CSEP 590
Data Compression
Autumn 2007

Sequitur

Sequitur

- Nevill-Manning and Witten, 1996.
- Uses a context-free grammar (without recursion) to represent a string.
- The grammar is inferred from the string.
- If there is structure and repetition in the string then the grammar may be very small compared to the original string.
- Clever encoding of the grammar yields impressive compression ratios.
- Compression plus structure!

Context-Free Grammars

- Invented by Chomsky in 1959 to explain the grammar of natural languages.
- Also invented by Backus in 1959 to generate and parse Fortran.
- Example:
  - terminals: b, e
  - non-terminals: S, A
  - Production Rules:
    $S \rightarrow SA$, $S \rightarrow A$, $A \rightarrow bSe$, $A \rightarrow be$
  - S is the start symbol

Context-Free Grammar Example

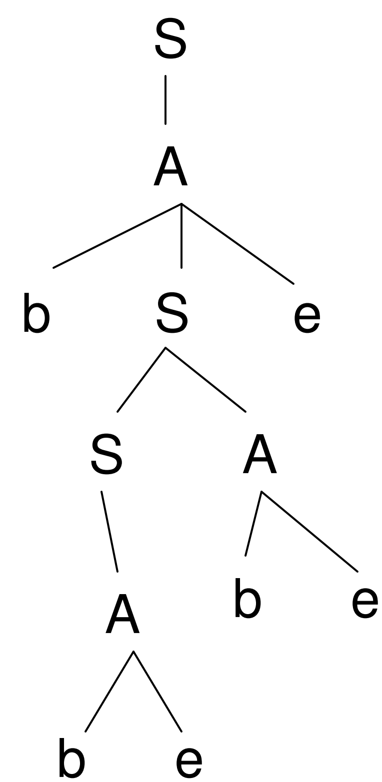
- $S \rightarrow SA$
  $S \rightarrow A$
  $A \rightarrow bSe$
  $A \rightarrow be$

Example: b and e matched as parentheses

derivation of bbebee

S
A
bSe
bSAe
bAAe
bbeAe
bbebee

hierarchical parse tree



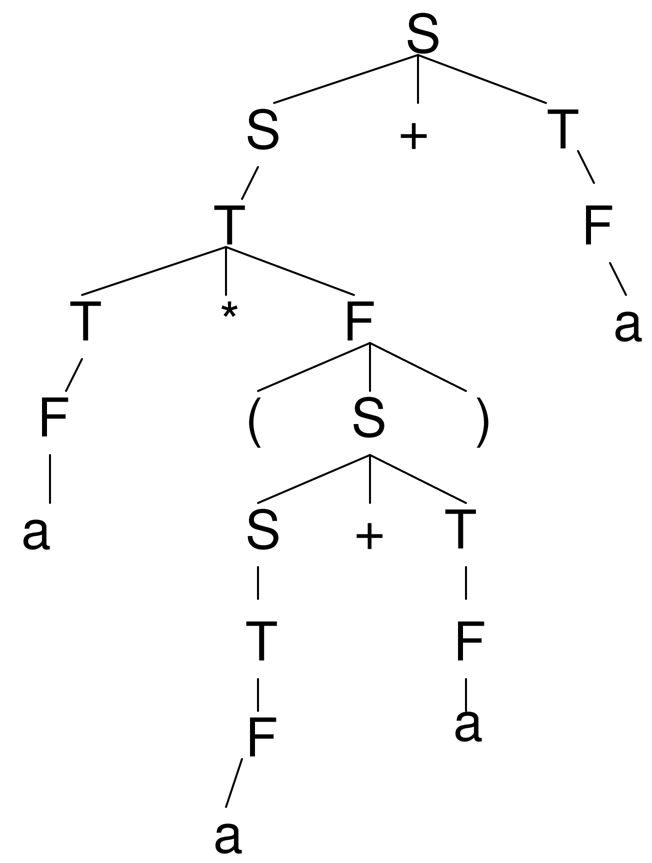
Arithmetic Expressions

- S → S + T
  S → T
  T → T*F
  T → F
  F → a
  F →(S)

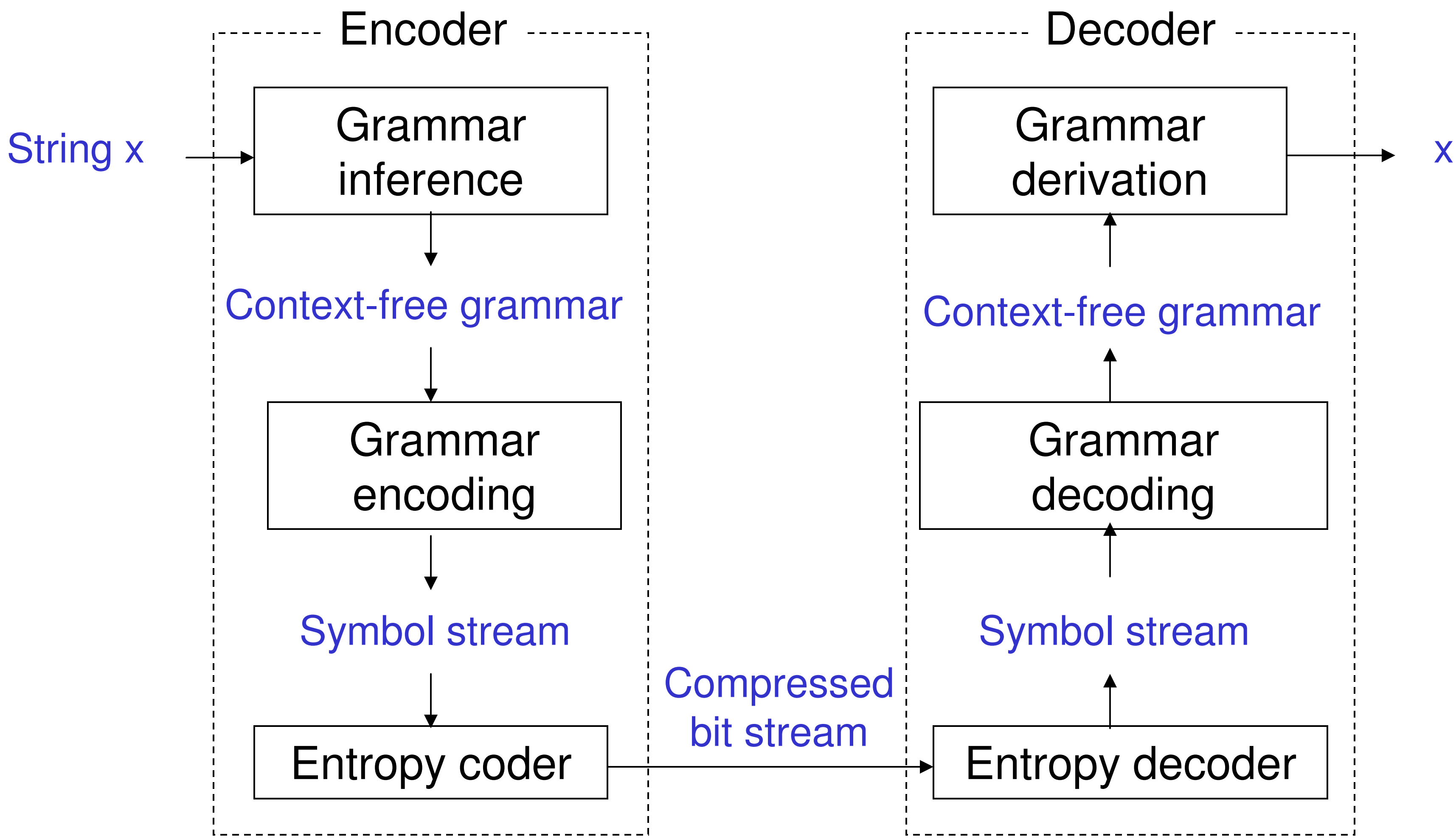
derivation of a * (a + a) + a

S
S+T
T+T
T*F+T
F*F+T
a*F+T
a*(S)+F
a*(S+F)+T
a*(T+F)+T
a*(F+F)+T
a*(a+F)+T
a*(a+a)+T
a*(a+a)+F
a*(a+a)+a

parse tree



Overview of Grammar Compression



Sequitur Principles

- Digram Uniqueness:
  - no pair of adjacent symbols (digram) appears more than once in the grammar.
- Rule Utility:
  - Every production rule is used more than once.
- These two principles are maintained as an invariant while inferring a grammar for the input string.

Sequitur Example (1)

bbebeebebebbebee

$S \rightarrow b$

Sequitur Example (2)

bbebeebebebbebee

$S \rightarrow bb$

Sequitur Example (3)

bbebeebebebbebee

S $\rightarrow$ bbe

Sequitur Example (4)

bbebeebebebbebee

S $\rightarrow$ bbeb

Sequitur Example (5)

bbebeebebebbebee

$S \rightarrow$ bbebe

Enforce digram uniqueness.
be occurs twice.
Create new rule $A \rightarrow$ be.

Sequitur Example (6)

bbebeebebebbebee

$S \rightarrow bAA$
$A \rightarrow be$

Sequitur Example (7)

bbebeebebebbebee

$S \rightarrow bAAe$
$A \rightarrow be$

Sequitur Example (8)

bbebeebebebbebee

$S \rightarrow bAAeb$
$A \rightarrow be$

Sequitur Example (9)

bbebeebebebbebee

S $\rightarrow$ bAAebe
A $\rightarrow$ be

Enforce digram uniqueness.
be occurs twice.
Use existing rule A $\rightarrow$ be.

Sequitur Example (10)

bbebeebebebbebee

$S \rightarrow bAAeA$
$A \rightarrow be$

Sequitur Example (11)

bbebeebebebbebee

$S \rightarrow$ bAAeAb
$A \rightarrow$ be

Sequitur Example (12)

bbebeebebebbebee

$S \rightarrow$ bAAeAbe
$A \rightarrow$ be

Enforce digram uniqueness.
be occurs twice.
Use existing rule $A \rightarrow$ be.

Sequitur Example (13)

bbebeebebebbebee

$S \rightarrow bAAeAA$
$A \rightarrow be$

Enforce digram uniqueness
AA occurs twice.
Create new rule $B \rightarrow AA$.

Sequitur Example (14)

bbebeebebebbebee

$S \rightarrow bBeB$
$A \rightarrow be$
$B \rightarrow AA$

Sequitur Example (15)

bbebeebebebbebee

$S \rightarrow bBeBb$
$A \rightarrow be$
$B \rightarrow AA$

Sequitur Example (16)

bbebeebebebbebee

$S \rightarrow$ bBeBbb
$A \rightarrow$ be
$B \rightarrow$ AA

Sequitur Example (17)

bbebeebebebbebee

S $\rightarrow$ bBeBbbe	Enforce digram uniqueness.
A $\rightarrow$ be	be occurs twice.
B $\rightarrow$ AA	Use existing rule A $\rightarrow$ be.

Sequitur Example (18)

bbebeebebebbebee

S → bBeBbA
A → be
B → AA

Sequitur Example (19)

bbebeebebebbebee

S -> bBeBbAb
A -> be
B -> AA

Sequitur Example (20)

bbebeebebebbebee

$S \rightarrow$ bBeBbAbe
$A \rightarrow$ be
$B \rightarrow$ AA

Enforce digram uniqueness.
be occurs twice.
Use existing rule $A \rightarrow$ be.

Sequitur Example (21)

bbebeebebebbebee

S → bBeBbAA	Enforce digram uniqueness.
A → be	AA occurs twice.
B → AA	Use existing rule B → AA.

Sequitur Example (22)

bbebeebebebbebee

$S \rightarrow$ bBeBbB
$A \rightarrow$ be
$B \rightarrow$ AA

Enforce digram uniqueness.
bB occurs twice.
Create new rule $C \rightarrow$ bB.

Sequitur Example (23)

<u>bbebeebebebbebe</u>e

$S \rightarrow$ CeBC
$A \rightarrow$ be
$B \rightarrow$ AA
$C \rightarrow$ bB

Sequitur Example (24)

bbebeebebebbebee

$S \rightarrow$ CeBCe
$A \rightarrow$ be
$B \rightarrow$ AA
$C \rightarrow$ bB

Enforce digram uniqueness.
Ce occurs twice.
Create new rule $D \rightarrow$ Ce.

Sequitur Example (25)

bbebeebebebbebee

$S \rightarrow DBD$
$A \rightarrow be$
$B \rightarrow AA$
$C \rightarrow bB$
$D \rightarrow Ce$

Enforce rule utility.
C occurs only once.
Remove $C \rightarrow bB$.

Sequitur Example (26)

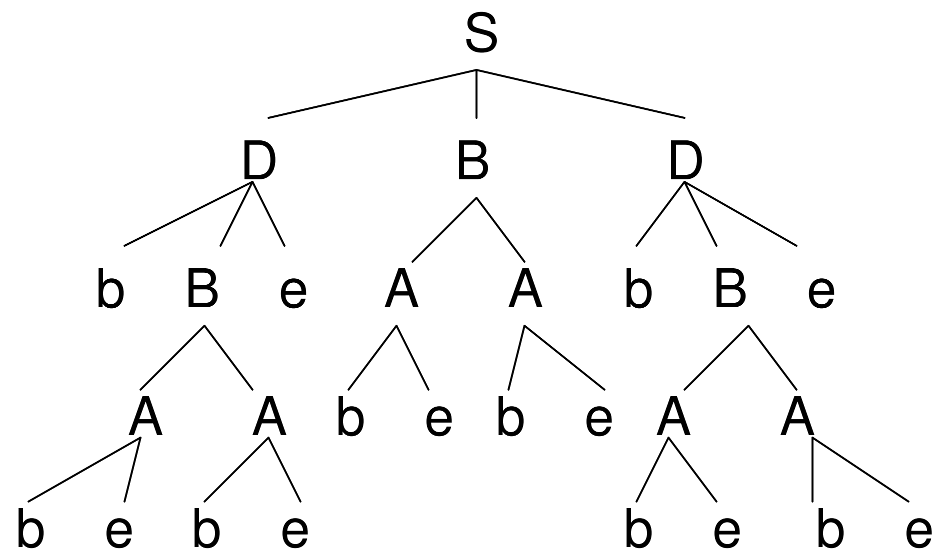
bbebeebebebbebee

$S \rightarrow DBD$
$A \rightarrow be$
$B \rightarrow AA$
$D \rightarrow bBe$

The Hierarchy

<u>bbebeebebebbebee</u>

$S \rightarrow DBD$
$A \rightarrow be$
$B \rightarrow AA$
$D \rightarrow bBe$



Is there compression? In this small example, probably not.

Sequitur Algorithm

Input the first symbol s to create the production $S \rightarrow s$;

repeat

match an existing rule:

$A \rightarrow \ldots.XY\ldots$ $\quad\longrightarrow\quad$ $A \rightarrow \ldots.B\ldots.$

$B \rightarrow XY$ $\quad\quad\quad\quad$ $B \rightarrow XY$

create a new rule:

$A \rightarrow \ldots.XY\ldots.$ $\quad\longrightarrow\quad$ $A \rightarrow \ldots.C\ldots.$

$B \rightarrow \ldots.XY\ldots.$ $\quad\quad\quad\quad$ $B \rightarrow \ldots.C\ldots.$

$C \rightarrow XY$

remove a rule:

$A \rightarrow \ldots.B\ldots.$ $\quad\longrightarrow\quad$ $A \rightarrow \ldots.\ X_1X_2\ldots X_k\ \ldots.$

$B \rightarrow X_1X_2\ldots X_k$

input a new symbol:

$S \rightarrow X_1X_2\ldots X_k$ $\quad\longrightarrow\quad$ $S \rightarrow X_1X_2\ldots X_k s$

until no symbols left

Exercise

Use Sequitur to construct a grammar for aaaaaaaaaa = a^{10}

Complexity

- The number of non-input sequitur operations applied $< 2n$ where n is the input length.
- Since each operation takes constant time, sequitur is a linear time algorithm

Amortized Complexity Argument

- Let m = # of non-input sequitur operations.
  Let n = input length. Show $m \leq 2n$.
- Let s = the sum of the right hand sides of all the production rules. Let r = the number of rules.
- We evaluate 2s - r.
- Initially 2s - r = 1 because s = 1 and r = 1.
- 2s - r > 0 at all times because each rule has at least 1 symbol on the right hand side.

Sequitur Rule Complexity

- Digram Uniqueness - match an existing rule.

$A \rightarrow ....XY...$		$A \rightarrow ....B....$	s	r	2s - r
$B \rightarrow XY$	$\longrightarrow$	$B \rightarrow XY$	-1	0	-2

- Digram Uniqueness - create a new rule.

$A \rightarrow ....XY....$		$A \rightarrow ....C....$	s	r	2s - r
$B \rightarrow ....XY....$	$\longrightarrow$	$B \rightarrow ....C....$	0	1	-1
		$C \rightarrow XY$			

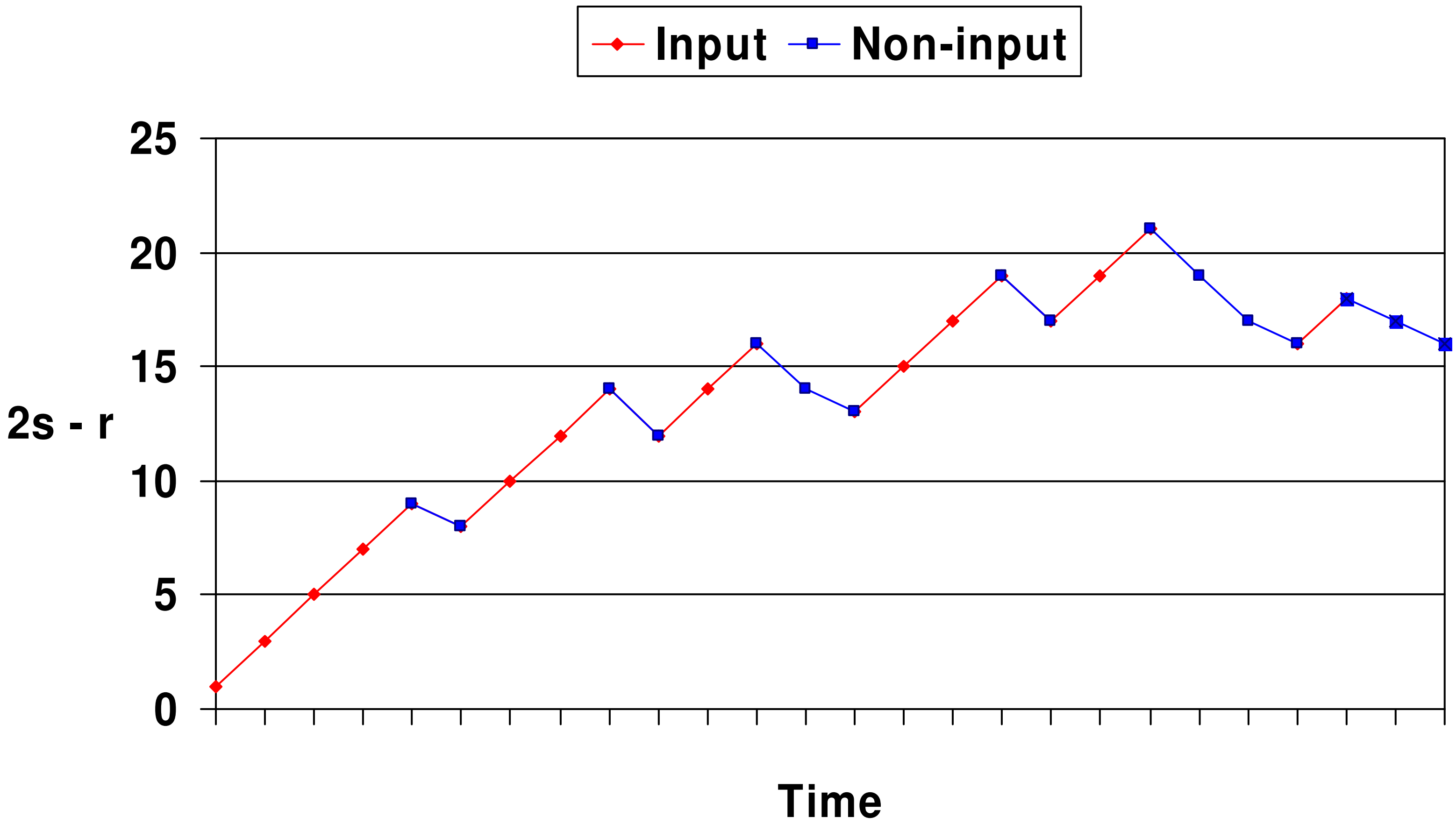
- Rule Utility - Remove a rule.

$A \rightarrow ....B....$			s	r	2s - r
$B \rightarrow X_1X_2...X_k$	$\longrightarrow$	$A \rightarrow .... X_1X_2...X_k ....$	-1	-1	-1

Amortized Complexity Argument

- 2s - r $\geq$ 0 at all times because each rule has at least 1 symbol on the right hand side.
- 2s - r increases by 2 for every input operation.
- 2s - r decreases by at least 1 for each non-input sequitur rule applied.
- n = number of input symbols
  m = number of non-input operations
- $2n - m \geq 0$. $m \leq 2n$.

Amortized Complexity Argument

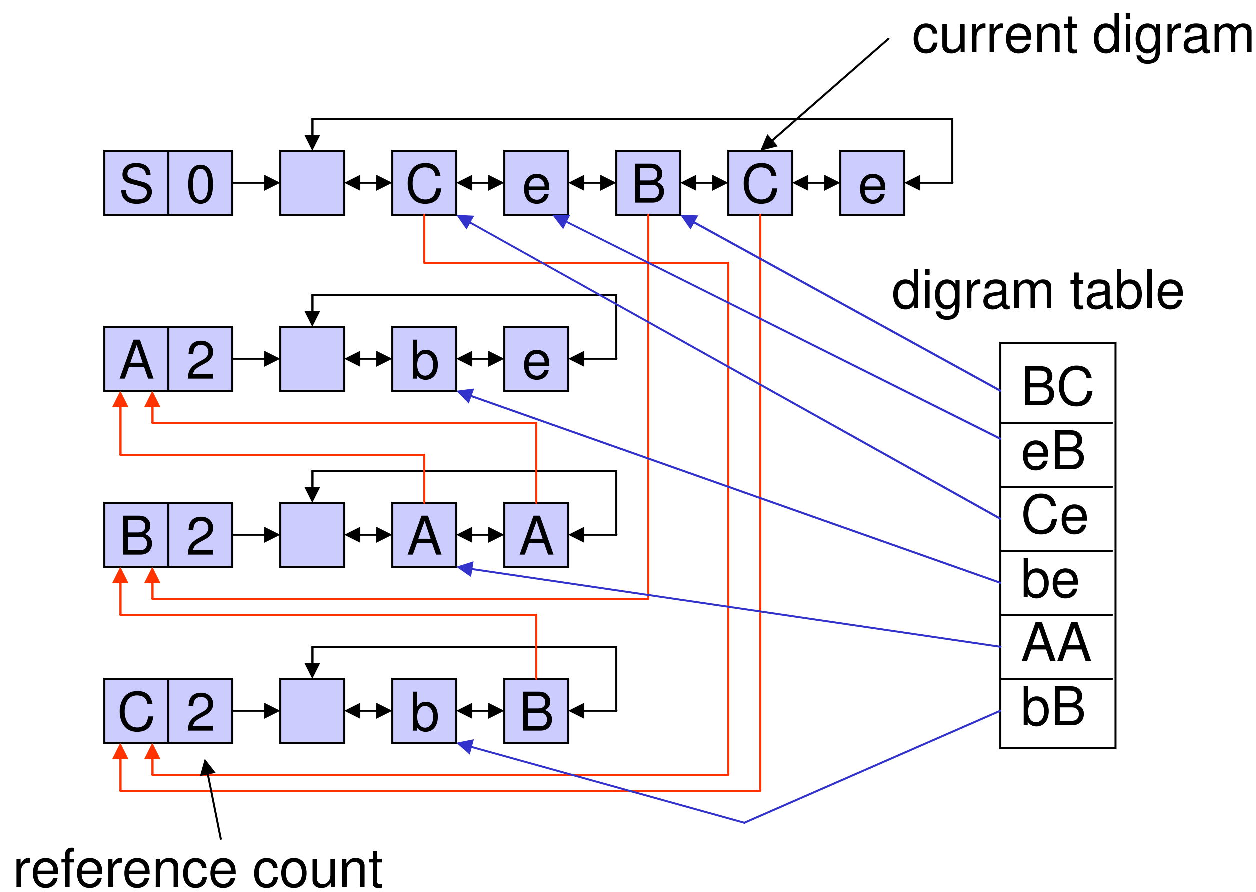


Linear Time Algorithm

- There is a data structure to implement all the sequitur operations in constant time.
  - Production rules in an array of doubly linked lists.
  - Each production rule has reference count of the number of times used.
  - Each nonterminal points to its production rule.
  - Digrams stored in a hash table for quick lookup.

Data Structure Example

S → CeBCe
A → be
B → AA
C → bB



Basic Encoding a Grammar

Grammar

$S \to DBD$
$A \to be$
$B \to AA$
$D \to bBe$

Symbol Code

b	000
e	001
A	010
B	011
D	100
#	101

No code for S needed

Grammar Code

D	B	D	#	b	e	#	A	A	#	b	B	e	
100	011	100	101	000	001	101	010	010	101	000	011	001	39 bits

$$|\text{Grammar Code}| = (s+r-1)\lceil \log_2(r+a) \rceil$$

r = number of rules
s = sum of right hand sides
a = number in original symbol alphabet

Better Encoding of the Grammar

- Nevill-Manning and Witten suggest a more efficient encoding of the grammar that uses LZ77 ideas.
  - Send the right hand side of the S production.
  - The first time a nonterminal is sent, its right hand side is transmitted instead.
  - The second time a nonterminal is sent as a triple [*i*, *j*, *d*], which says the right hand side starts at position *j* in production rule *i* and is *d* long. A new production rule is then added to a dictionary.
  - Subsequently, the nonterminal is represented by the index of the production rule.

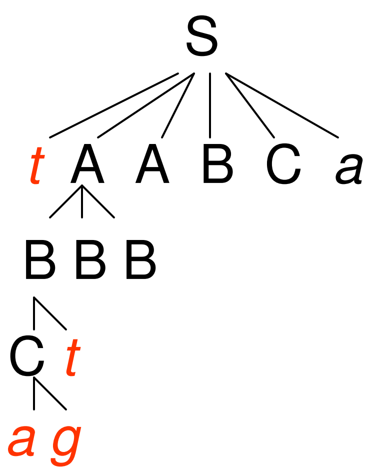
Transmission Example

S → *t*AABC*a*
A → BBB
B → C*t*
C → *ag*

T = Transmitted

T *tagt*

X_0 *tagt* $l_0 = 4$



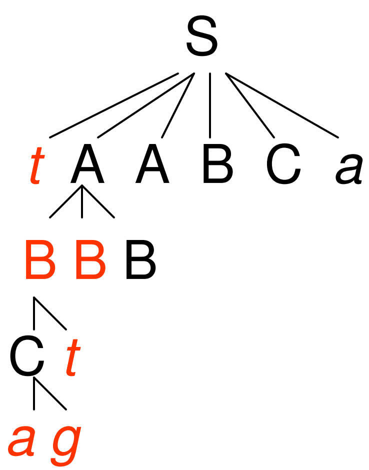
Transmission Example

$S \rightarrow tAABCa$
$A \rightarrow BBB$
$B \rightarrow Ct$
$C \rightarrow ag$

T = Transmitted

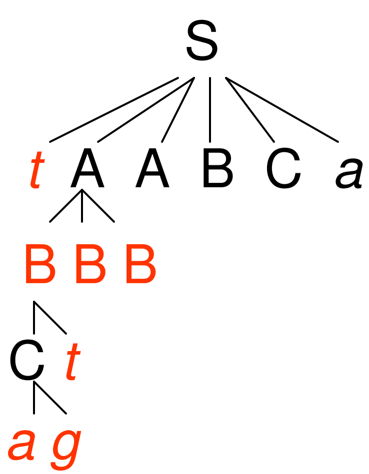
T *tagt* [0, 1, 3]

X_0 $t\,X_1X_1$ $l_0 = 3$
X_1 agt $l_1 = 3$



Transmission Example

$S \rightarrow tAABCa$
$A \rightarrow BBB$
$B \rightarrow Ct$
$C \rightarrow ag$



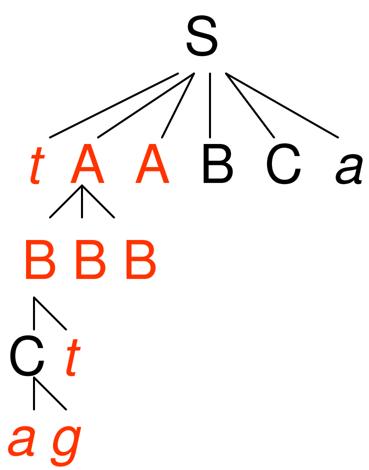
T = Transmitted

T *tagt* [0, 1, 3] 1

X_0 $t\, X_1 X_1 X_1$ $l_0 = 4$
X_1 agt $l_1 = 3$

Transmission Example

$S \rightarrow t$AABCa
$A \rightarrow$ BBB
$B \rightarrow$ Ct
$C \rightarrow ag$



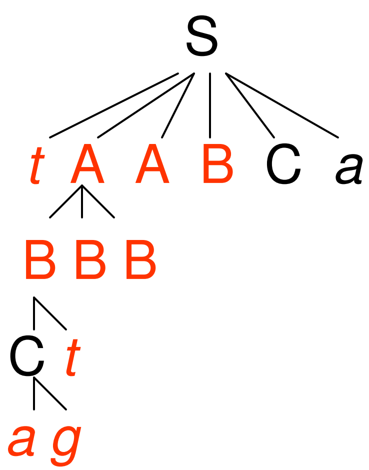
T = Transmitted

T *tagt* [0, 1, 3] 1 [0, 1, 3]

X_0 $t\,X_2X_2$ $l_0 = 3$
X_1 agt $l_1 = 3$
X_2 $X_1X_1X_1$ $l_2 = 3$

Transmission Example

$S \rightarrow tAABCa$
$A \rightarrow BBB$
$B \rightarrow Ct$
$C \rightarrow ag$



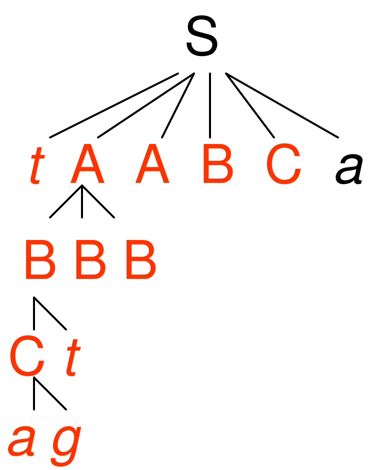
T = Transmitted

T *tagt* [0, 1, 3] 1 [0, 1, 3] 1

X_0	$t\, X_2X_2X_1$	$l_0 = 4$
X_1	agt	$l_1 = 3$
X_2	$X_1X_1X_1$	$l_2 = 3$

Transmission Example

$S \rightarrow tAABCa$
$A \rightarrow BBB$
$B \rightarrow Ct$
$C \rightarrow ag$



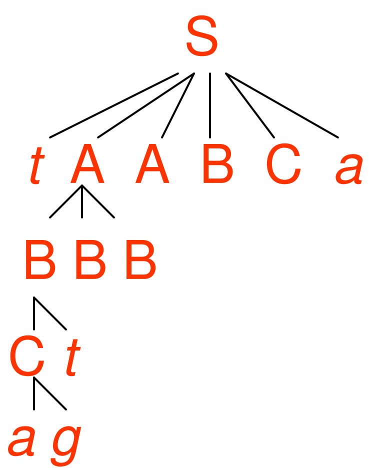
T = Transmitted

T *tagt* [0, 1, 3] 1 [0, 1, 3] 1 [1, 0, 2]

X_0 $t X_2 X_2 X_1 X_3$ $l_0 = 5$
X_1 $X_3 t$ $l_1 = 2$
X_2 $X_1 X_1 X_1$ $l_2 = 3$
X_3 ag $l_3 = 2$

Transmission Example

$S \rightarrow tAABCa$
$A \rightarrow BBB$
$B \rightarrow Ct$
$C \rightarrow ag$



T = Transmitted

T   *tagt* [0, 1, 3] 1 [0, 1, 3] 1 [1, 0, 2] *a*

X_0	$t\, X_2 X_2 X_1\, X_3\, a$	$l_0 = 6$
X_1	$X_3 t$	$l_1 = 2$
X_2	$X_1 X_1 X_1$	$l_2 = 3$
X_3	ag	$l_3 = 2$

Kieffer-Yang Improvement

- Kieffer and Yang developed a theoretical framework for studying these types of grammars in 2000.
  - KY is universal; it achieves entropy in the limit
- Add to sequitur Reduction Rule 5:

S → AB		S → AA	
A → CD		A → CD	Adding this constraint makes sequitur universal.
B → *a*E	⇒	~~B → *a*E~~	
C → *ab*		C → *ab*	
D → *cd*		D → *cd*	
E → *b*D		~~E → *b*D~~	

<A> = <B> = *abcd*

Compression Quality

- Neville-Manning and Witten 1997

	size	comp	gzip	sequitur	PPMC	bzip2
bib	111261	3.35	2.51	2.48	2.12	1.98
book	768771	3.46	3.35	2.82	2.52	2.42
geo	102400	6.08	5.34	4.74	5.01	4.45
obj2	246814	4.17	2.63	2.68	2.77	2.48
pic	513216	0.97	0.82	0.90	0.98	0.78
progc	38611	3.87	2.68	2.83	2.49	2.53

= First; = Second; = Third.

Files from the Calgary Corpus
Units in bits per character (8 bits)
compress - based on LZW
gzip - based on LZ77
PPMC - adaptive arithmetic coding with context
bzip2 – Burrows-Wheeler block sorting

Notes on Sequitur

- Yields compression and hierarchical structure simultaneously.
- With clever encoding is competitive with the best of the standards.
- The grammar size is not close to approximation algorithms
  - Upper = $O((n/\log n)^{3/4})$; Lower = $\Omega(n^{1/3})$. (Lehman, 2002)
- *But!* Practical linear time encoding and decoding.

Other Grammar Based Methods

- Longest Match
- Most frequent digram
- Match producing the best compression