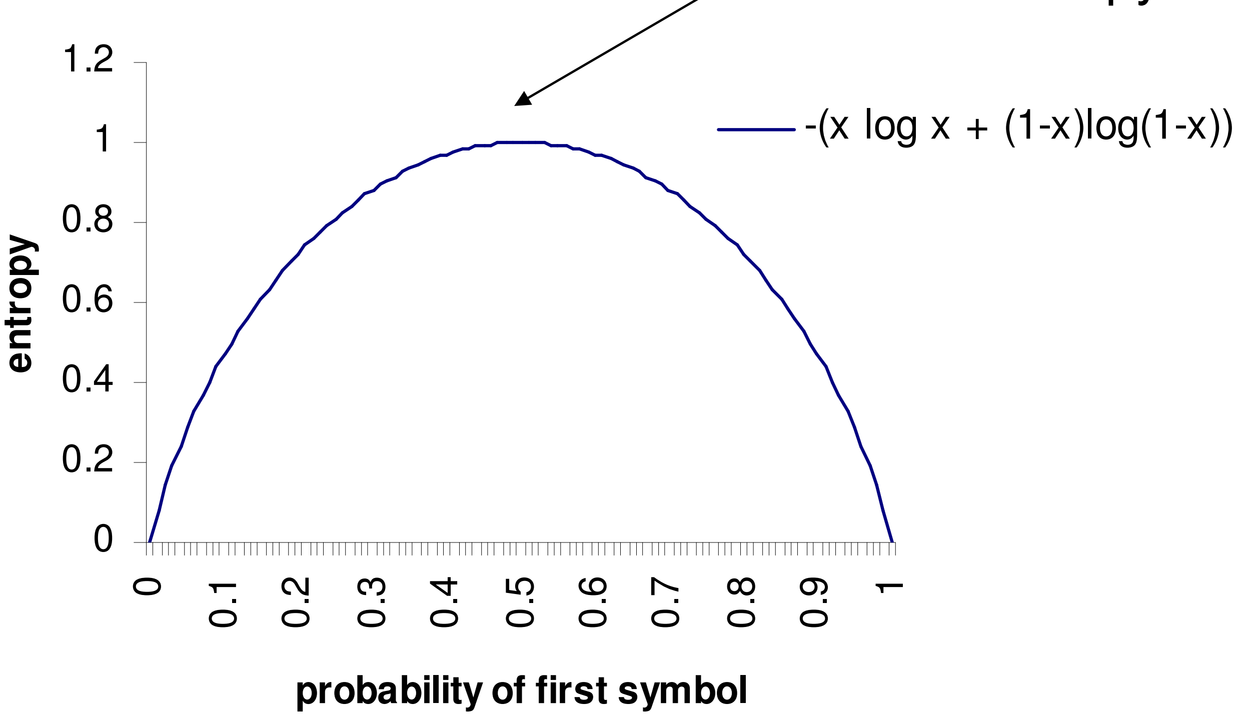
An Extreme Case

- {a, b, c} with a 1, b 0, c 0
  - H = ?

Entropy Curve

- Suppose we have two symbols with probabilities x and 1-x, respectively.

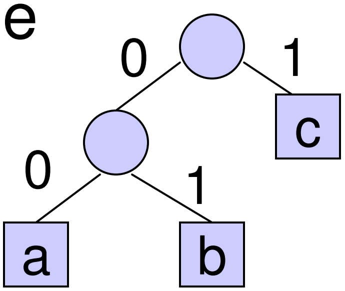
maximum entropy at .5



A Simple Prefix Code

- {a, b, c} with a 1/8, b 1/4, c 5/8.
- A prefix code is defined by a binary tree
- Prefix code property
  - no output is a prefix of another

binary tree



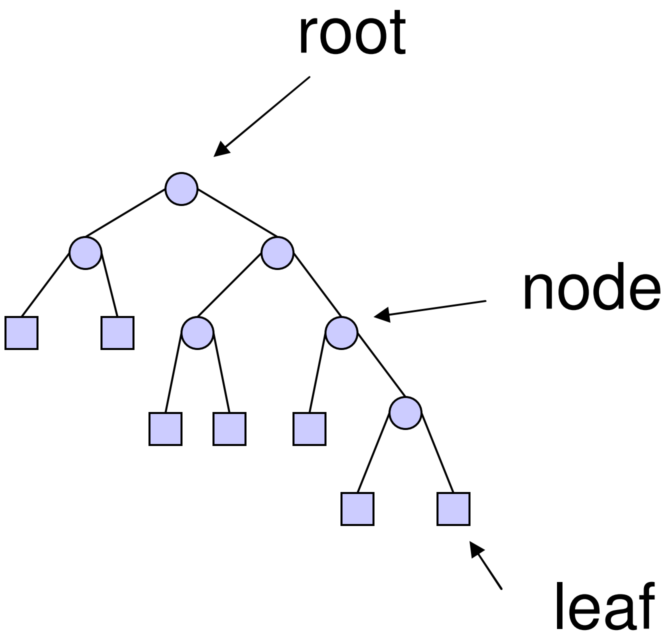
input	output
a	00
b	01
c	1

code

ccabccbccc

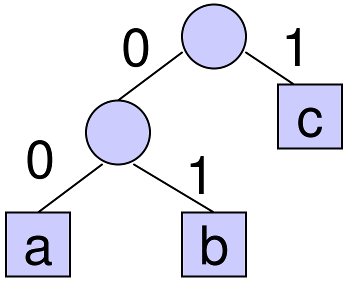
1 1 00 01 1 1 01 1 1 1

Binary Tree Terminology



1. Each node, except the root, has a unique parent.
2. Each internal node has exactly two children.

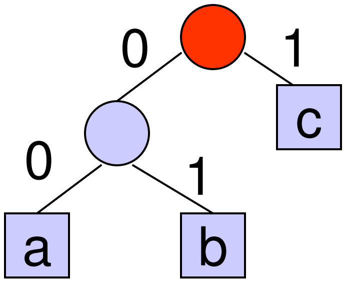
Decoding a Prefix Code



```
repeat
start at root of tree
  repeat
    if read bit = 1 then go right
    else go left
  until node is a leaf
  report leaf
until end of the code
```

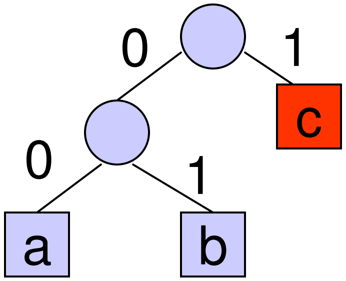
11000111100

Decoding a Prefix Code



<u>1</u>1000111100

Decoding a Prefix Code



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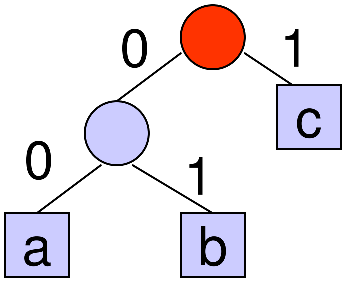
c

Decoding a Prefix Code



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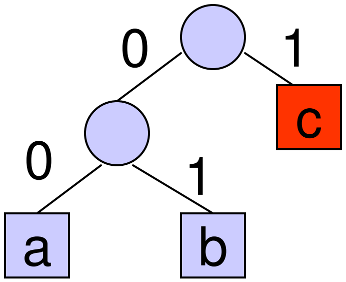
c

Decoding a Prefix Code



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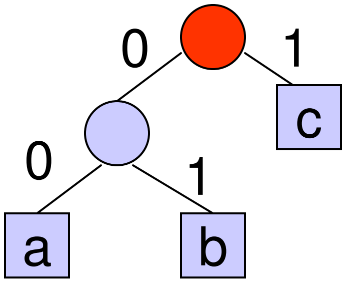

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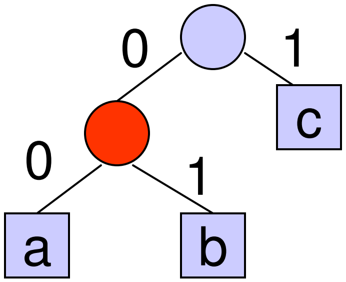

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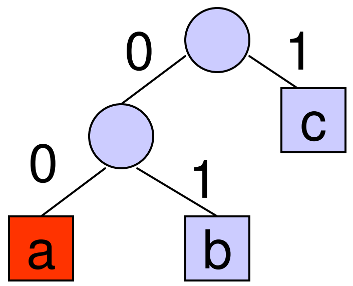
cc

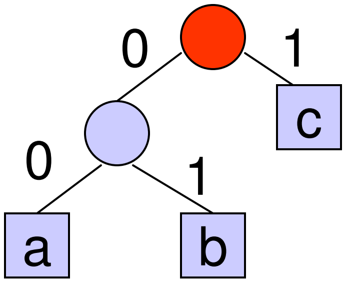
Decoding a Prefix Code



11000111100

cca

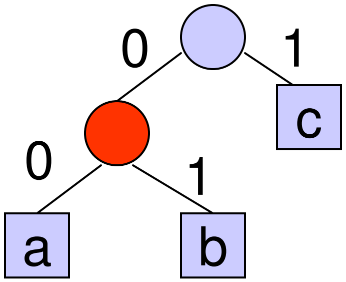
Decoding a Prefix Code



11000111100

cca

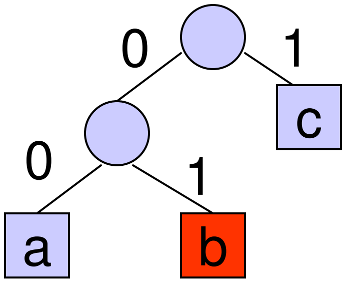
Decoding a Prefix Code



11000111100

cca

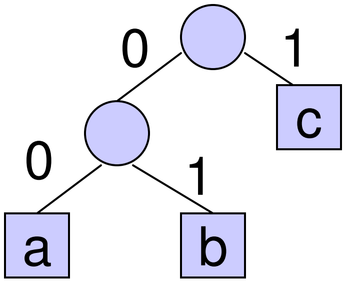
Decoding a Prefix Code



11000<u>1</u>11100

ccab

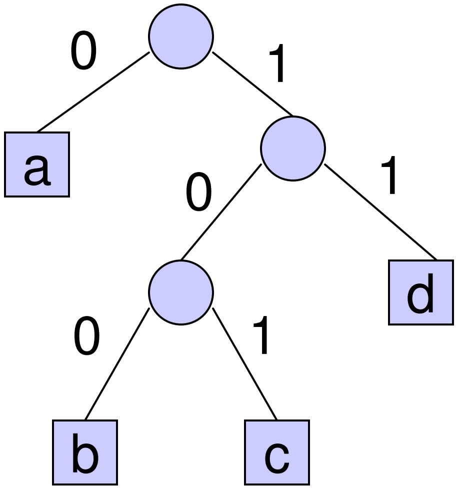
Decoding a Prefix Code



11000111100

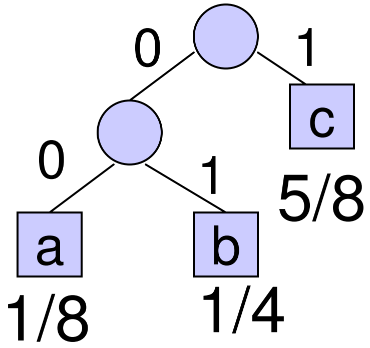
ccabccca

Exercise Encode/Decode



- Player 1: Encode a symbol string
- Player 2: Decode the string
- Check for equality

How Good is the Code



bit rate = (1/8)2 + (1/4)2 + (5/8)1 = 11/8 = 1.375 bps
Entropy = 1.3 bps
Standard code = 2 bps

(bps = bits per symbol)

Design a Prefix Code 1

- abracadabra
- Design a prefix code for the 5 symbols {a,b,r,c,d} which compresses this string the most.

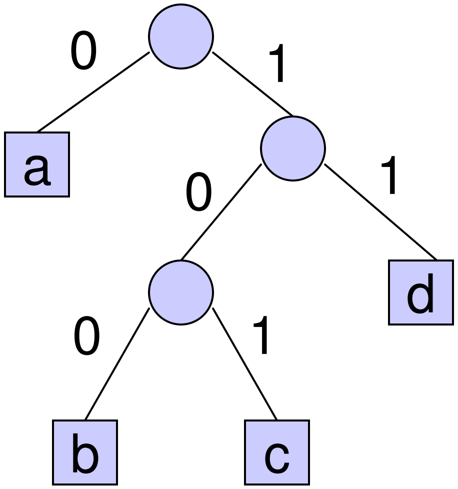
Design a Prefix Code 2

- Suppose we have n symbols each with probability 1/n. Design a prefix code with minimum average bit rate.
- Consider n = 2,3,4,5,6 first.

Huffman Coding

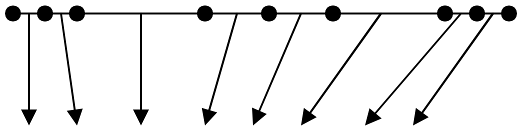
- Huffman (1951)
- Uses frequencies of symbols in a string to build a variable rate prefix code.
  - Each symbol is mapped to a binary string.
  - More frequent symbols have shorter codes.
  - No code is a prefix of another.
- Example:

a	0
b	100
c	101
d	11



Variable Rate Code Example

- Example: a 0, b 100, c 101, d 11
- Coding:
  - aabddcaa = 16 bits
  - 0 0 100 11 11 101 0 0= 14 bits
- Prefix code ensures unique decodability.
  - 00100111110100



  - a a b d d c a a

Cost of a Huffman Tree

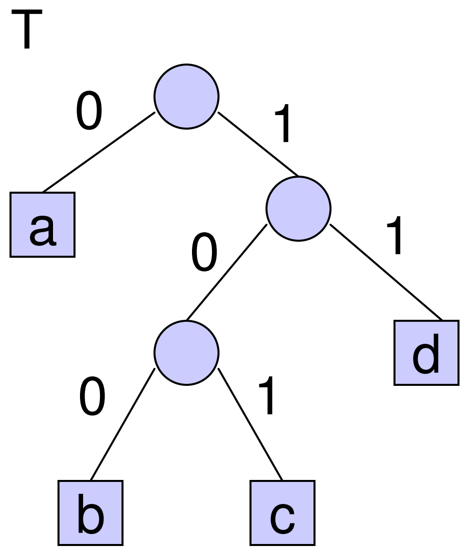
- Let $p_1, p_2, \ldots, p_m$ be the probabilities for the symbols $a_1, a_2, \ldots, a_m$, respectively.
- Define the cost of the Huffman tree T to be

$$C(T) = \sum_{i=1}^{m} p_i r_i$$

  where r_i is the length of the path from the root to a_i.
- C(T) is the expected length of the code of a symbol coded by the tree T. C(T) is the bit rate of the code.

Example of Cost

- Example: a 1/2, b 1/8, c 1/8, d 1/4



$$C(T) = \underset{a}{1 \times 1/2} + \underset{b}{3 \times 1/8} + \underset{c}{3 \times 1/8} + \underset{d}{2 \times 1/4} = 1.75$$

Huffman Tree

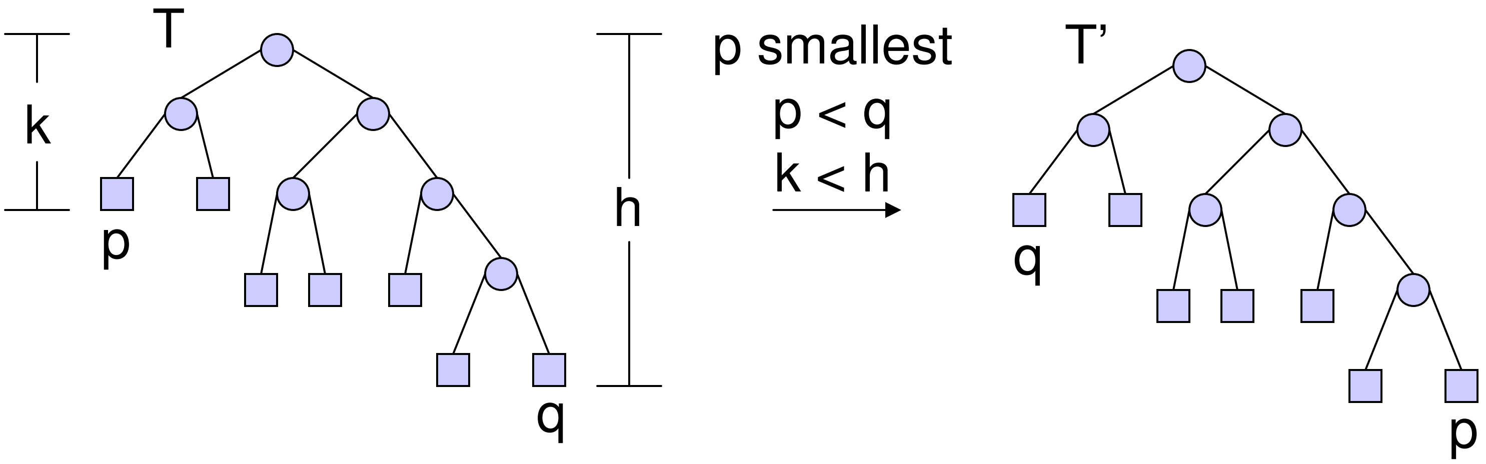
- Input: Probabilities $p_1, p_2, \ldots , p_m$ for symbols $a_1, a_2, \ldots ,a_m$, respectively.
- Output: A tree that minimizes the average number of bits (bit rate) to code a symbol. That is, minimizes

$$HC(T) = \sum_{i=1}^{m} p_i r_i \quad \text{bit rate}$$

where r_i is the length of the path from the root to a_i. This is the Huffman tree or Huffman code

Optimality Principle 1

- In a Huffman tree a lowest probability symbol has maximum distance from the root.
  - If not exchanging a lowest probability symbol with one at maximum distance will lower the cost.



p smallest

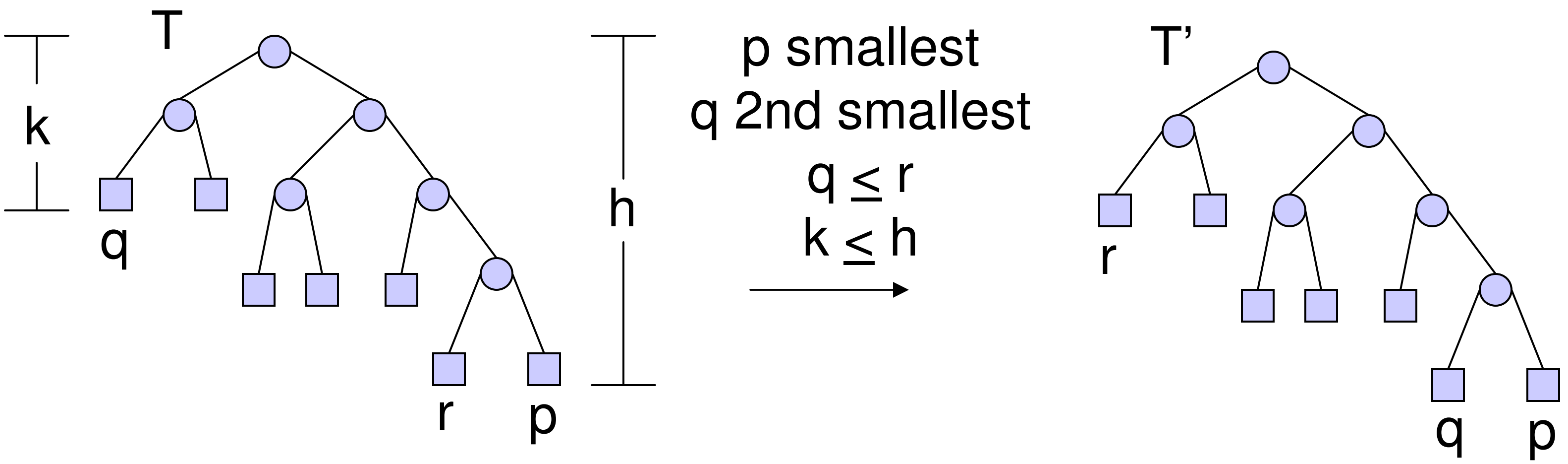
$p < q$

$k < h$

$$C(T') = C(T) + hp - hq + kq - kp = C(T) - (h-k)(q-p) < C(T)$$

Optimality Principle 2

- The second lowest probability is a sibling of the smallest in some Huffman tree.
  - If not, we can move it there not raising the cost.



p smallest

q 2nd smallest

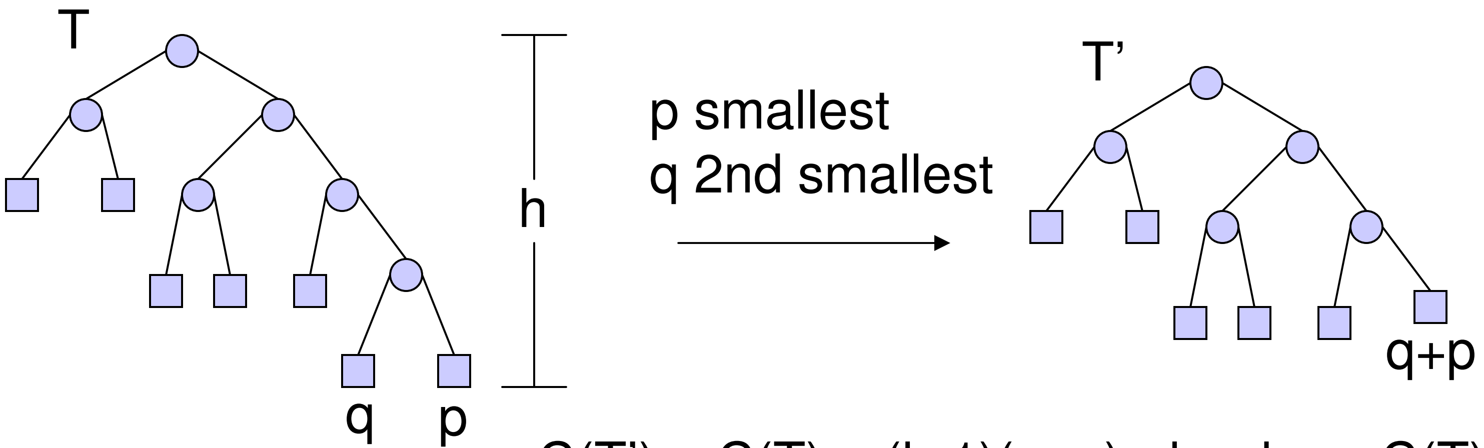
$q \leq r$

$k \leq h$

$$C(T') = C(T) + hq - hr + kr - kq = C(T) - (h-k)(r-q) \leq C(T)$$

Optimality Principle 3

- Assuming we have a Huffman tree T whose two lowest probability symbols are siblings at maximum depth, they can be replaced by a new symbol whose probability is the sum of their probabilities.
  - The resulting tree is optimal for the new symbol set.



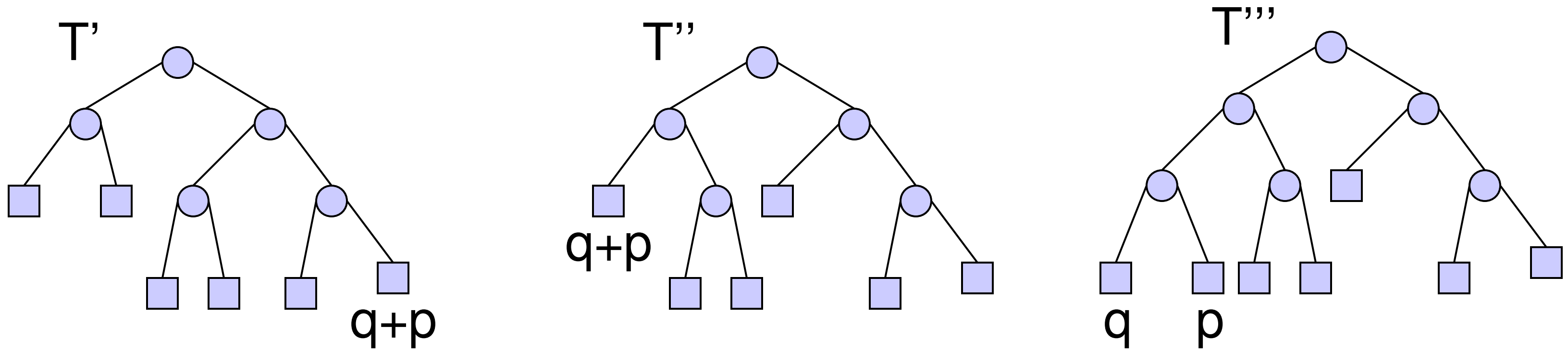
p smallest

q 2nd smallest

$$C(T') = C(T) + (h-1)(p+q) - hp - hq = C(T) - (p+q)$$

Optimality Principle 3 (cont’)

- If T’ were not optimal then we could find a lower cost tree T’’. This will lead to a lower cost tree T’’’ for the original alphabet.



C(T’’’) = C(T’’) + p + q < C(T’) + p + q = C(T) which is a contradiction

Recursive Huffman Tree Algorithm

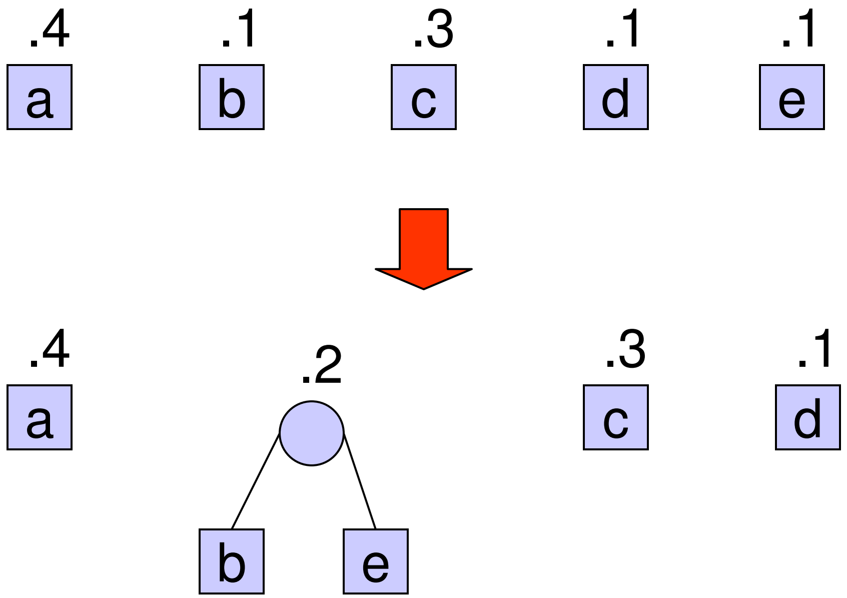
1. If there is just one symbol, a tree with one node is optimal. Otherwise
2. Find the two lowest probability symbols with probabilities p and q respectively.
3. Replace these with a new symbol with probability p + q.
4. Solve the problem recursively for new symbols.
5. Replace the leaf with the new symbol with an internal node with two children with the old symbols.

Iterative Huffman Tree Algorithm

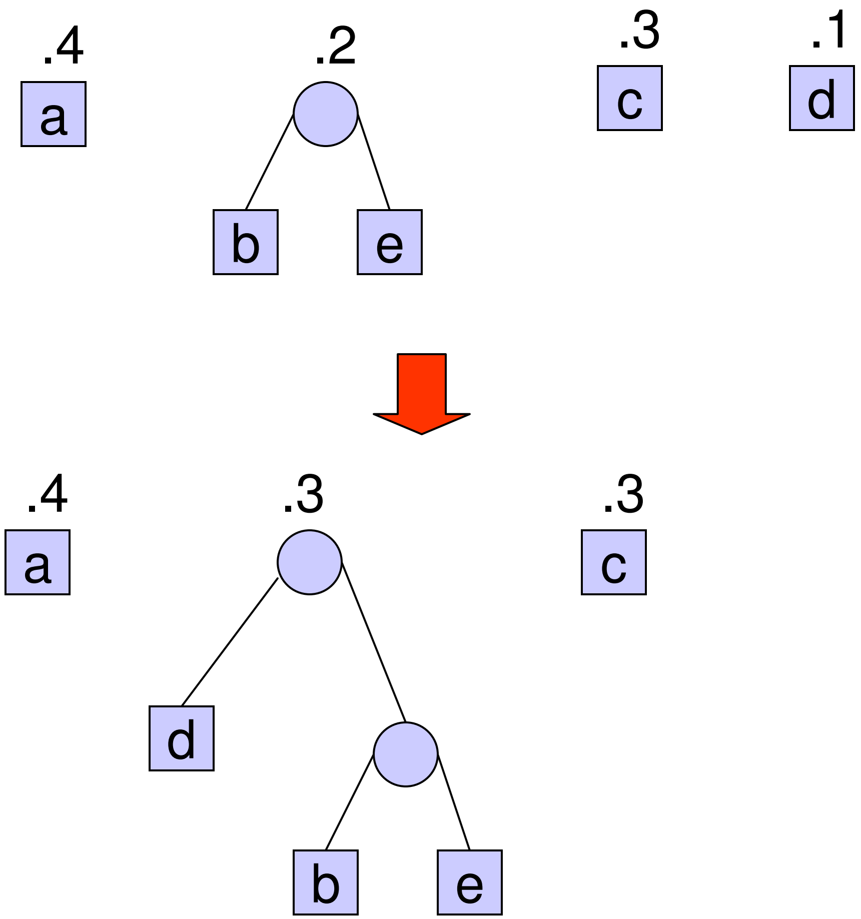
```
form a node for each symbol $a_i$ with weight $p_i$;
insert the nodes in a min priority queue ordered by probability;
while the priority queue has more than one element do
    min1 := delete-min;
    min2 := delete-min;
    create a new node n;
    n.weight := min1.weight + min2.weight;
    n.left := min1;
    n.right := min2;
    insert(n)
return the last node in the priority queue.
```

Example of Huffman Tree Algorithm (1)

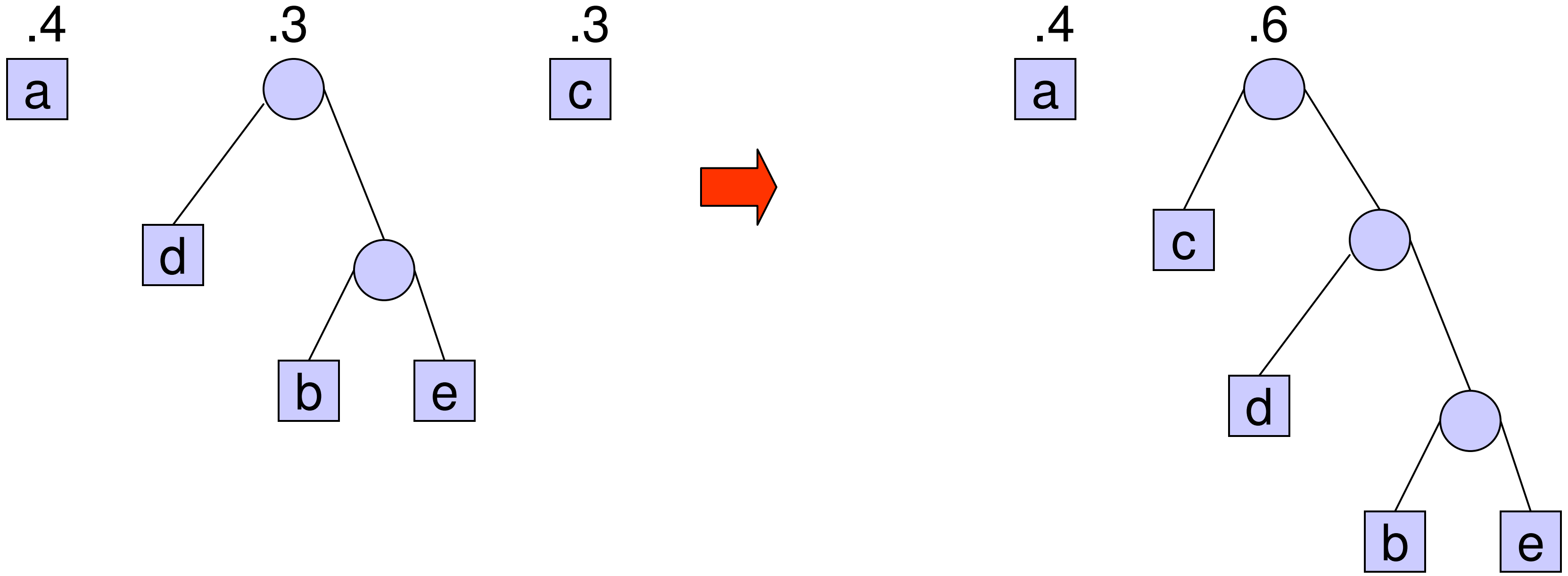
- P(a) =.4, P(b)=.1, P(c)=.3, P(d)=.1, P(e)=.1



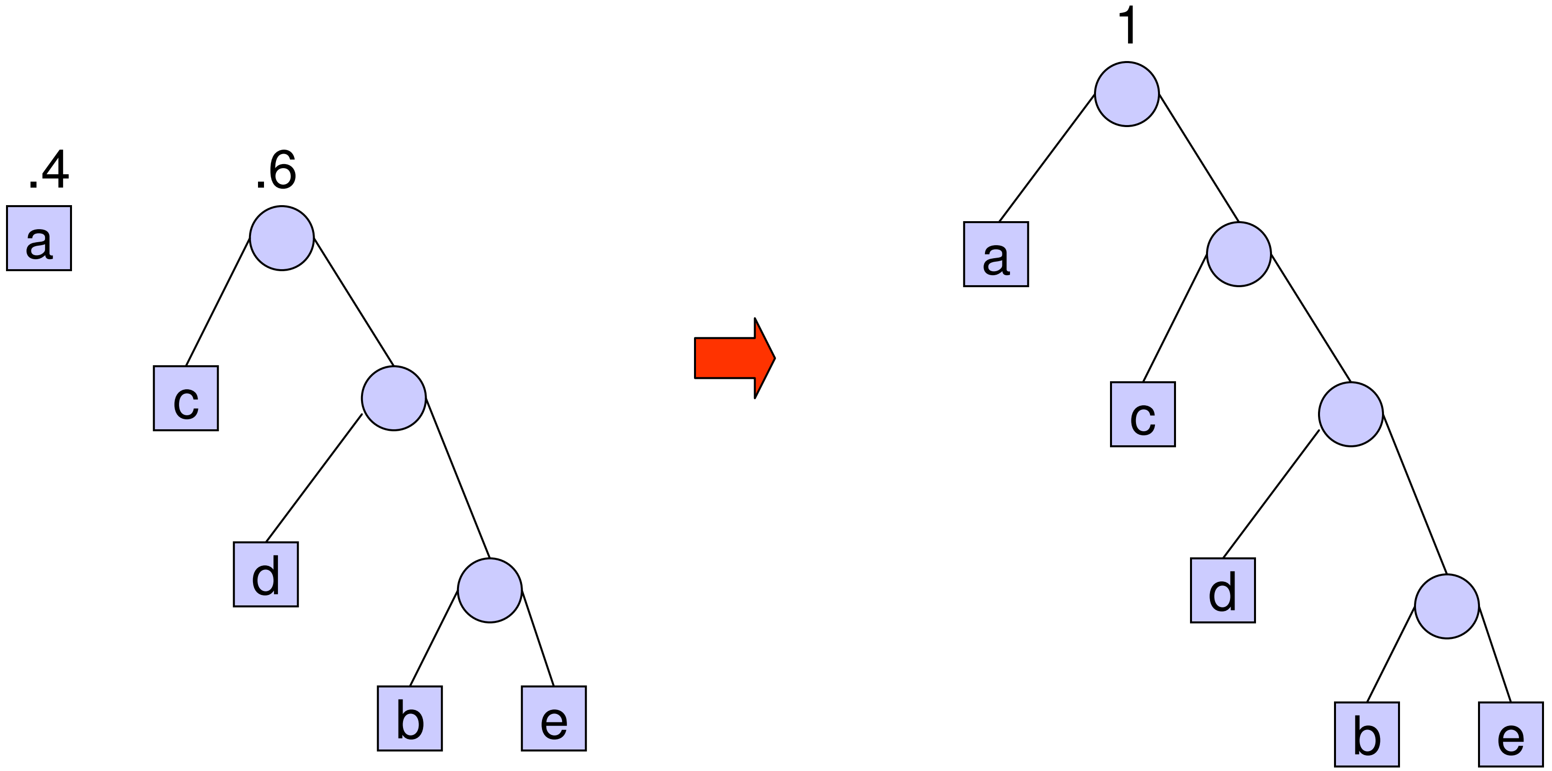
Example of Huffman Tree Algorithm (2)



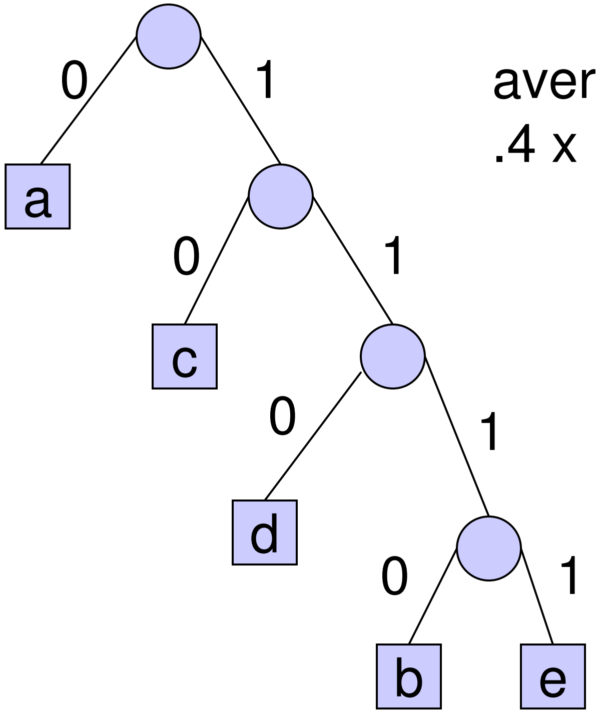
Example of Huffman Tree Algorithm (3)



Example of Huffman Tree Algorithm (4)



Huffman Code



average number of bits per symbol is
.4 x 1 + .1 x 4 + .3 x 2 + .1 x 3 + .1 x 4 = 2.1

a	0
b	1110
c	10
d	110
e	1111

Optimal Huffman Code vs. Entropy

- P(a) =.4, P(b)=.1, P(c)=.3, P(d)=.1, P(e)=.1

Entropy

$$H = -(.4 \times \log_2(.4) + .1 \times \log_2(.1) + .3 \times \log_2(.3) + .1 \times \log_2(.1) + .1 \times \log_2(.1))$$

$= 2.05$ bits per symbol

Huffman Code

$$HC = .4 \times 1 + .1 \times 4 + .3 \times 2 + .1 \times 3 + .1 \times 4$$

$= 2.1$ bits per symbol

pretty good!

In Class Exercise

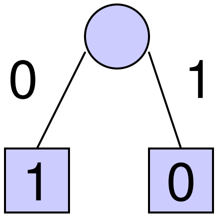
- P(a) = 1/2, P(b) = 1/4, P(c) = 1/8, P(d) = 1/16, P(e) = 1/16
- Compute the Optimal Huffman tree and its average bit rate.
- Compute the Entropy
- Compare
- Hint: For the tree change probabilities to be integers: a:8, b:4, c:2, d:1, e:1. Normalize at the end.

Quality of the Huffman Code

- The Huffman code is within one bit of the entropy lower bound.

$$H \le HC \le H + 1$$

- Huffman code does not work well with a two symbol alphabet.
  - Example: P(0) = 1/100, P(1) = 99/100
  - HC = 1 bits/symbol



  - H = -((1/100)*log$_2$(1/100) + (99/100)log$_2$(99/100))
    = .08 bits/symbol

Powers of Two

- If all the probabilities are powers of two then

$$HC = H$$

- Proof by induction on the number of symbols.

  Let $p_1 \leq p_2 \leq \ldots \leq p_n$ be the probabilities that add up to 1

  If $n = 1$ then $HC = H$ (both are zero).

  If $n > 1$ then $p_1 = p_2 = 2^{-k}$ for some k, otherwise the sum cannot add up to 1.

  Combine the first two symbols into a new symbol of probability $2^{-k} + 2^{-k} = 2^{-k+1}$.

Powers of Two (Cont.)

By the induction hypothesis

$$
\begin{aligned}
&HC(p_1+p_2,p_3,\ldots,p_n)=H(p_1+p_2,p_3,\ldots,p_n)\\
&=\text{-}(p_1+p_2)\log_2(p_1+p_2)-\sum_{i=3}^{n}p_i\log_2(p_i)\\
&=-2^{-k+1}\log_2(2^{-k+1})-\sum_{i=3}^{n}p_i\log_2(p_i)\\
&=-2^{-k+1}(\log_2(2^{-k})+1)-\sum_{i=3}^{n}p_i\log_2(p_i)\\
&=-2^{-k}\log_2(2^{-k})-2^{-k}\log_2(2^{-k})-\sum_{i=3}^{n}p_i\log_2(p_i)-2^{-k}-2^{-k}\\
&=-\sum_{i=1}^{n}p_i\log_2(p_i)-(p_1+p_2)\\
&=H(p_1,p_2,\ldots,p_n)-(p_1+p_2)
\end{aligned}
$$

Powers of Two (Cont.)

By the previous page,

$$HC(p_1 + p_2, p_3, \ldots, p_n) = H(p_1, p_2, \ldots, p_n) - (p_1 + p_2)$$

By the properties of Huffman trees (principle 3),

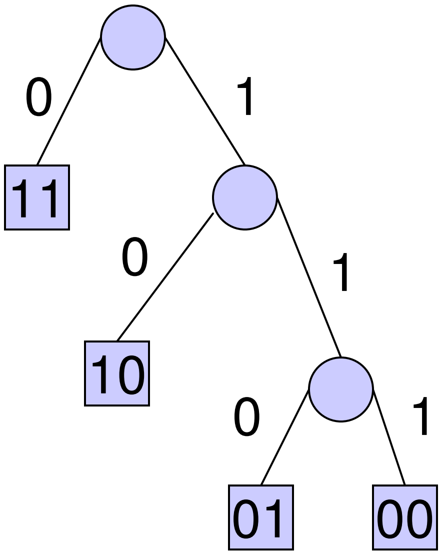
$$HC(p_1, p_2, \ldots, p_n) = HC(p_1 + p_2, p_3, \ldots, p_n) + (p_1 + p_2)$$

Hence,

$$HC(p_1, p_2, \ldots, p_n) = H(p_1, p_2, \ldots, p_n)$$

Extending the Alphabet

- Assuming independence P(ab) = P(a)P(b), so we can lump symbols together.
- Example: P(0) = 1/100, P(1) = 99/100
  - P(00) = 1/10000, P(01) = P(10) = 99/10000, P(11) = 9801/10000.



HC = 1.03 bits/symbol (2 bit symbol)
= .515 bits/bit

Still not that close to H = .08 bits/bit

Quality of Extended Alphabet

- Suppose we extend the alphabet to symbols of length k then

$$H \leq HC \leq H + 1/k$$

- Pros and Cons of Extending the alphabet
  - + Better compression
  - - 2^k symbols
  - - padding needed to make the length of the input divisible by k

Huffman Codes with Context

- Suppose we add a one symbol context. That is in compressing a string $x_1x_2...x_n$ we want to take into account x_{k-1} when encoding x_k.
  - New model, so entropy based on just independent probabilities of the symbols doesn't hold. The new entropy model (2nd order entropy) has for each symbol a probability for each other symbol following it.
  - Example: {a,b,c}

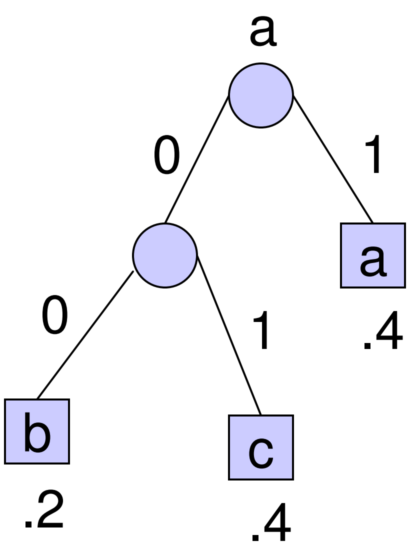
		next		
		a	b	c
prev	a	.4	.2	.4
	b	.1	.9	0
	c	.1	.1	.8

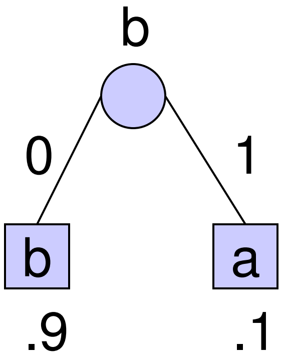
Multiple Codes

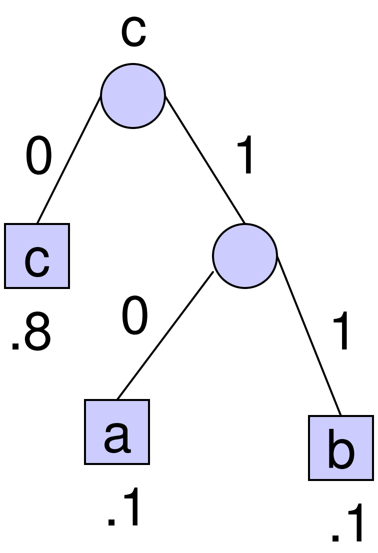
next

		a	b	c
prev	a	.4	.2	.4
	b	.1	.9	0
	c	.1	.1	.8

Code for first symbol	
a	00
b	01
c	10







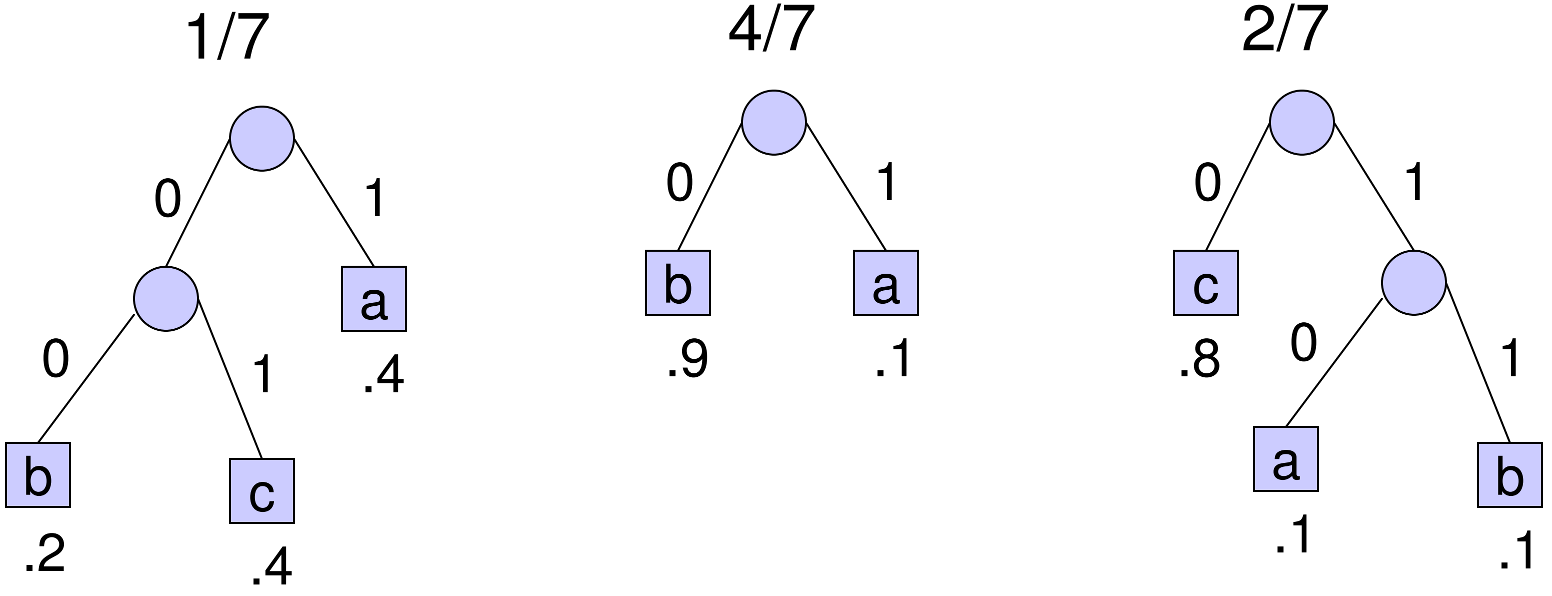
a b b a c c

$\underline{00}\ \underline{00}\ \underline{0}\ \underline{1}\ \underline{01}\ \underline{0}$

Average Bit Rate for Code

- $P(a) = .4\,P(a) + .1\,P(b) + .1\,P(c)$
  $P(b) = .2\,P(a) + .9\,P(b) + .1\,P(c)$
  $1 = P(a) + P(b) + P(c)$
- $0 = -.6\,P(a) + .1\,P(b) + .1\,P(c)$
  $0 = .2\,P(a) - .1\,P(b) + .1\,P(c)$
  $1 = P(a) + P(b) + P(c)$
- $P(a) = 1/7$, $P(b) = 4/7$, $P(c) = 2/7$

Average Bit Rate for Code



ABR = 1/7 (.6 x 2 + .4) + 4/7 (1) + 2/7 (.2 x 2 +.8)
= 8/7 = 1.14 bps

Complexity of Huffman Code Design

- Time to design Huffman Code is O(n log n) where n is the number of symbols.
  - Each step consists of a constant number of priority queue operations (2 deletemin's and 1 insert)

Approaches to Huffman Codes

1. Frequencies computed for each input
   - Must transmit the Huffman code or frequencies as well as the compressed input
   - Requires two passes
2. Fixed Huffman tree designed from training data
   - Do not have to transmit the Huffman tree because it is known to the decoder.
   - H.263 video coder
3. Adaptive Huffman code
   - One pass
   - Huffman tree changes as frequencies change

Run-Length Coding

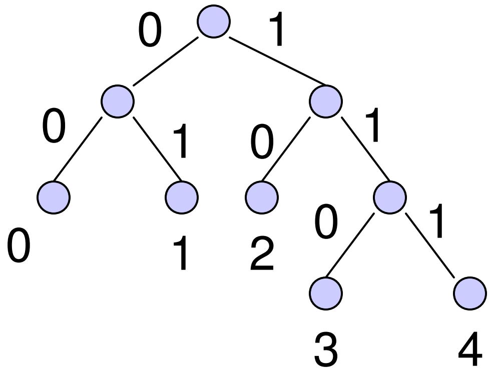
- Lots of 0’s and not too many 1’s.
  - Fax of letters
  - Graphics
- Simple run-length code
  - Input
    00000010000000001000000000010001001.....
  - Symbols
    6 9 10 3 2 ...
  - Code the bits as a sequence of integers
  - Problem: How long should the integers be?

Golomb Code of Order m
Variable Length Code for Integers

- Let $n = qm + r$ where $0 \leq r < m$.
  - Divide m into n to get the quotient q and remainder r.
- Code for n has two parts:

1. q is coded in unary
2. r is coded as a fixed prefix code

Example: m = 5



code for r

Example

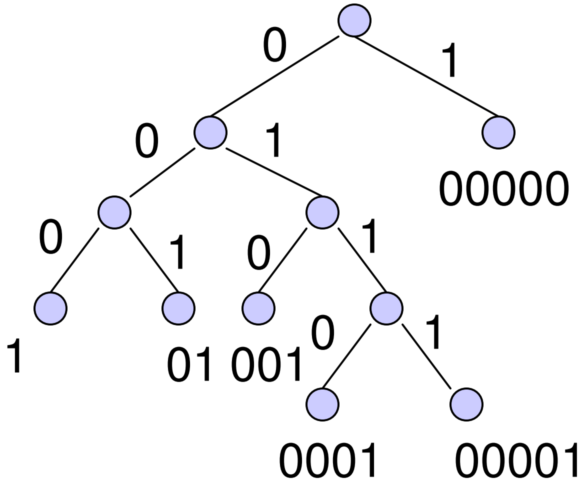
- n = qm + r is represented by:

$$\overbrace{11\cdots1}^{q}0\hat{r}$$

  - where $\hat{r}$ is the fixed prefix code for r
- Example (m = 5):

2	6	9	10	27
010	1001	10111	11000	11111010

Alternative Explanation Golomb Code of order 5



input	output
00000	1
00001	0111
0001	0110
001	010
01	001
1	000

Variable length to variable length code.

Run Length Example: m = 5

00000010000000001000000000010001001.....

1

00000010000000001000000000010001001.....

001

00000010000000001000000000010001001.....

1

00000010000000001000000000010001001.....

0111

In this example we coded 17 bits in only 9 bits.

Choosing m

- Suppose that 0 has the probability p and 1 has probability 1-p.
- The probability of 0^n1 is $p^n(1-p)$. The Golomb code of order

$$m = \left\lceil {}^{-1}\!/\!_{\log_2 p} \right\rceil$$

is optimal.

- Example: p = 127/128.

$$m = \left\lceil {}^{-1}\!/\!_{\log_2 (127/128)} \right\rceil = 89$$

Average Bit Rate for Golomb Code

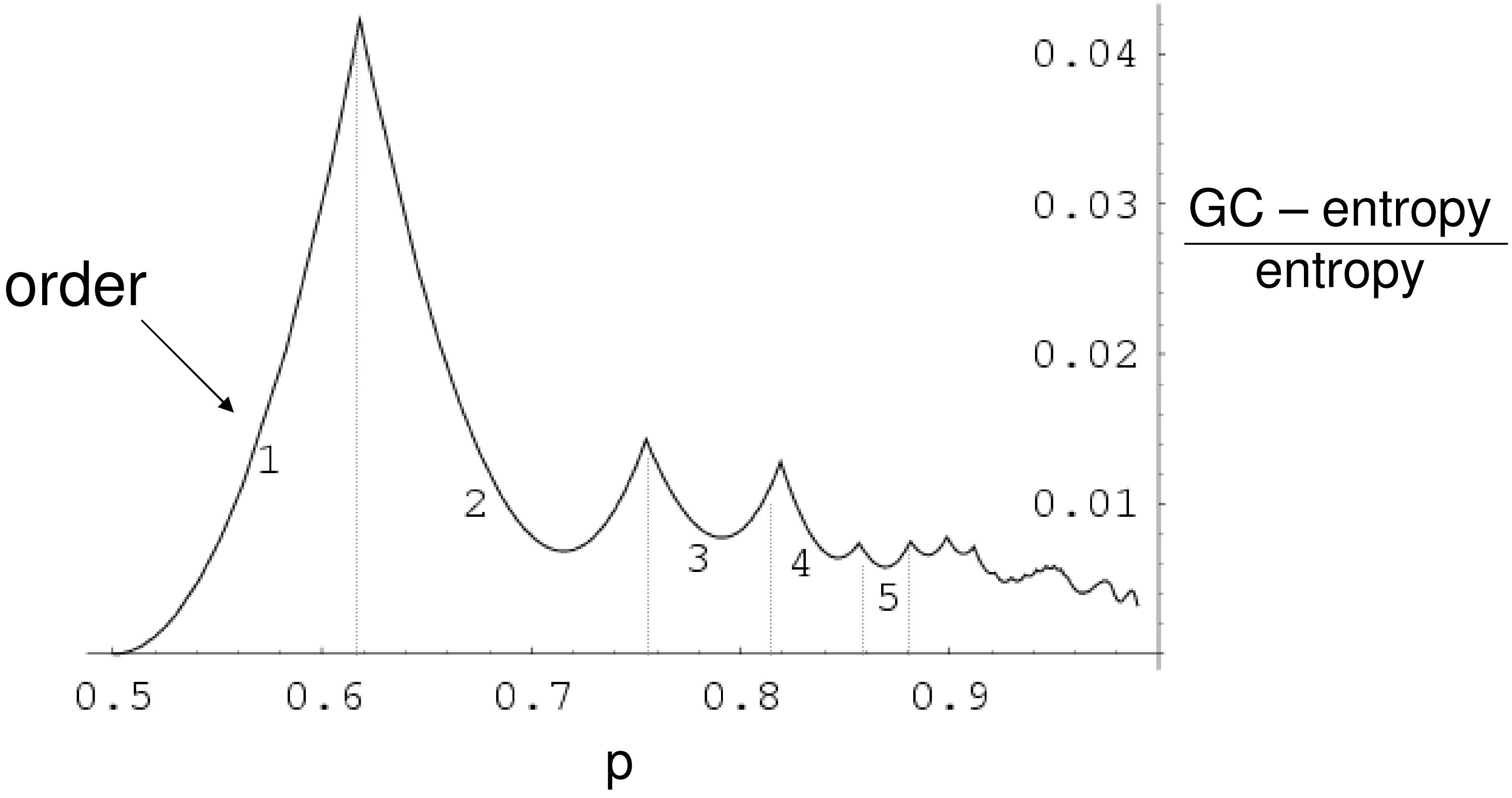
$$\text{Average Bit Rate} = \frac{\text{Average output code length}}{\text{Average input code length}}$$

- m = 4 as an example. With p as the probability of 0.

$$\text{ABR} = \frac{p^4 + 3p^3(1-p) + 3p^2(1-p) + 3p(1-p) + 3(1-p)}{4p^4 + 4p^3(1-p) + 3p^2(1-p) + 2p(1-p) + (1-p)}$$

ouput	1	011	010	001	000
input	0000	0001	001	01	1
weight	p^4	$p^3(1-p)$	$p^2(1-p)$	$p(1-p)$	$1-p$

Comparison of GC with Entropy



Notes on Golomb codes

- Useful for binary compression when one symbol is much more likely than another.
  - binary images
  - fax documents
  - bit planes for wavelet image compression
- Need a parameter (the order)
  - training
  - adaptively learn the right parameter
- Variable-to-variable length code
- Last symbol needs to be a 1
  - coder always adds a 1
  - decoder always removes a 1

Tunstall Codes

- Variable-to-fixed length code
- Example

input	output
a	000
b	001
ca	010
cb	011
cca	100
ccb	101
ccc	110

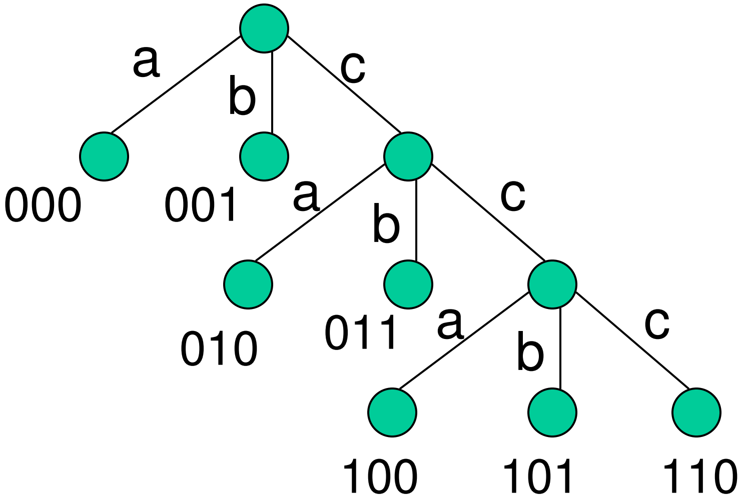
a	b	cca	cb	ccc	...
000	001	110	011	110	...

Tunstall code Properties

1. No input code is a prefix of another to assure unique encodability.
2. Minimize the number of bits per symbol.

Prefix Code Property

a	000
b	001
ca	010
cb	011
cca	100
ccb	101
ccc	110



Unused output code is 111.

Use for unused code

- Consider the string “cc”, if it occurs at the end of the data. It does not have a code.
- Send the unused code and some fixed code for the cc.
- Generally, if there are k internal nodes in the prefix tree then there is a need for k-1 fixed codes.

Designing a Tunstall Code

- Suppose there are m initial symbols.
- Choose a target output length n where $2^n > m$.

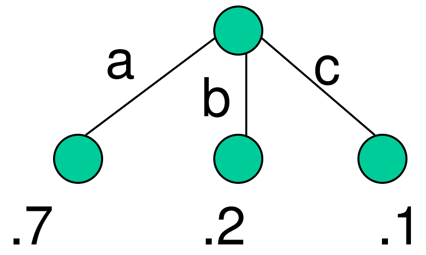
1. Form a tree with a root and m children with edges labeled with the symbols.
2. If the number of leaves is $> 2^n – m$ then halt.*
3. Find the leaf with highest probability and expand it to have m children.** Go to 2.

* In the next step we will add m-1 more leaves.

** The probability is the product of the probabilities of the symbols on the root to leaf path.

Example

- P(a) = .7, P(b) = .2, P(c) = .1
- n = 3



Example

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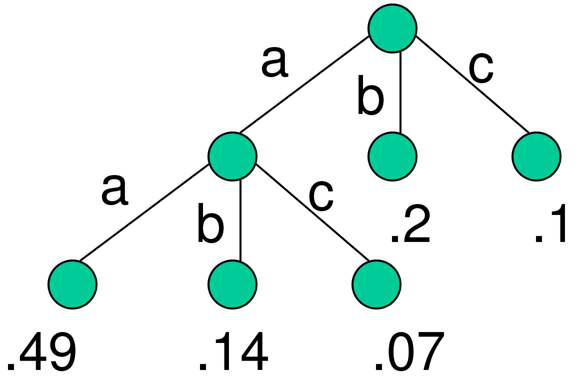
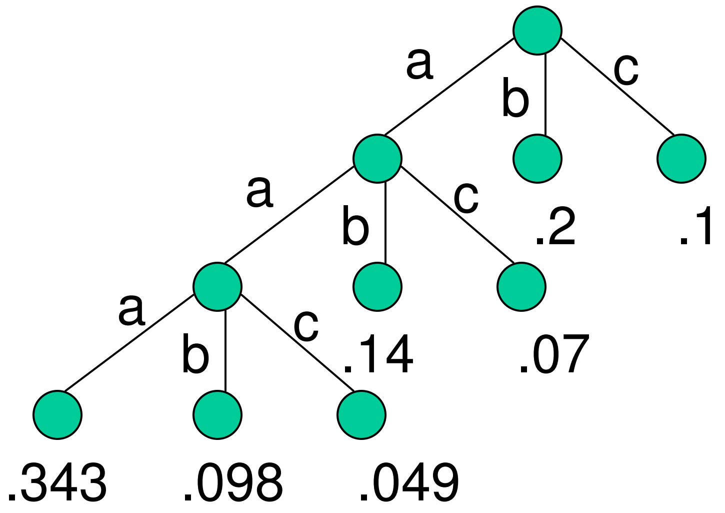


Example

- P(a) = .7, P(b) = .2, P(c) = .1
- n = 3



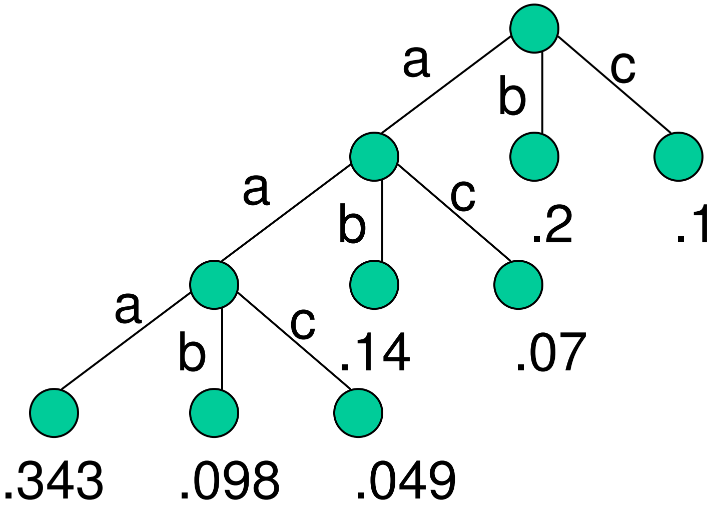
aaa	000
aab	001
aac	010
ab	011
ac	100
b	101
c	110

Bit Rate of Tunstall

- The length of the output code divided by the average length of the input code.
- Let p_i be the probability of, and r_i the length of input code i ($1 \leq i \leq s$) and let n be the length of the output code.

$$\text{Average bit rate} = \frac{n}{\sum_{i=1}^{s} p_i r_i}$$

Example



aaa	.343	000
aab	.098	001
aac	.049	010
ab	.14	011
ac	.07	100
b	.2	101
c	.1	110

ABR = 3/[3 (.343 + .098 + .049) + 2 (.14 + .07) + .2 + .1]

= 1.37 bits per symbol

Entropy = 1.16 bits per symbol

Notes on Tunstall Codes

- Variable-to-fixed length code
- Error resilient
  - A flipped bit will introduce just one error in the output
  - Huffman is not error resilient. A single bit flip can destroy the code.