

# CSE-P590a

## Robotics

### **Probabilistic Robotics**

Probabilities

Bayes rule

Bayes filters

# Probabilistic Robotics

Key idea: Explicit representation of uncertainty

(using the calculus of probability theory)

- Perception = state estimation
- Action = utility optimization

# Discrete Random Variables

- $X$  denotes a **random variable**.
- $X$  can take on a countable number of values in  $\{x_1, x_2, \dots, x_n\}$ .
- $P(X=x_i)$ , or  $P(x_i)$ , is the **probability** that the random variable  $X$  takes on value  $x_i$ .
- $P(\cdot)$  is called **probability mass function**.
- E.g.  $P(\text{Room}) = \langle 0.7, 0.2, 0.08, 0.02 \rangle$

# Joint and Conditional Probability

- $P(X=x \text{ and } Y=y) = P(x,y)$
- If  $X$  and  $Y$  are independent then
$$P(x,y) = P(x) P(y)$$
- $P(x | y)$  is the probability of  $x$  given  $y$ 
$$P(x | y) = P(x,y) / P(y)$$
$$P(x,y) = P(x | y) P(y)$$
- If  $X$  and  $Y$  are independent then
$$P(x | y) = P(x)$$

# Law of Total Probability, Marginals

## Discrete case

$$\sum_x P(x) = 1$$

$$P(x) = \sum_y P(x, y)$$

$$P(x) = \sum_y P(x | y)P(y)$$

## Continuous case

$$\int p(x) dx = 1$$

$$p(x) = \int p(x, y) dy$$

$$p(x) = \int p(x | y)p(y) dy$$

# Events

- $P(+x, +y)$  ?
- $P(+x)$  ?
- $P(-y \text{ OR } +x)$  ?
- Independent?

$P(X, Y)$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

# Marginal Distributions

$P(X, Y)$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

$$P(x) = \sum_y P(x, y)$$

$$P(y) = \sum_x P(x, y)$$

$P(X)$

X	P
+x	
-x	

$P(Y)$

Y	P
+y	
-y	

# Conditional Probabilities

$P(X, Y)$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

- $P(+x \mid +y)$  ?
- $P(-x \mid +y)$  ?
- $P(-y \mid +x)$  ?



# Bayes Formula

$$P(x, y) = P(x | y)P(y) = P(y | x)P(x)$$

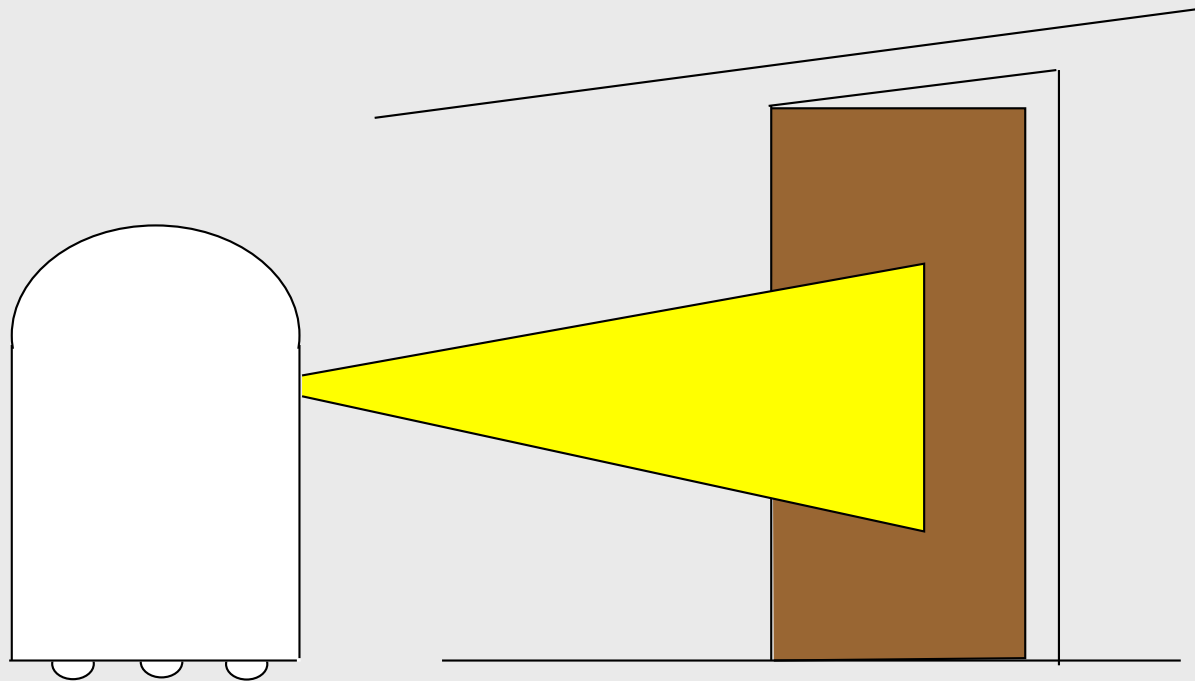
$\Rightarrow$

$$P(x | y) = \frac{P(y | x) P(x)}{P(y)} = \frac{\text{likelihood} \cdot \text{prior}}{\text{evidence}}$$

- Often **causal** knowledge is easier to obtain than **diagnostic** knowledge.
- Bayes rule allows us to use causal knowledge.

# Simple Example of State Estimation

- Suppose a robot obtains measurement  $z$
- What is  $P(open|z)$ ?



# Example

$$P(z | open) = 0.6 \quad P(z | \neg open) = 0.3$$

$$P(open) = P(\neg open) = 0.5$$

$$P(open | z) = \frac{P(z | open)P(open)}{P(z | open)p(open) + P(z | \neg open)p(\neg open)}$$

$$P(open | z) = \frac{0.6 \cdot 0.5}{0.6 \cdot 0.5 + 0.3 \cdot 0.5} = \frac{2}{3} = 0.67$$

- $z$  raises the probability that the door is open.

# Normalization

$$P(x|y) = \frac{P(y|x) P(x)}{P(y)} = \eta P(y|x) P(x)$$

$$\eta = P(y)^{-1} = \frac{1}{\sum_{x'} P(y|x')P(x')}$$

Algorithm:

$$\forall x : \text{aux}_{x|y} = P(y|x) P(x)$$

$$\eta = \frac{1}{\sum_x \text{aux}_{x|y}}$$

$$\forall x : P(x|y) = \eta \text{aux}_{x|y}$$

# Conditioning

- Bayes rule and background knowledge:

$$P(x | y, z) = \frac{P(y | x, z) P(x | z)}{P(y | z)}$$

$$P(x | y) \stackrel{?}{=} \int P(x | y, z) P(z) dz$$

$$\stackrel{?}{=} \int P(x | y, z) P(z | y) dz$$

$$\stackrel{?}{=} \int P(x | y, z) P(y | z) dz$$

# Conditioning

- Bayes rule and background knowledge:

$$P(x | y, z) = \frac{P(y | x, z) P(x | z)}{P(y | z)}$$

$$P(x | y) = \int P(x | y, z) P(z | y) dz$$

# Conditional Independence

$$P(x, y | z) = P(x | z)P(y | z)$$

- Equivalent to

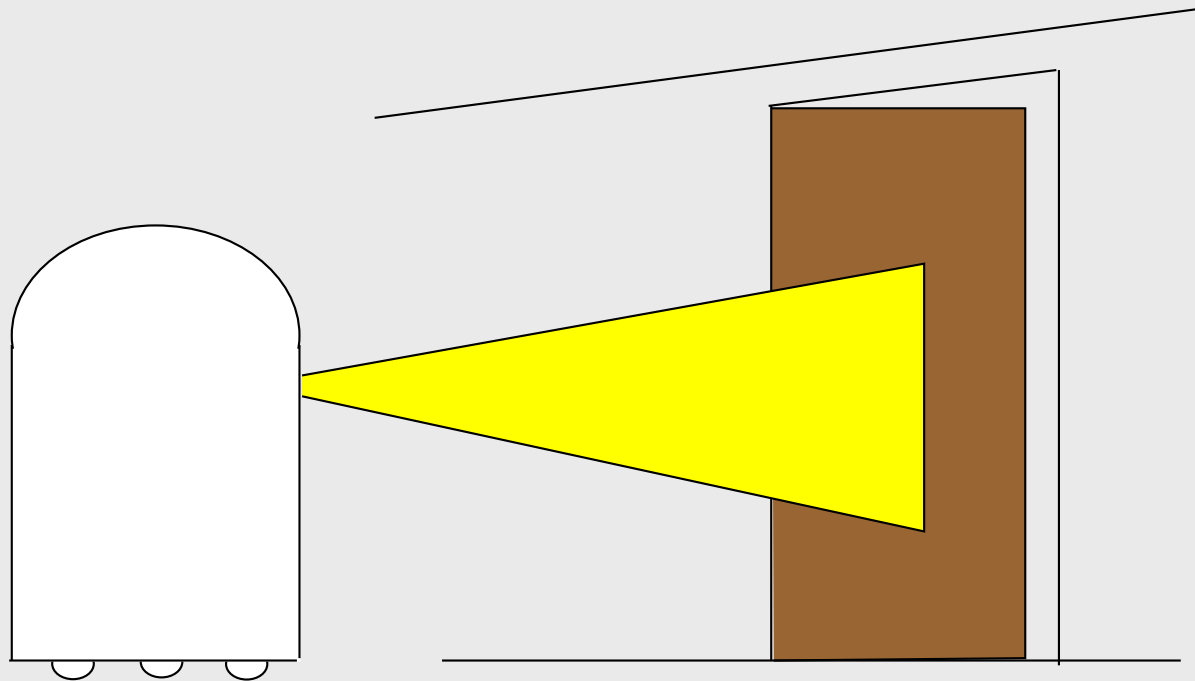
$$P(x | z) = P(x | z, y)$$

and

$$P(y | z) = P(y | z, x)$$

# Simple Example of State Estimation

- Suppose our robot obtains another observation  $z_2$ .
- What is  $P(\text{open} | z_1, z_2)$ ?





# Recursive Bayesian Updating

$$P(x | z_1, \dots, z_n) = \frac{P(z_n | x, z_1, \dots, z_{n-1}) P(x | z_1, \dots, z_{n-1})}{P(z_n | z_1, \dots, z_{n-1})}$$

**Markov assumption:**  $z_n$  is conditionally independent of  $z_1, \dots, z_{n-1}$  given  $x$ .

$$\begin{aligned} P(x | z_1, \dots, z_n) &= \frac{P(z_n | x) P(x | z_1, \dots, z_{n-1})}{P(z_n | z_1, \dots, z_{n-1})} \\ &= \eta P(z_n | x) P(x | z_1, \dots, z_{n-1}) \\ &= \eta_{1\dots n} \prod_{i=1\dots n} P(z_i | x) P(x) \end{aligned}$$

## Example: Second Measurement

$$P(z_2 \mid open) = 0.5 \quad P(z_2 \mid \neg open) = 0.6$$

$$P(open \mid z_1) = 2/3 \quad P(\neg open \mid z_1) = 1/3$$

$$\begin{aligned} P(open \mid z_2, z_1) &= \frac{P(z_2 \mid open) P(open \mid z_1)}{P(z_2 \mid open) P(open \mid z_1) + P(z_2 \mid \neg open) P(\neg open \mid z_1)} \\ &= \frac{\frac{1}{2} \cdot \frac{2}{3}}{\frac{1}{2} \cdot \frac{2}{3} + \frac{1}{5} \cdot \frac{1}{3}} = \frac{5}{8} = 0.625 \end{aligned}$$

- $z_2$  lowers the probability that the door is open.

# Bayes Filters: Framework

- **Given:**

- Stream of observations  $z$  and action data  $u$ :

$$d_t = \{u_1, z_2 \dots, u_{t-1}, z_t\}$$

- Sensor model  $P(z|x)$ .

- Action model  $P(x|u, x')$ .

- Prior probability of the system state  $P(x)$ .

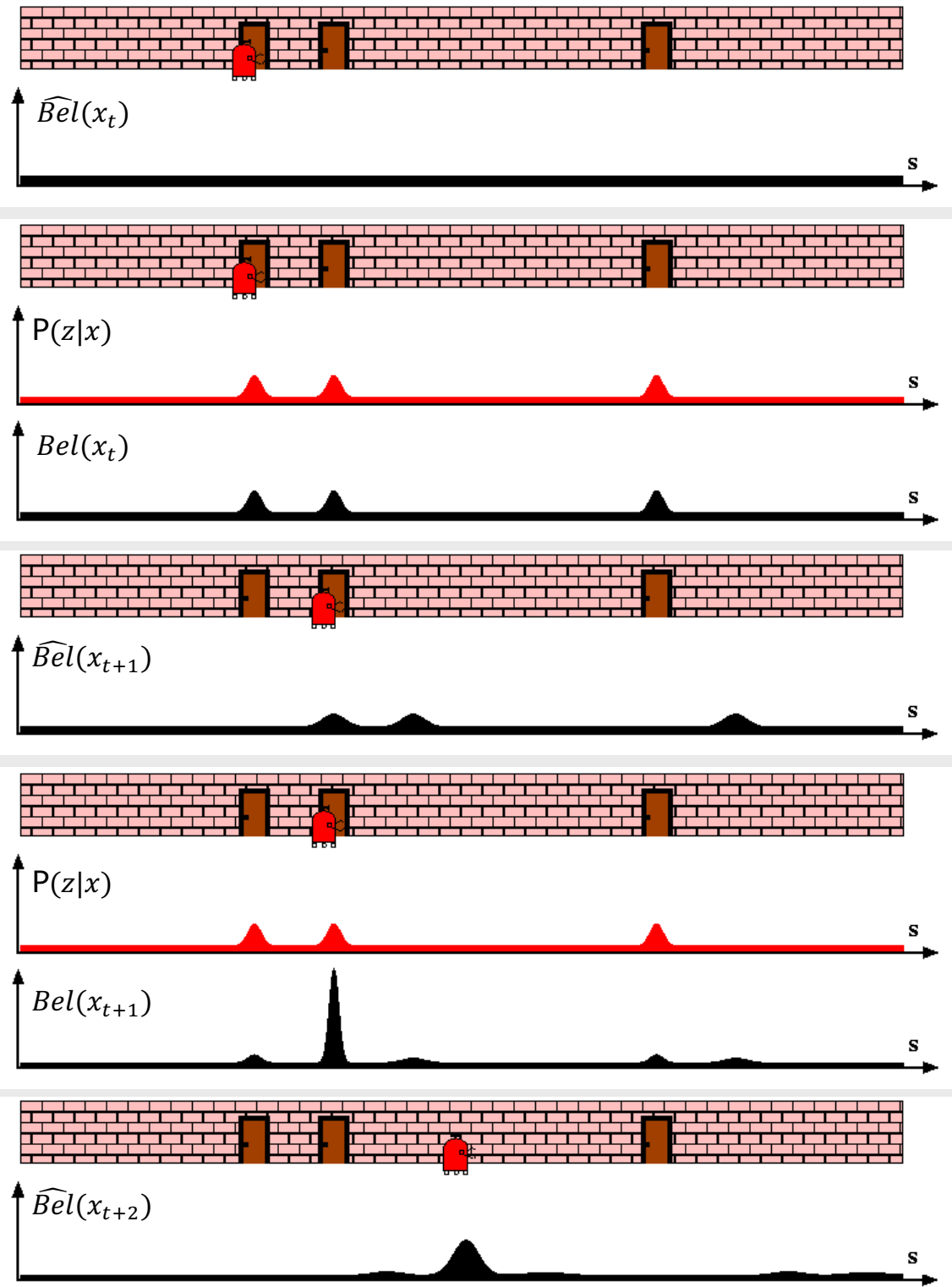
- **Wanted:**

- Estimate of the state  $X$  of a dynamical system.

- The posterior of the state is also called **Belief**:

$$Bel(x_t) = P(x_t | u_1, z_2 \dots, u_{t-1}, z_t)$$

# Bayes Filters for Robot Localization



# Bayes Filters

$z$  = observation  
 $u$  = action  
 $x$  = state

$$Bel(x_t) = P(x_t | u_1, z_1, \dots, u_t, z_t)$$

**Bayes** 
$$= \eta P(z_t | x_t, u_1, z_1, \dots, u_t) P(x_t | u_1, z_1, \dots, u_t)$$

**Markov** 
$$= \eta P(z_t | x_t) P(x_t | u_1, z_1, \dots, u_t)$$

**Total prob.**

$$= \eta P(z_t | x_t) \int P(x_t | u_1, z_1, \dots, u_t, x_{t-1}) P(x_{t-1} | u_1, z_1, \dots, u_t) dx_{t-1}$$

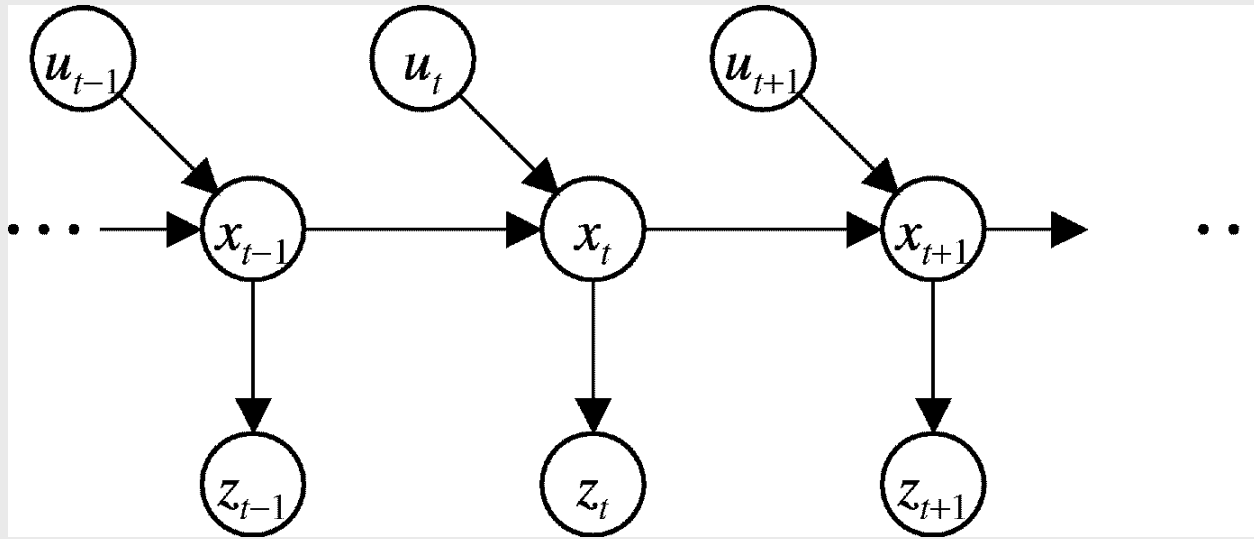
**Markov** 
$$= \eta P(z_t | x_t) \int P(x_t | u_t, x_{t-1}) P(x_{t-1} | u_1, z_1, \dots, u_t) dx_{t-1}$$

$$= \eta P(z_t | x_t) \int P(x_t | u_t, x_{t-1}) Bel(x_{t-1}) dx_{t-1}$$

$$Bel(x_t) = \eta P(z_t | x_t) \int P(x_t | u_t, x_{t-1}) Bel(x_{t-1}) dx_{t-1}$$

1. Algorithm **Bayes\_filter**(  $Bel(x), d$  ):
2.  $n=0$
3. If  $d$  is a **perceptual** data item  $z$  then
4.     For all  $x$  do
5.          $Bel'(x) = P(z | x)Bel(x)$
6.          $\eta = \eta + Bel'(x)$
7.     For all  $x$  do
8.          $Bel'(x) = \eta^{-1}Bel'(x)$
9. Else if  $d$  is an **action** data item  $u$  then
10.     For all  $x$  do
11.          $Bel'(x) = \int P(x | u, x') Bel(x') dx'$
12. Return  $Bel'(x)$

# Markov Assumption



$$p(z_t \mid x_{0:t}, z_{1:t-1}, u_{1:t}) = p(z_t \mid x_t)$$

$$p(x_t \mid x_{1:t-1}, z_{1:t-1}, u_{1:t}) = p(x_t \mid x_{t-1}, u_t)$$

## Underlying Assumptions

- Static world
- Independent noise
- Perfect model, no approximation errors

# Bayes Filters are Familiar!

$$Bel(x_t) = \eta P(z_t | x_t) \int P(x_t | u_t, x_{t-1}) Bel(x_{t-1}) dx_{t-1}$$

- Kalman filters
- Particle filters
- Hidden Markov models
- Dynamic Bayesian networks
- Partially Observable Markov Decision Processes (POMDPs)



# Summary

- Bayes rule allows us to compute probabilities that are hard to assess otherwise.
- Under the Markov assumption, recursive Bayesian updating can be used to efficiently combine evidence.
- Bayes filters are a probabilistic tool for estimating the state of dynamic systems.