Plan for Today

- Fair division basics
- Intro to auctions

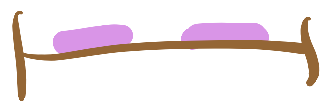
2 people who want to split a heterogeneous divisible good.

Player 1 is told to divide the cake into 2 pieces she values equally

Player 2 picks their favorite.

<u>Model:</u> good is $[0,1]$

$v_i(S)$ is the value that player i has for piece of cake S.

- $v_i[0,1] = 1$
- value is additive on disjoint intervals 
- values are divisible
  $\forall\, c \in [0,1]$ and X, there is a Y in X s.t. $v_i(Y) = c\, v_i(X)$

Are the players incentivized to follow rules of I cut, U choose protocol?

An allocation $A = (A_1, A_2, \dots, A_n)$

A_i piece that player i gets

is Pareto optimal if $\exists$ no alternative allocation $B = (B_1, \dots, B_n)$ s.t. every agent is at least as happy & some agent is strictly happier.

I cut you choose not Pareto optimal

- give the whole cake $\to$ 1 is PO.

How to define fairness?

1. both players are equally happy
   I cut y choose doesn't satisfy this?
2. allocation $(A_1, \dots, A_n)$ is proportional if $v_i(A_i) \geq \frac{1}{n}$ $\forall i$
3. envy-free if $\forall i, j$ $v_i(A_i) \geq v_i(A_j)$

EF $\Rightarrow$ proportional.

$v_i(A_j) \leq v_i(A_i)$ $\forall j$

$$1 = \underbrace{\sum_{j=1}^{n} v_i(A_j)}_{} \leq n\, v_i(A_i) \Rightarrow v_i(A_i) \geq \frac{1}{n}$$

1	$\boxed{\frac{1}{3}}$	$\frac{2}{3}$	0
	0	$\boxed{\frac{1}{3}}$	$\frac{2}{3}$
	$\frac{2}{3}$	0	$\boxed{\frac{1}{3}}$

proportional but not EF

Proportional Allocation

Moving-knife Algorithm for fair division of a cake among n people

- Move a knife continuously over the cake from left to right until some player yells "Stop!"
- Give that player the piece of cake to the left of the knife.
- Iterate with the other $n-1$ players and the remaining cake.

Claim: Procedure is proportional

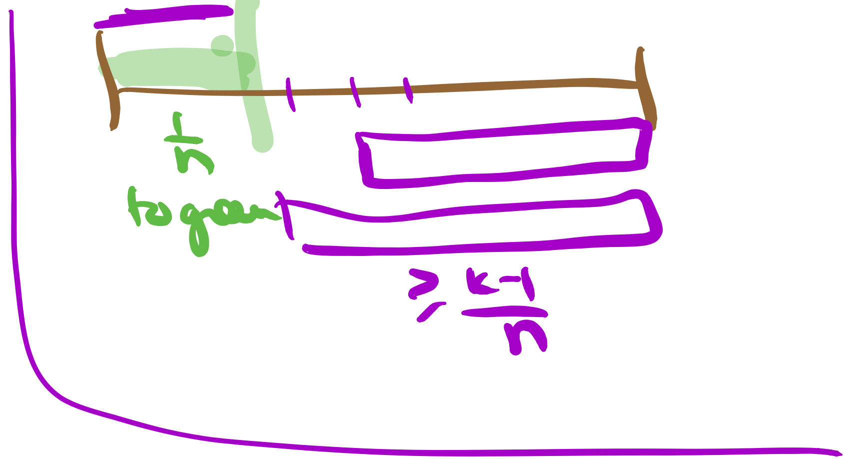
Proof: whenever there are k players left, remaining cake is worth at least $\frac{k}{n}$ to each of them.

Base case: $k=n$ ✓

Inductive Step: k left & IH holds

Complexity:

restrict access to value fns V_i through 2 types of queries



① $\text{Eval}_i([x,y])$ returns $V_i[x,y]$

② $\text{Cut}_i(x, c)$ returns y s.t. $V_i([x,y]) = c$

n^2 queries for Moving Knife.

$\exists$ simple D& Conquer $O(n \log n)$ queries

$\Omega(n \log n)$ queries are necessary.

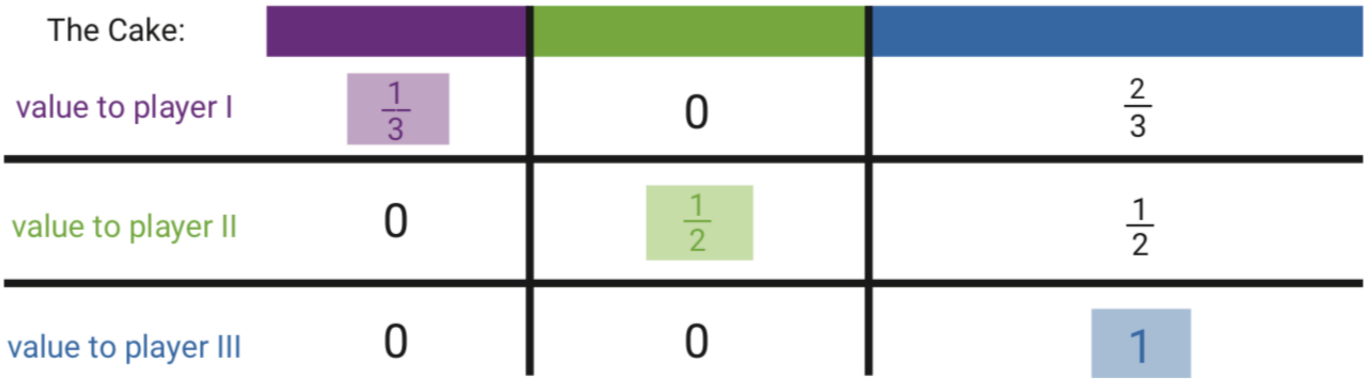
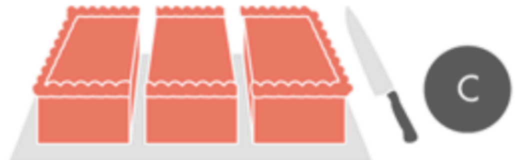


FIGURE 11.3. This figure shows an example of how the Moving-knife Algorithm might evolve with three players. The knife moves from left to right. Player I takes the first piece, then II, then III. In the end, player I is envious of player III.

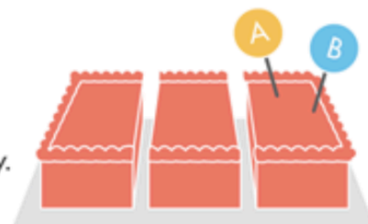
CAKE CUTTING FOR THREE

1. Alice, Bob and Charlie want to share a cake so that none of them envies other pieces.

2. Charlie cuts the cake into three pieces that are equally valuable from his perspective.



3. Alice and Bob identify their first choices.
If they identify the same choice, things get tricky.



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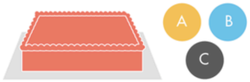

(2) Charlie cuts the cake into three pieces that are equally valuable from his perspective.

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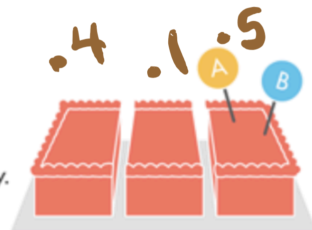
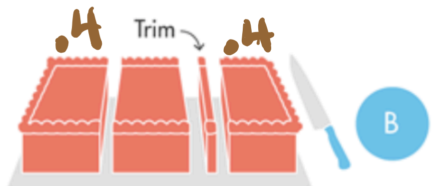
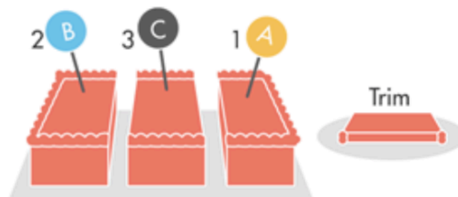


(4) Bob trims his preferred piece to match his second most preferred piece.



(5) Putting the trim to one side they choose in this order: Alice first*, Bob second and Charlie last.



It is envy free

...for Alice, because she got first choice.

...for Bob, because his second choice was equally valuable.

...for Charlie, because the three original slices were equal to him.

*If Alice doesn't choose the trimmed piece, then Bob must take it. Alice and Bob then trade places for the rest of the process.

(6) To divvy up the trimmed slice, first Bob cuts the trim into three pieces that are equally valuable from his perspective.

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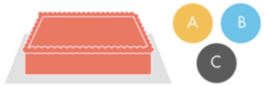

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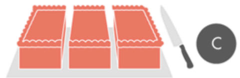

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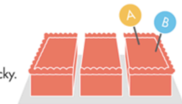

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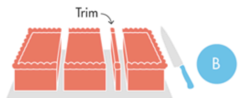

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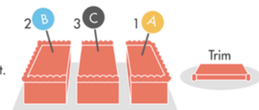

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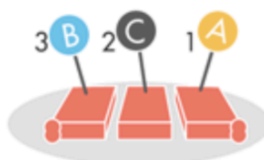

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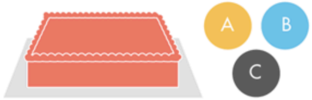

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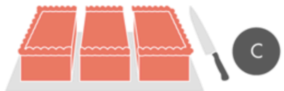

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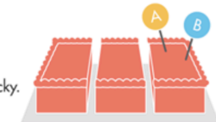

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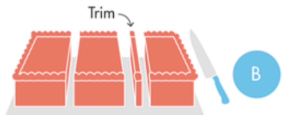

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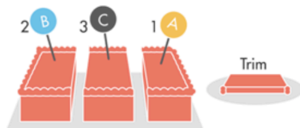

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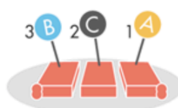

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1960 $n=3$

1995 [BT]

2016 [AM]

$n=4$ 203 cuts

all n

$n^{n^{n^{n^{n}}}}$ cuts

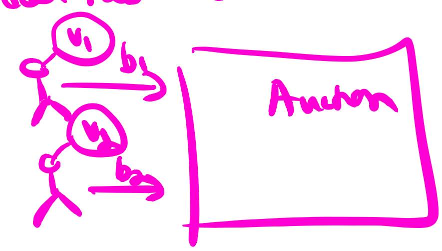
$\Omega(n^2)$

Auctions

Single-item auction
n bidders

Bidders

- each bidder has a value (private) v_i



rules specify who wins & how much each person pays

1. First price auction
   highest bidder wins
   pay their bid

2. 2nd price auction
   highest bidder wins
   pay 2nd highest bid

3. All-pay auction
   highest bidder wins
   every pays what they bid

$$\text{bidder utility} = \begin{cases} v_i - p_i & \text{if they win item \& pay } p_i \\ -p_i & \text{if they lose \& pay } p_i \end{cases}$$

$80 \downarrow 55$ $60 \downarrow \boxed{58}$ $\rightarrow 2$

Thm In a 2^{nd} price auction, its a dominant strategy to bid truthfully

Pf Fix bids of everyone else. B max of these

$$\max(v_i - B, 0)$$

$$b_i := v_i$$

---

never regret participating & bid truthfully
"individually rational"

The item goes to the bidder who values it most. "welfare maximizing".

no other allocation can have higher total utility to the bidders + auctioneer.

sum of utility of bidders + auctioneer $= \sum_i (v_i \mathbb{1}_{i \text{ wins}} - p_i) + \sum_i p_i$

$= \sum_i v_i \mathbb{1}_{i \text{ wins}}$

---

k identical items, each bidder has a value v_i

each bidder only wants one.

Design auction ~~that~~ is (truthful, IR, welfare maximizing)

Try 1: charge everyone second highest bid X
not IR

Try 2: Charge the ith highest bidder the (i+1)st bid.

10

8

5   7

3

9

Fix: Change the winners the $(k+1)^{st}$ highest bid.

1^{st} price auction:

for every player

β_i : mapping from values to bids for player i.

assume that each player's value $v_i \sim F$ and those probability distr's are public.

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2 players V_1, $V_2 \sim \mathrm{Unif}[0,100]$

you're player 1; $\boxed{\beta(V_2) = \frac{V_2}{2}}$

How should you bid when your value is v

$\beta(v) = b$

$$u(b|v) = (v-b)\Pr(\text{win})$$
$$= (v-b)\Pr\left(b > \frac{V_2}{2}\right)$$
$$= (v-b)\Pr(V_2 < 2b)$$
$$= (v-b)\frac{2b}{100}$$

$\max \quad (v-b)b$

$$\boxed{b = \frac{v}{2}}$$

Bayes-Nash Equilibrium BNE

$(\beta_1, \beta_2, \dots, \beta_n)$ is a BNE

if $\forall v_i, \; E[u_i(\beta_i(v), \beta_{-i}(V_{-i}))]$

$\geq E[u_i(b', \beta_{-i}(V_{-i}))] \quad \forall b'$

$V_1, V_2 \sim U[0,100]$. 1st price

$$E[\text{auctioneer revenue}] = E\left[\max\left(\frac{V_1}{2}, \frac{V_2}{2}\right)\right]$$

$$= \frac{1}{2} \underbrace{E[\max(V_1, V_2)]}_{2/3\, 100} = \frac{100}{3}$$

2nd price

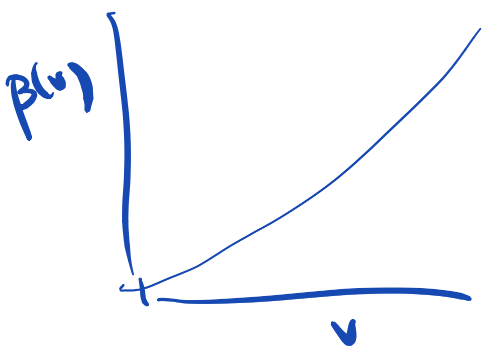
$$E[\text{auctioneer revenue}] = E[\min(V_1, V_2)] = \frac{100}{3}$$

Thm (Revenue equivalence)

Set of buyers & values all drawn i.i.d ~ F

Then any auction mechanism in which the object goes to the bidder with the highest value (and the bidder with min possible value has utility 0) has the same expected revenue.

[exp payment of bidder w/ value v is same in any such auction]



2nd price auction, 2 players, U[0,1]

player with value v

E(payment) = $\frac{v^2}{2}$?

$= E[V_2 \mid V_2 \le v] \Pr(V_2 \le v)$

$\quad\; \frac{v}{2} \qquad\qquad\quad v$

1st price auction

E[payment of bidder w/ value v]

$= \beta(v)\Pr(\text{wins})$

$\quad \frac{v}{2} \qquad\quad v$

$= \frac{v^2}{2}$

All pay auction: E[pay. v] $v_1, v_2 \sim U[0,1]$

$= \beta_{all}(v) = \frac{v^2}{2}$

$\int_0^1 \frac{v^2}{2}\,dv$