Plan for Today

- Price of anarchy
  - Selfish routing
  - Network formation games
  - A market sharing game
- Brief discussion of second homework
- Brief discussion of fair allocation

Due dates:

Feb 2 Strategy design 1
Feb 5 Exercise Set 2
Feb 9 Project proposal
gerrymandering → Feb 16 Strategy design 2
auctions → Feb 23 Strategy design 3
Bitcoin → Mar 1 Strategy design 4
Mar 10 project presentations
Mar 15 project writeup
Mar 19 peer reviews

Price of Anarchy

- Explores games that arise "in the wild", such as in Internet settings.

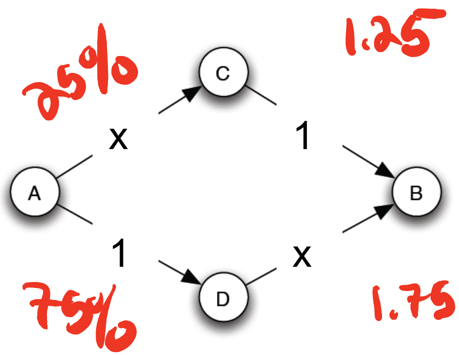
- Tries to understand the impact of selfish behavior on society by comparing the overall performance attained in equilibrium when players behave selfishly to the performance that could be attained if decisions were made by a centralized authority.

Selfish Routing [Roughgarden, Tardos]

- Model network as directed graph.
- We assume network users are selfish -- in equilibrium each user will choose a route that minimizes their travel time, given what everyone else is doing.
- What will the traffic be in equilibrium?

1 unit of traffic going frm A to B

in equilibrium split 50-50.



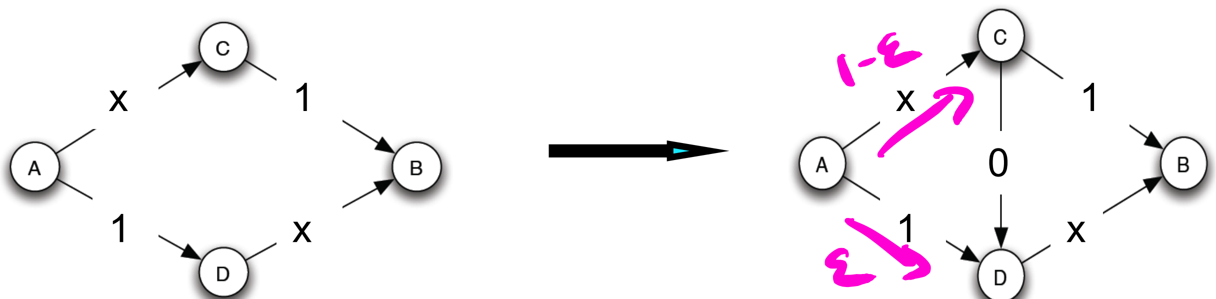
Average latency= 1.5

Edge labels: latency as function of fraction of traffic

Braess's Paradox

- Small changes can lead to counterintuitive behavior.
- Example: Government builds a new, very fast highway.

Edge labels: latency as function of fraction of traffic



Average Latency

in Nash Eq = 3/2

Avg latency in Nash eq is 2

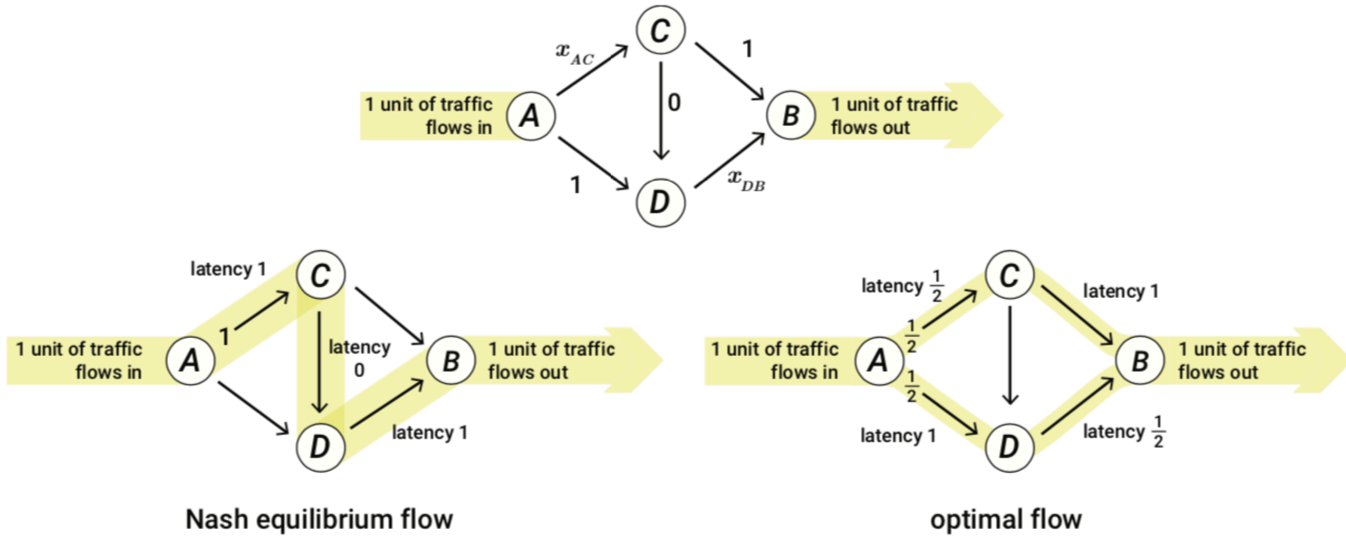
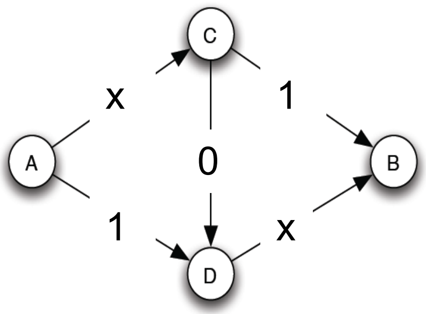


FIGURE 8.2. The Braess Paradox: Each link in the top figure is labeled with a latency function $\ell(x)$ which describes the travel time on that edge as a function of the fraction x of traffic using that edge. These figures show the effect of adding a 0 latency road from C to D: The travel time on each of $\gamma_C = A - C - B$ and $\gamma_D = A - D - B$ is always at least the travel time on the new route $\gamma = A - C - D - B$. Moreover, if a positive fraction of the traffic takes route γ_C (resp. γ_D), then the travel time on γ is *strictly* lower than that of γ_C (resp. γ_D). Thus, the unique Nash equilibrium is for all the traffic to go on the path γ, as shown in the bottom left figure. In this equilibrium, the average travel time the drivers experience is 2, as shown on the bottom left. On the other hand, if the drivers could be forced to choose routes that would minimize the average travel time, it would be reduced to 3/2, the social optimum, as shown on the bottom right.

Price of Anarchy
[Koutsoupias, Papadimitriou]

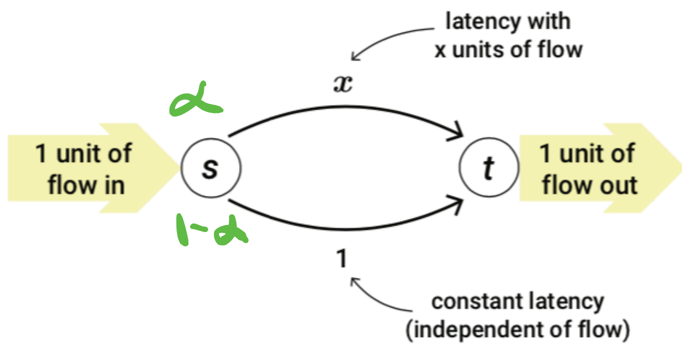
- How bad can selfishness be for society?



Selfish equilibrium: 2

Social optimum: 3/2

$$\text{Price of Anarchy} = \frac{\text{Avg latency at } \overset{\text{Nash}}{\cancel{\text{selfish}}} \text{ equilibrium}}{\text{Avg latency at social optimum}} = \frac{2}{3/2} = \frac{4}{3}$$



$\alpha \cdot \alpha$
$+ (1-\alpha) \cdot 1$
if $\alpha = \frac{1}{2}$
$\frac{1}{4} + \frac{1}{2} = \frac{3}{4}$

NE/dom strategy for all to take upper path

avg latency 1

OPT

$$\text{PoA} = \frac{1}{3/4} = 4/3$$

NE: latency 1

$\varepsilon \cdot 1 + (1-\varepsilon)^d (1-\varepsilon)$

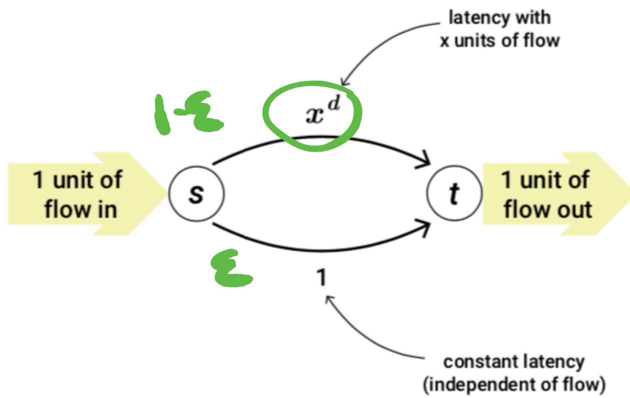


FIGURE 8.6. With the given latency functions, the optimal flow routes x units of flow on the upper link (and thus $1-x$ on the lower link) so as to minimize the average latency, which is $x \cdot x^d + (1-x)$. The Nash equilibrium flow routes all the flow on the upper link. The resulting price of anarchy is approximately 1.6 for $d = 2$, approximately 1.9 for $d = 3$, and is asymptotic to $d/\ln d$ as d tends to infinity.

Price of Anarchy

- How bad can selfishness be for society?

- Previous example is worst case (any network, affine latency functions)

$x + .3$ $\xrightarrow{}$ $.5x + 10$

$$\text{Price of Anarchy} = \frac{\text{Avg latency at selfish equilibrium}}{\text{Avg latency at social optimum}}$$

is always at most 4/3

if all cost fns are of form $ax+b$, $a,b \geq 0$

<u>Model</u>: directed graph with source s & target t.

- every edge e has cost fn $c_e : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$

  $c_e(x)$: latency on edge e when x units of flow use it

- cost fns are non-decreasing and continuous.

Our goal: get a handle on worst-case Price of anarchy as a fn of class $\mathcal{C}$ of cost fns we allow.

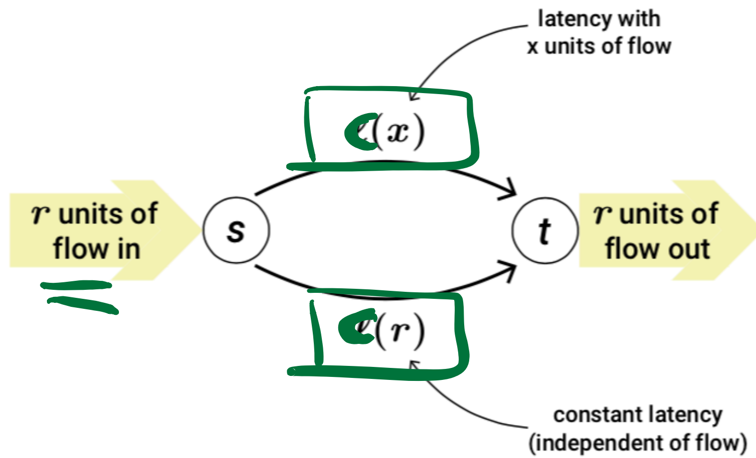
$$\mathcal{C} = \{ ax+b \mid a,b \geq 0 \}$$

<u>Thm</u> worst case for class $\mathcal{C}$ is 2-node networks (Pigou like networks)

Pigou-Like Networks

Pick 2 parameters

- r total flow from s to t
- $c(x) \in \mathcal{C}$



$$PoA(c(x), r) = \frac{\overset{\text{for those}}{\text{NE latency}}}{\text{OPT latency}} = \max_{0 \le x \le r} \frac{r \cdot c(r)}{x\,c(x) + (r-x)\,c(r)}$$

$$= \max_{x \ge 0} \frac{r\,c(r)}{x\,c(x) + (r-x)\,c(r)}$$

any r, any $c \in \mathcal{C}$

$$\text{PoA for Pigou} = \alpha(\mathcal{C}) = \max_{c \in \mathcal{C}} \max_{r \ge 0} PoA(c(x), r)$$

Theorem: $\forall$ class $\mathcal{C}$, $\forall$ network with cost fns in $\mathcal{C}$, Price of anarchy $\le \alpha(\mathcal{C})$

Equilibria preliminaries

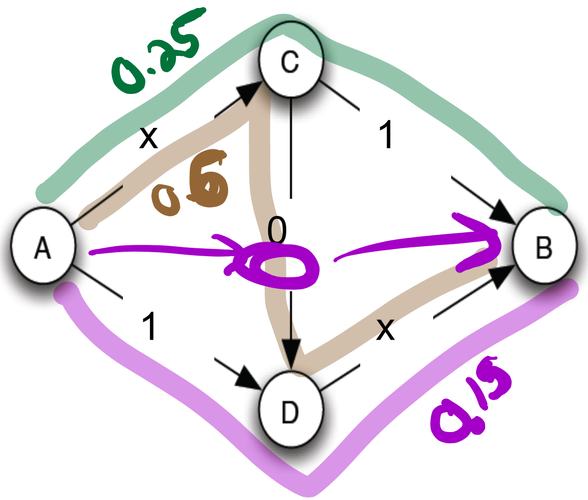
s-t network, r units of flow

f_P: flow on path P (path from s to t)

flow $f := \{f_P\}_{P \in \mathcal{P}_{s,t}}$

$$\sum_{P \in \mathcal{P}_{s,t}} f_P = r$$

$$f_e = \sum_{\substack{\text{paths } P \\ \text{that contain} \\ \text{edge } e}} f_P$$



Equilibrium flow f_P

If $f_P > 0$ then the latency on that path achieves min latency among all paths

equilibria exist and are unique in the sense of latency

$C(f)$: avg latency of drivers 'using that flow'

Path by Path: $C(f) = \sum_P \underline{C_p(f)}\, f_P$

latency on that path

$$C(f) = \sum_{e \in E} f_e\, C_e(f_e)$$

Part I: If we freeze the latencies on each edge at their cost in the eq flow, then, the eq flow is optimal

Suppose that $\boxed{f \text{ is eq flow}}$ and if $f_p > 0$ then $\boxed{C_p(f) = L}$

$\sum_p f_p\, C_p(f) = r \cdot L \leftarrow$

Suppose that $\boxed{f^* \text{ is optimal flow}}$

$\rightarrow \sum_p f_p^*\, C_p(f) \geq rL \rightarrow$ $\qquad \underline{\sum f_p^* = r}$

$\boxed{\sum_e (f_e^* - f_e)\, C_e(f_e) \geq 0} \Leftarrow$

Part II:

$$\frac{r\,c(r)}{x\,c(x) + (r-x)\,c(r)} \leq \alpha(\mathcal{C}) \quad \forall r, \text{ any } c(x) \in \mathcal{C}, \text{ any } x$$

Focus on e.

plug in

c_e for c

f_e for r

f_e^* for x

$$\frac{f_e\, c_e(f_e)}{f_e^*\, c_e(f_e^*) + (f_e - f_e^*)\, c_e(f_e)} \leq \alpha(\mathcal{C})$$

$$\underbrace{\sum_e f_e^*\, c_e(f_e^*)}_{C(f^*)} \geq \sum_e \left(\frac{1}{\alpha(\mathcal{C})} f_e\, c_e(f_e) + (f_e^* - f_e)\, c_e(f_e) \right)$$

$$C(f^*) \geq \frac{1}{\alpha(\mathcal{C})} \sum_e f_e\, c_e(f_e) + \sum_e (f_e^* - f_e)\, c_e(f_e)$$

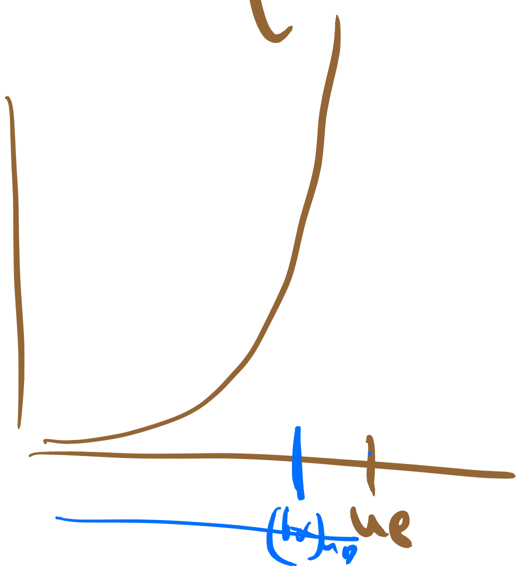
$C(f)$ ≥ 0

$$C(f^*) \geq \frac{1}{\alpha(\mathcal{C})} C(f)$$

$$\boxed{\frac{C(f)}{C(f^*)} \leq \alpha(\mathcal{C})}$$

Overprovisioning.

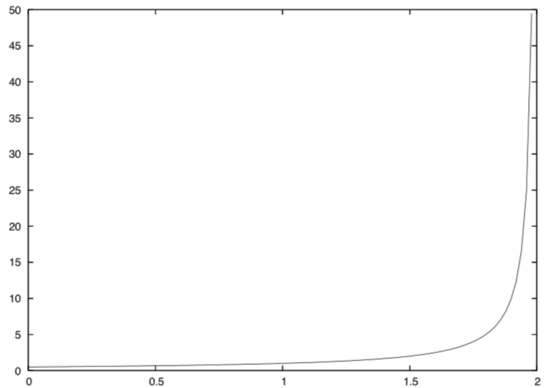
$$c_e(x) = \begin{cases} \dfrac{1}{u_e - x} & x < u_e \\ \infty & x \geq u_e \end{cases}$$



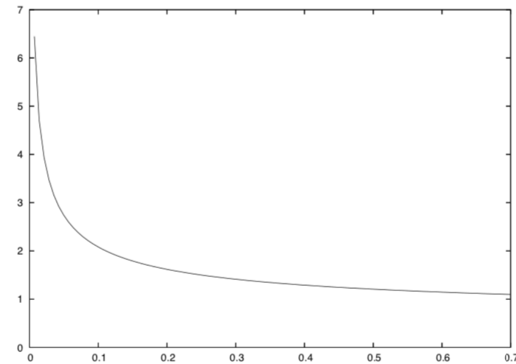
Suppose network overprovisioned in the sense that

$$f_e \leq (1-\alpha)u_e$$

$$\text{PoA} \leq \frac{1}{2}\left(1 + \frac{1}{\sqrt{\alpha}}\right)$$



(a) M/M/1 delay function



(b) Extra capacity vs. POA curve

Figure 1: Modest overprovisioning guarantees near-optimal routing. The left-hand figure displays the per-unit cost $c(x) = 1/(u - x)$ as a function of the load x for an edge with capacity $u = 2$. The right-hand figure shows the worst-case price of anarchy as a function of the fraction of unused network capacity.

Definition 5.1. A k-player finite **extensive-form game** is defined by a finite, rooted tree T.

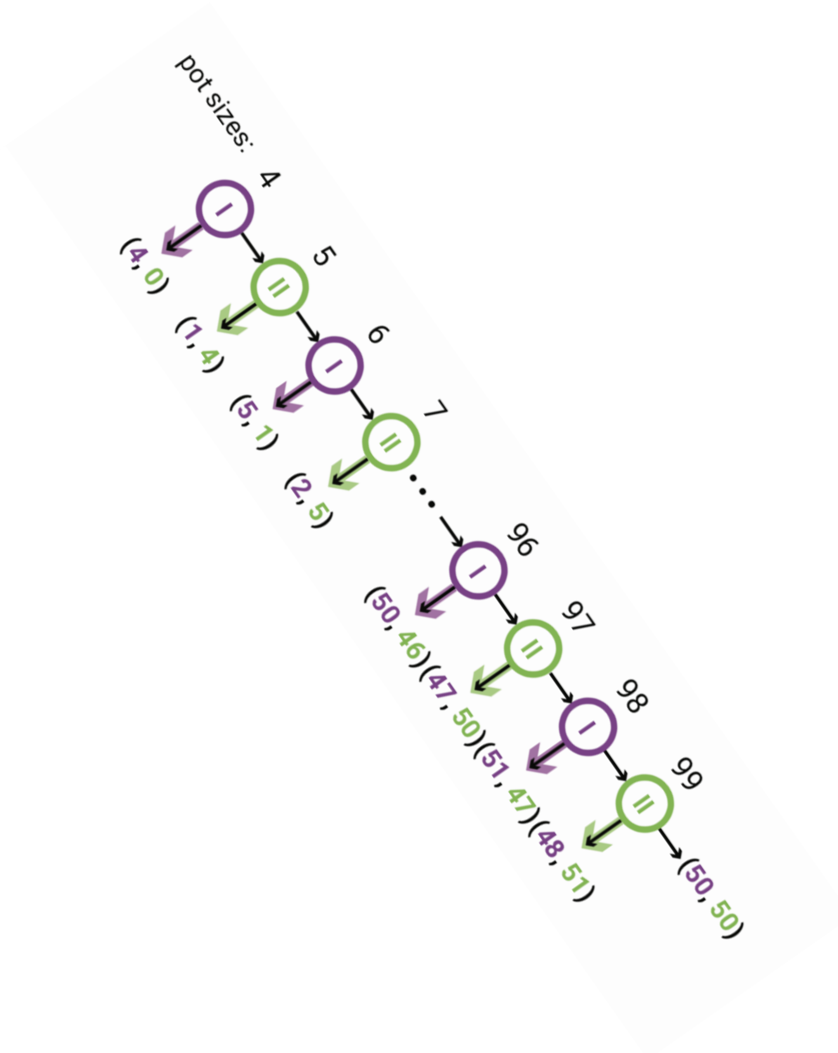
- Each node in T represents a possible state in the game, with leaves representing terminal states.
- Each internal (nonleaf) node v in T is associated with one of the players, indicating that it is his turn to play if/when v is reached.
- The edges from an internal node to its children are labeled with **actions**, the possible moves the corresponding player can choose from when the game reaches that state.
- Each leaf/terminal state results in a certain payoff for each player.

A **pure strategy** for a player in an extensive-form game specifies an action to be taken at each of that player's nodes.

A **mixed strategy** is a probability distribution over pure strategies.

The kind of equilibrium that is computed by backward induction is called a **subgame-perfect equilibrium** because the behavior in each **subgame**, is also an equilibrium.


pot sizes:
4
5
6
7
96
97
98
99
I
II
I
II
I
II
I
II
(50, 50)
(4, 0)
(1, 4)
(5, 1)
(2, 5)
(50, 46)
(47, 50)
(51, 47)
(48, 51)



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(51, 47)
(48, 51)
(50, 50)

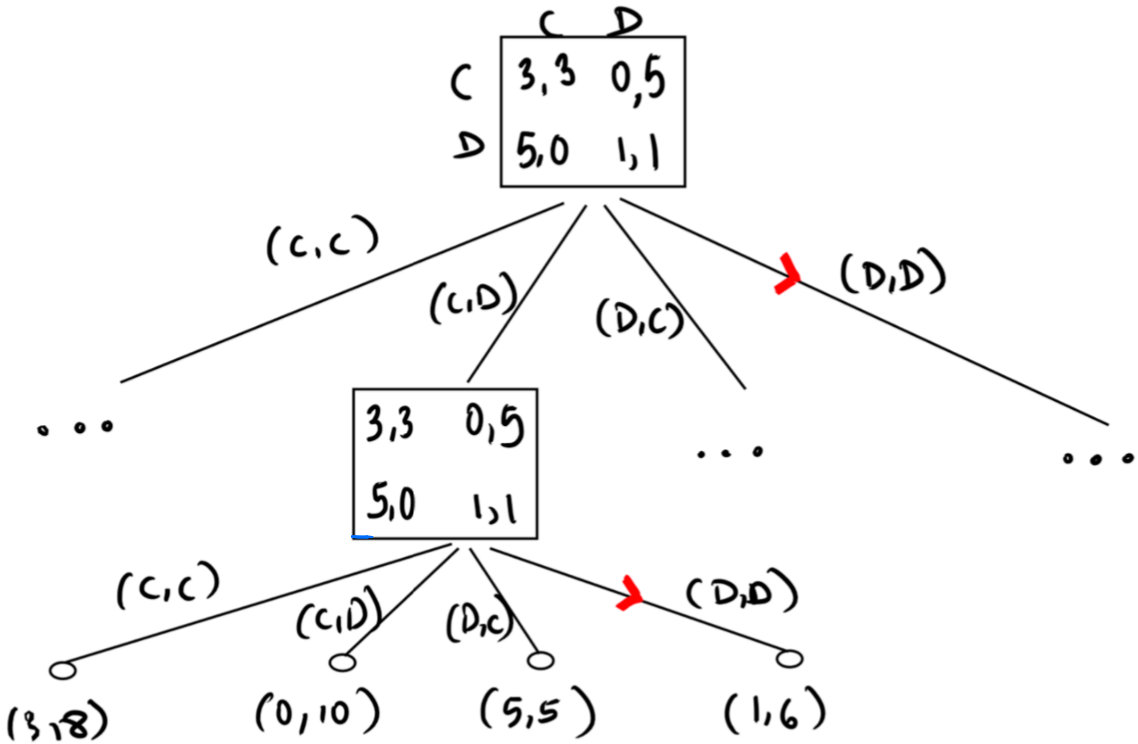
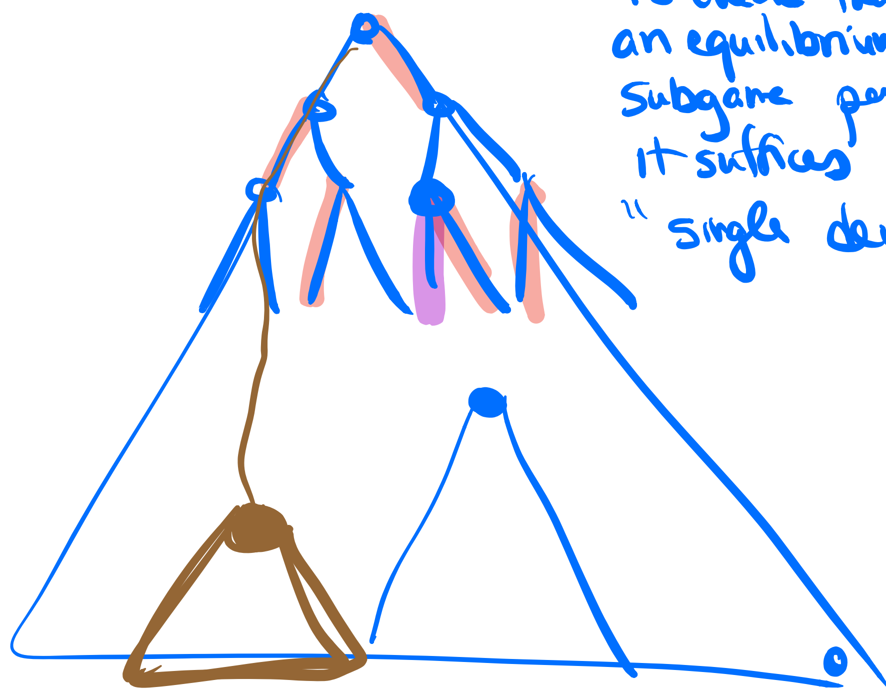


Figure 4.20.: The unique subgame-perfect equilibrium in a two-period repeated Prisoners' Dilemma. The arrows indicate the equilibrium strategy.

Single deviation principle

For finite games (or for discounted)

To check that an equilibrium is subgame perfect, it suffices to check "single deviations"



A strategy is a subgame perfect eq iff ∄ no single deviations that improve payoff.

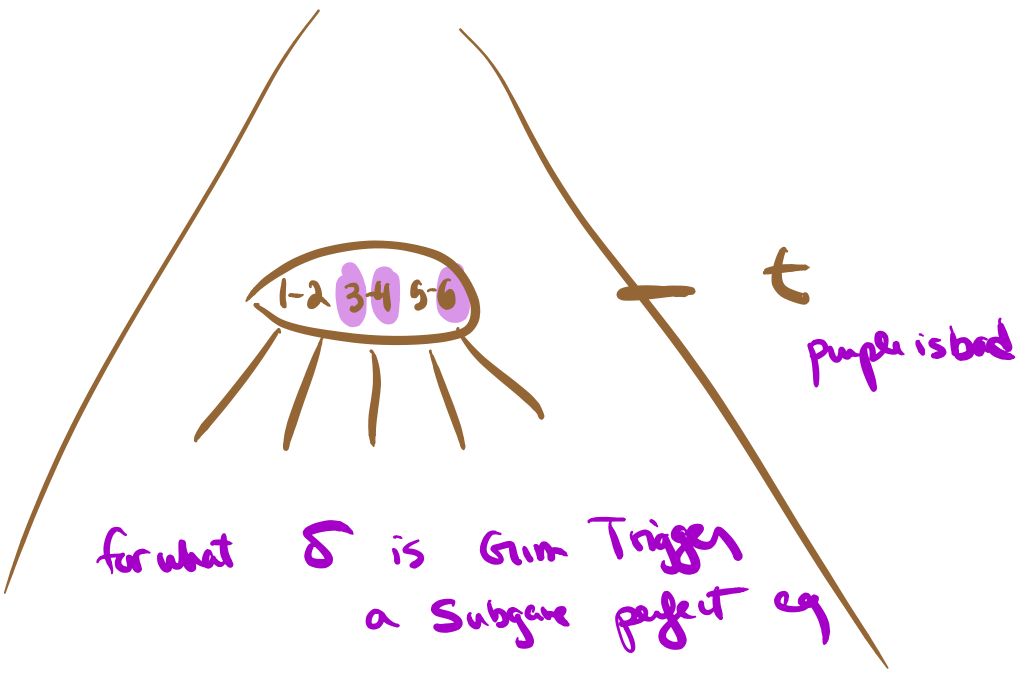
2n players. - pair them up at random each step

$$\boxed{\sum_{t=1}^{\infty}\left(\delta^{t} \text{ payoff in round } t\right)}$$

everybody either good or bad.

Grim Trigger:

- if both good, play C (you or your opponent are)
- if opponent bad, play D
- become bad if they defect against good agent



for what δ is Grim Trigger a subgame perfect eq

Network formation games

set of companies jointly forming a comm network

company i needs to be sure there is a path from s_i to t_i

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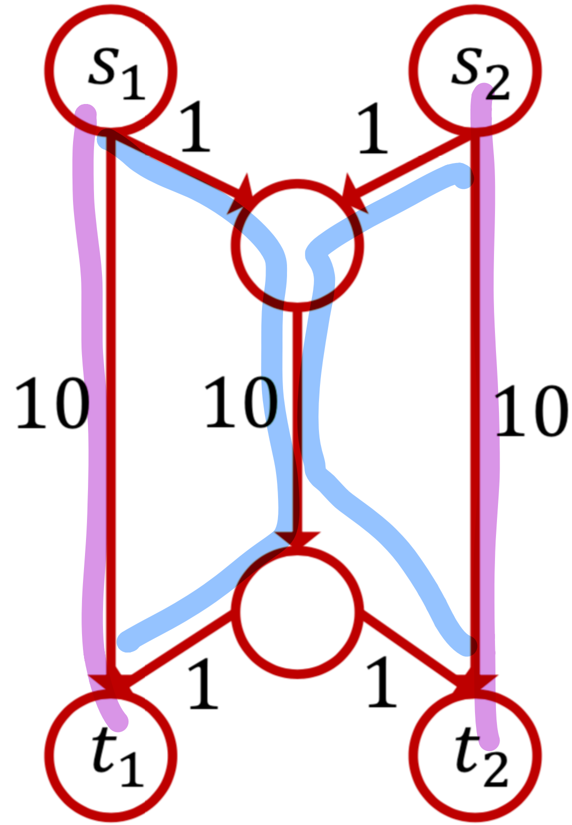
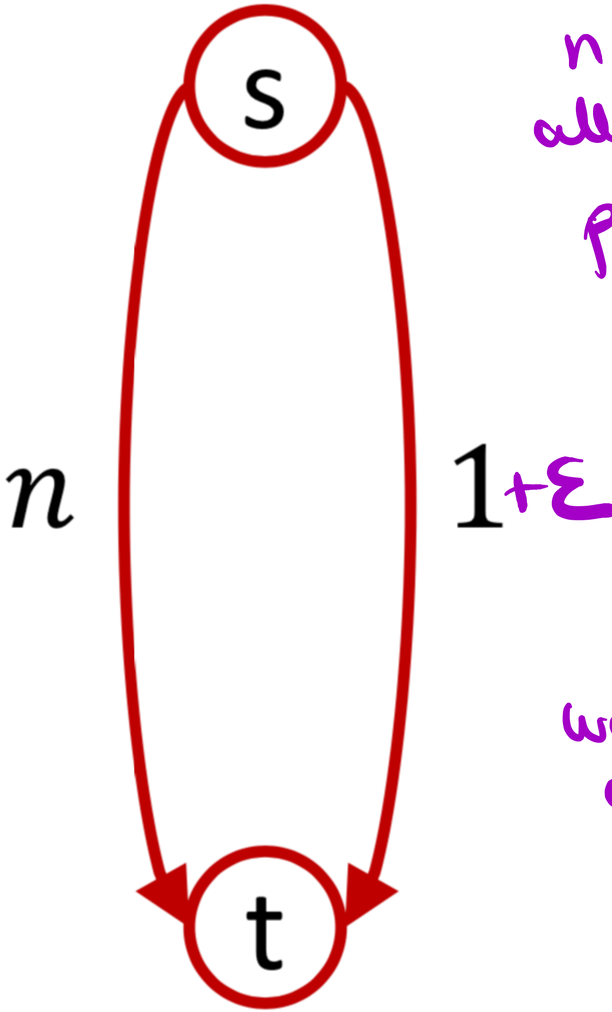
each company is trying to min their cost, given other choices.

Social opt: min cost network that connects all the pairs



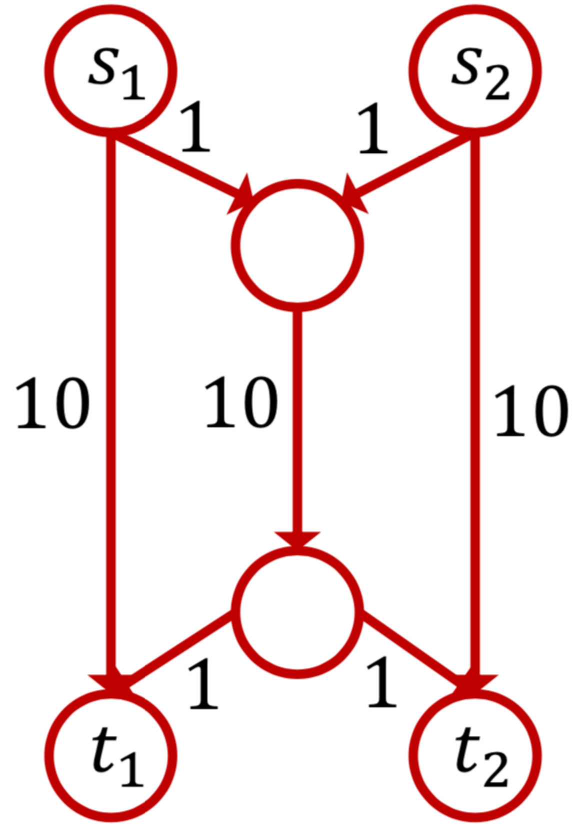


n players
all greedy a
path fm s to t

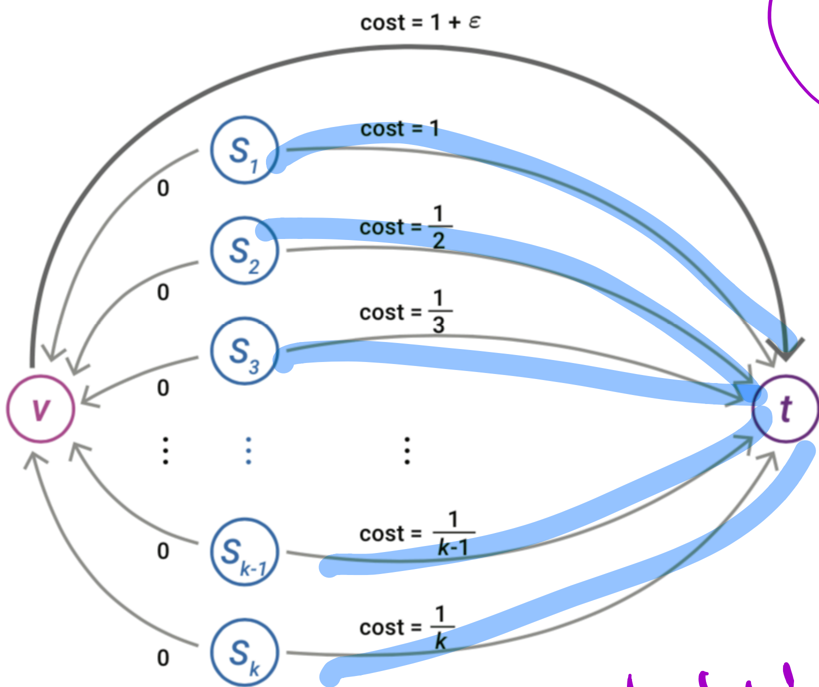
worst case Price
of anarchy
$\leq n = \#$players

$\exists$ an equilibrium

where $\dfrac{\text{cost in eq}}{\text{OPT cost}} \le \log(n)$



k players
log k



$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{k}$$

$$\int_1^k \frac{1}{x}\,dx$$

$$\simeq \log k$$

Market Sharing Game.

k NBA teams
each city j has
"profit potential" v_j
multiple teams → same city
split potential.

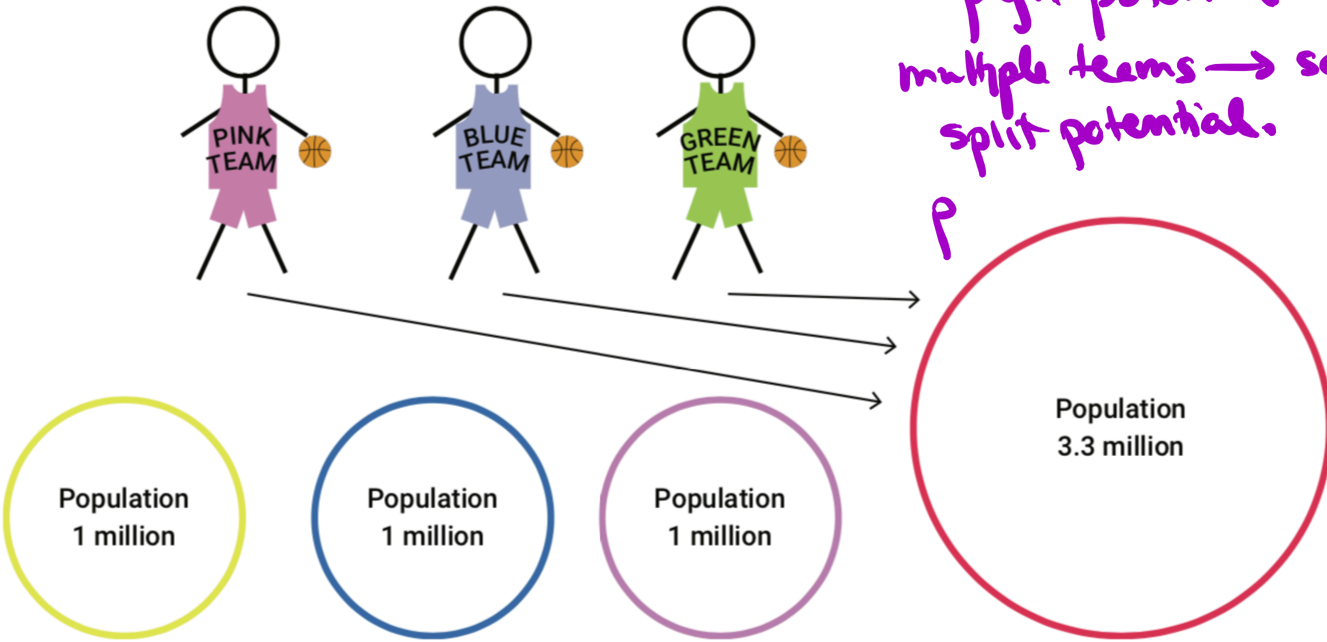


FIGURE 8.10. Three basketball teams are deciding which city to locate in when four choices are available. It is a Nash equilibrium for all of them to locate in the largest city where they will each have a utility of 1.1 million. If one of the teams were to switch to one of the smaller cities, that team's utility would drop to 1 million.

payoff to value NBA team = $\dfrac{\text{value of city selected}}{\#\text{teams in that city}}$

$$V(S) = \sum_{j \in S} v_j$$

set of cities selected

Claim: Price of anarchy: $\dfrac{\text{OPT value}}{\text{Nash value}} \leq 2.$

$$V(\text{Nash}) = \sum_i u_i(c_i, c_{-i}) \geq \boxed{\sum_i u_i(c_i', c_{-i})} \quad \forall c_i'$$

in Nash team i chooses city c_i

$v_1 \geq v_2 \geq v_3 \geq \cdots v_k \geq \qquad \geq v_n$

k players OPT value $= v_1 + v_2 + \ldots + v_k$

1	2	3	4	5	6	7
1	2	3	4	1	1	2

Nash 1 1 1 2 2 3 4

= Value of OPT − Value of Nash

$\Rightarrow$ Value(OPT) ≤ 2(Value of Nash)