Graceful Degradation over Packet Erasure Channels through Forward Error Correction

Alexander E. Mohr
Electrical Engineering
University of Washington
Seattle, WA 98195–2500
amohr@isdl.ee.washington.edu

Eve A. Riskin
Electrical Engineering
University of Washington
Seattle, WA 98195–2500
riskin@isdl.ee.washington.edu

Richard E. Ladner [*]
Computer Sci. and Engr.
University of Washington
Seattle, WA 98195–2350
ladner@cs.washington.edu

Abstract

We present an algorithm that assigns unequal amounts of forward error correction to progressive data so as to provide graceful degradation as packet losses increase. We use the SPIHT coder to compress images in this work, but our algorithm can protect any progressive compression scheme. The algorithm can also use almost any function as a model of packet loss conditions. We find that for an exponential packet loss model with a mean of 20% and a total rate of 0.2 bpp, good image quality can be obtained, even when 40% of transmitted packets are lost.

1 Introduction

The Internet is a widely deployed network of computers that allows the exchange of data packets. In traversing the network, a packet is sent from computer to computer until it arrives at its destination. Because of congestion, however, packets are often discarded, causing loss of data and most likely decoding failure if the lost data are not replaced. When each packet is assigned a unique sequence number, it is known which packets are received and which are lost. If the underlying transport protocol does not assign a sequence number, one or two bytes of the payload can be used to provide one. The sequence number allows the receiver to sort the packets according to their transmission order: any gaps in the sequence are known to be lost packets and out-of-order packets can be positioned correctly. The receiver takes whatever action it deems best to decode the data.

In networks in which packets are discarded at random, there is no way to specify the importance of a particular packet. However, the data that we transmit usually vary in importance. If we transmit a portrait of a face, for example, data that let us recognize the person are more important than data that show the texture of a few strands of hair. If the network is unable to transmit all of the data, then we would like it to discard the part describing the hair and retain the part that allows recognition of the face. Such a coding strategy needs to quantify the importance of different chunks of data and protect important data more than it protects less-important data. A compression algorithm that provides a progressive ordering of the data will enable us to determine which data are most important to the signal quality.

In this paper, we describe an algorithm to assign unequal amounts of forward error correction (FEC) to images that are first compressed with an unmodified progressive algorithm (such as SPIHT [1]) and then transmitted over lossy packet networks. Our scheme is modular in that

[*]This work was supported by U. S. Army Research Office Grant DAAH004-96-1-0255 and an NSF Young Investigator Award. This paper appears in Proc. of DCC 1999.

we can use any progressive compression algorithm and have graceful degradation with increasing packet loss rate. We focus on those channels without feedback whose variable loss rates can be modeled by a probability mass function (PMF). Notable examples are Asynchronous Transfer Mode (ATM) networks and UDP-based transport on the Internet.

1.1 Set Partitioning in Hierarchical Trees

An example of a progressive image compression algorithm is SPIHT [1]. This algorithm is an extension of Shapiro's Embedded Zerotree Wavelet algorithm [2]. These two new algorithms are a significant breakthrough in lossy image compression in that they give substantially higher compression ratios than prior lossy compression techniques including JPEG [3], vector quantization [4], and the discrete wavelet transform [5] combined with quantization. In addition, the algorithms allow for progressive transmission [6] (meaning coarse approximations of an image can be reconstructed quickly from beginning parts of the bitstream), require no training, and are of low computational complexity.

1.2 Joint Source/Channel Coding Using SPIHT

Joint source/channel coding is an area that has attracted a significant amount of research effort. Despite the fact that Shannon's separation theorem [7] states that for a noisy channel, the source and channel coders can be independently designed and cascaded with the same results as given by a joint source/channel coder, complexity considerations have led numerous researchers develop joint source/channel coding techniques [8, 9]. To date, a majority of this effort has been for fixed rate codes because they do not suffer from the synchronization problems that occur with variable rate codes. (Notable exceptions that have considered joint source/channel coding schemes for variable rate codes include work on reversible variable length codes that can be decoded in both directions [10]. However, these codes can still have problems with synchronization.)

SPIHT has the advantage of yielding impressive compression ratios for still images, but images compressed with SPIHT are vulnerable to data loss. Furthermore, because SPIHT produces an *embedded* or *progressive* bitstream, meaning that the later bits in the bitstream refine earlier bits, the earlier bits are needed for the later bits to even be useful. However, SPIHT's excellent compression performance is leading researchers to consider the problem of transmitting images compressed with SPIHT over lossy channels and networks.

1.3 Prior Work on Transmitting SPIHT Over Lossy Channels

Sherwood and Zeger [11] protected images compressed with SPIHT against noise from the memoryless binary symmetric channel with RCPC codes [12] with good results. They extended this work to images transmitted over the Gilbert-Elliott channel (a fading channel) in [13]. In the latter case, they implement a product code of RCPC and Reed-Solomon codes and find that this outperforms the work in [11] even for the binary symmetric channel.

Rogers and Cosman [14] used a fixed-length packetization scheme called packetized zerotree wavelet (PZW) compression to transmit images compressed with a modified SPIHT over lossy packet networks. The algorithm does not use any channel coding, and hence can be considered joint source/channel coding. They implemented a scheme to fit as many complete wavelet trees (i.e. one coefficient from the lowest frequency wavelet subband along with all its descendants) as possible into a packet. The algorithm degrades gracefully in the presence of packet loss because the packets are independent. If a packet is lost, they attempt to reconstruct the lowest frequency coefficients from the missing trees of wavelet coefficients by interpolating from neighboring low frequency coefficients that have been correctly received by the decoder. To simplify their algorithm, they used fewer levels of wavelet decomposition and removed the arithmetic coder from the SPIHT algorithm. The modification of the SPIHT algorithm caused a decrease of about 1.1 dB in the PSNR at 0.209 bits per pixel for the case of a channel without losses.

These two schemes were combined into a hybrid scheme in [15]. The authors consider the case where, in addition to packet loss, packets can arrive with bit errors in them. They use channel coding to correct bit errors and PZW to conceal packet losses. If they can not correct all of the bit errors in a packet, they consider the packet to be erased. The hybrid scheme shows resilience to packet loss, bit errors, and error bursts. It is still based on the modified SPIHT algorithm used in [14], which does not perform as well as the original SPIHT algorithm.

1.4 Forward Error Correction for Packet Erasure Channels

Priority Encoding Transmission (PET) [16] is an algorithm that assigns FEC, according to priorities specified by the user, to message fragments (also specified by the user) sent over lossy packet networks. Each of these fragments is protected from packet erasures by added FEC. Priorities can range from high, which means that the message fragment can be recovered if relatively few packets are received by the decoder, to low, meaning that most or all packets need to be received to recover the message fragment. This recovery is possible by treating the message fragment as the coefficients of a polynomial in a Galois field and evaluating it at a number of additional points, thus creating redundant data [17]. The receiver can recover the message fragment by interpolation from any subset of the transmitted packets, so long as it receives a fraction of packets at least as large as the priority of the message fragment.

In the PET algorithm, each message fragment is assigned a fixed position within each packet. For figure 1, the first fragment M_1 and its FEC F_1 consist of the first L_1 bytes of each packet, the second fragment M_2 and its FEC F_2 consist of bytes from (L_1+1) to (L_2) of each packet, and M_3 and F_3 consist of the remaining bytes of each packet. PET determines the value of L_i for each fragment and the total number of packets N, making the assumption that the number of fragments is much smaller than the number of bytes in each packet, and constrained by the user-specified priorities.

The PET algorithm does not specify how to choose the priorities to assign to the various message fragments: this assignment is left to the user. It defines priorities as the fraction of transmitted packets that must be received to decode the message; thus a high priority is represented by a low percentage. Leicher [18] applied PET to video compressed with MPEG and transmitted over packet loss channels. He used a simple three-class system in which M_1 was the intraframe (I) frames and had priority 60%, M_2 was the forward-only predicted (P) frames and had priority 80%, and M_3 was the forward-backward predicted (B) frames and had priority 95%. Thus, he can recover the I frames from 60% of the packets, the I and P frames from 80% of the packets, and all the data from 95% of the packets. This is diagrammed in Figure 1.

In related work, Davis, Danskin, and Song [19] presented fast lossy Internet image transmission (FLIIT) which is a joint source/channel coding algorithm that, like PET, assigns different levels of FEC to different types of data, but it considers distortion-rate tradeoffs in its assignments. They begin with a 5-level discrete wavelet transform, create an embedded bitstream by quantizing each subband's coefficients in bit planes, apply entropy coding, and pack the bitstream from each subband into 64-byte blocks. To do bit allocation, they determine the reduction in distortion due to each block, similar to work in [20]. They then compare the greatest decrease in distortion from those blocks with the addition of a block of FEC data to

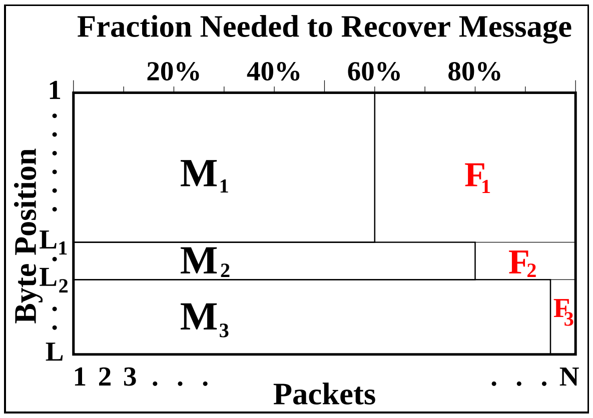


Figure 1: In Leicher's application of PET to MPEG [18], he applied 60% priority to message fragment M_1 (the I frames), 80% priority to M_2 (the P frames), and 95% priority to M_3 (the B frames). Each message and its associated FEC use the same range of bytes in every packet.

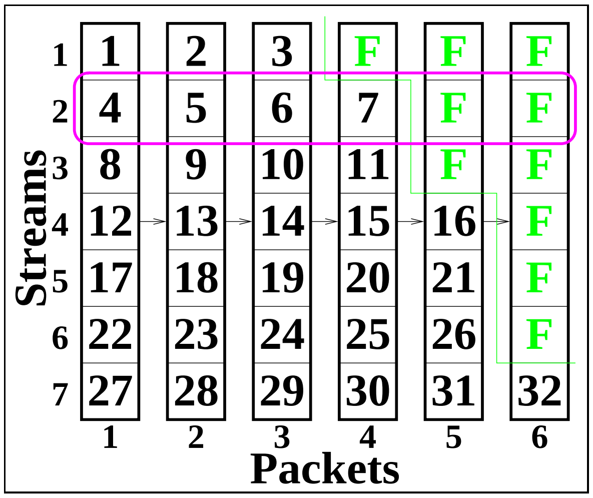


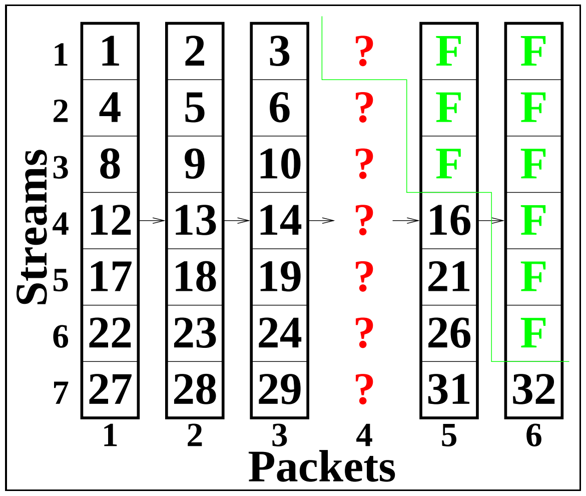
Figure 2: Each of the rows is a stream and each of the columns is a packet. A stream contains one byte from each packet. Here, a message of 32 bytes of data (numbers 1-32) and ten bytes of FEC (F) is divided into seven streams and sent in six packets.

the already-allocated blocks. They allocate the block of data or block of FEC that decreases the expected distortion the most. They only consider three simple cases of assigning FEC to a block: no protection, protection that consists of one FEC block shared among a group of blocks, and replication of the block. They find that, as expected, it is advantageous to apply more FEC to the coarse/low frequency wavelet scales and to the most significant bit planes of the quantization.

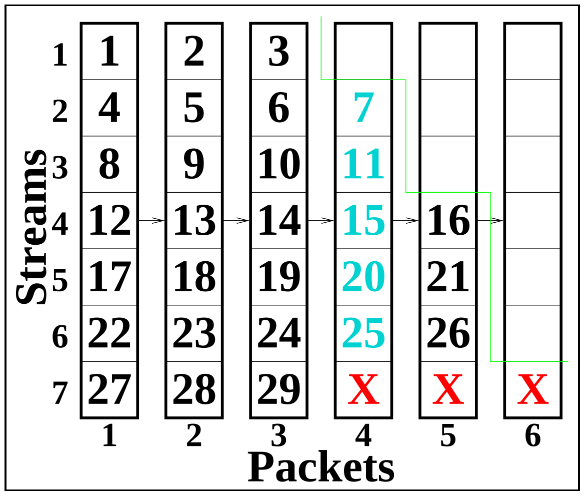
2 An Algorithm for Assigning Forward Error Correction

While the algorithms in [13, 14, 15, 19] yield good results for memoryless and fading channels and for lossy packet networks, there are additional ways to transmit compressed images over lossy networks such that image quality gracefully degrades with increasing packet loss. Specifically, we will protect images transmitted over lossy channels with unequal amounts of FEC in a manner similar to the PET scheme, but we will consider the increase in PSNR due to each data byte when assigning our protection.

In our approach to using PET to assign unequal amounts of FEC to progressive data, we remove PET's restriction that the number of message fragments be much less than the number of bytes in each packet. Instead, we use a number of message fragments equal to the number of available bytes in each packet and have our algorithm dynamically choose the length and content of each message fragment. We add FEC to each message fragment to protect against packet loss such that the fragment and the FEC form a *stream*. The message is divided into L streams such that each stream has one byte of each of N packets. In Figure 2, each of the $L = 7$ rows is a stream and each of the $N = 6$ columns is a packet. For a given stream i, for $i = 1, 2, \ldots, L$, containing both data bytes and FEC bytes, as long as the number of lost packets is less than or equal to the number of FEC bytes, the entire stream can be decoded



(a)



(b)

Figure 3: Demonstration of how much data can be recovered when one of six packets is lost. Here, stream one is unaffected by the loss, streams two through five use FEC to recover from the loss, and in stream seven, only the bytes up to the lost packet are useful to the decoder.

[16]. Figure 2 shows one possible way to send a message of 32 bytes of data (numbers 1-32) and ten bytes of FEC (F). Notice that in the figure, more bytes of FEC are applied to the earlier parts of the message and fewer are used for the later parts of the message. For SPIHT, which produces an embedded bitstream, the earlier parts of the message should have the highest priority because they are most important to the overall quality of the reproduction.

Figure 3 shows the case where one packet out of six is lost, and five are received correctly. In this case, the first six streams can be recovered since they contain five or fewer data bytes. The last stream can not be decoded since it contains six bytes of data and no FEC. We point out that bytes 27-29 from the seventh stream are useful since they were received correctly, but for an embedded bitstream, bytes 31 and 32 are not useful without byte 30. Similarly, if two packets are lost, bytes 1-11 are guaranteed to be recovered and bytes 12-15 may or may not be recovered. In messages of practical length, however, those few extra bytes have only a small effect on image quality. Analogous to progressive transmission [6], even if severe packet loss occurred, we could recover a lower fidelity version of the image from the earlier streams that are decoded correctly. Each additional stream that is successfully decoded improves the quality of the received message, as long as all previous streams are correctly decoded.

2.1 Notation

Assume we have a message M, which is simply a sequence of data bytes to be transmitted. For example, this could be a still image compressed with SPIHT to 0.5 bits per pixel. If, instead of sending M, we send a prefix of M and some FEC, we can still maintain the same overall bit rate. We let m_i equal the number of data bytes assigned to stream i and let $f_i = N - m_i$ equal the number of FEC bytes assigned to stream i. We define an L-dimensional FEC vector whose entries are the length of FEC assigned to each stream as:

$$\bar{f} = (f_1, f_2, \ldots, f_L).$$

For a given $\bar{f}$, we divide M into fragments $M_i(\bar{f})$ and define $M_i(\bar{f})$ to be the sequence of data bytes in the ith stream. That is, $M_i(\bar{f})$ includes the bytes of message M from position $\sum_{j=1}^{i-1} m_j + 1$ to position $\sum_{j=1}^{i} m_j$; $i = 2, 3, \ldots, L$, with $M_1(\bar{f})$ composed of m_1 bytes of stream 1. We denote a prefix of M containing the first j fragments for redundancy vector $\bar{f}$ as:

$$M(j, \bar{f}) = M_1(\bar{f}) M_2(\bar{f}) \ldots M_j(\bar{f}).$$

We now introduce the *incremental PSNR* of stream i:

$$g_i(\bar{f}) = \text{PSNR}[M(i, \bar{f})] - \text{PSNR}[M(i-1, \bar{f})].$$

The quantity $g_i(\bar{f})$ is the amount by which the PSNR increases when the receiver decodes fragment i, given that all fragments prior to i have already been decoded. We set $g_1(\bar{f})$ to be the difference in PSNR between the case in which $M_1(\bar{f})$ is received and the case in which no information is received.

Because the data are progressive, we require that $f_i \geq f_{i+1}; i = 1, 2, \ldots, L-1$; that is, the FEC assigned to the streams is non-increasing with i. With this requirement, if $M_i(\bar{f})$ can be decoded, then $M_1(\bar{f}), M_2(\bar{f}), \ldots, M_{i-1}(\bar{f})$ can also be decoded.

The packet loss model we use is given by a PMF $p_n; n = 0, 1, \ldots, N$, such that p_n is the probability that n packets are lost. To simplify calculations, we determine the probability that k or fewer packets are lost, and thus the CDF is $c(k) = \sum_{n=0}^{k} p_n; k = 0, 1, \ldots, N$. The quantity $c(f_i)$ is the probability that receiver can decode stream i.

We can now calculate the expected PSNR of the received message as a function of $\bar{f}$ by summing over the L streams:

$$G(\bar{f}) = \sum_{i=1}^{L} c(f_i) g_i(\bar{f}).$$

In designing our algorithm to assign FEC, we seek the $\bar{f}$ that maximizes $G(\bar{f})$ subject to our packet loss model p_n.

2.2 Channel Loss Model

To determine the FEC vector $\bar{f}$, we need an estimate of the channel loss model p_n that a message is likely to encounter. In keeping with our modular design philosophy, we assume that channel loss behavior can be modeled by an *estimator* that outputs a PMF that indicates the likelihood that a particular number of packets is lost. How the estimator derives its estimate of current channel conditions is not addressed by this paper.

2.3 The Algorithm

Finding the globally optimal assignment of FEC data to each stream is computationally prohibitive for a useful amount of data. We therefore developed a hill-climbing algorithm that makes limited assumptions about the data, but is computationally tractable. As mentioned in Section 2.1, we constrain $f_i \geq f_{i+1}$. Additionally, we assume that a single byte missing from the progressive bitstream causes all later bytes to become useless. Finally, the PMF should be reasonably well-behaved: the more it deviates from a unimodal function, the larger the search distance must be to capture the irregularities.

We initialize each stream to contain only data bytes, such that $m_i = N$ and $f_i = 0; i = 1, 2, \ldots, L$. In each iteration, our algorithm examines a number of possible assignments equal to $2QL$, where Q is the search distance (maximum number of FEC bytes that can be added

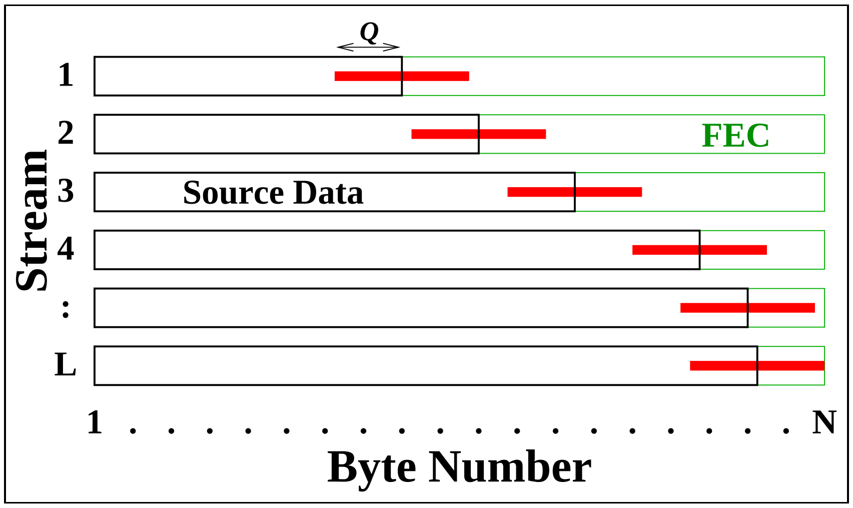


Figure 4: At each iteration of the optimization algorithm, Q bytes of data can be added or subtracted to any of the L streams.

or subtracted to a stream in one iteration) and L is the number of streams. We determine $G(\bar{f})$ after adding or subtracting 1 to Q bytes of FEC data to each stream (see Figure 4), while satisfying our constraint $f_i \geq f_{i+1}$. We choose the $\bar{f}$ corresponding to the highest $G(\bar{f})$, update the allocation of FEC data to all affected streams, and repeat the search until none of the cases examined improves the expected PSNR. This algorithm will find a local maximum that we believe is quite close to the global maximum and, in some cases, may be identical to the global maximum.

Note that for every byte of FEC data that we add to a stream, one byte of data needs to be removed. When changing the FEC assignment, we start at the first stream affected by the new allocation, move its last data byte to the next stream, move the last data byte of this stream to the next stream, and so on. This causes a cascade of data bytes to move down the streams until the last data byte from stream L is discarded. This part of the algorithm makes use of our assumption that the compressed sequence is progressive, because the data byte that we discard is among the least important in the embedded bitstream.

3 Results

For these experiments, we used the standard 512×512 gray-scale Lenna image compressed with SPIHT. We chose a total bit rate of 0.2 bits per pixel for the combination of data and FEC bytes and used an exponential packet loss model with a mean loss rate of 20%. Because ATM packets have a payload length of 48 bytes and one byte is required for a sequence number, we place 47 bytes of data in each packet and send 137 packets, giving a total payload size of 6576 bytes, of which 6439 are data. Including the sequence number, the bit rate is 0.201 bits per pixel. Excluding it, the bit rate is 0.197 bits per pixel. Convergence of the algorithm is typically reached in about 27 iterations and requires 7.3 seconds of CPU time on a Sun SPARCstation 10 workstation or 0.5 seconds on an Intel Pentium II 300MHz workstation.

We had our algorithm maximize expected PSNR for two cases: unequal erasure protection, in which the algorithm could freely allocate FEC to the compressed data; and equal erasure protection, in which the algorithm was constrained to assign FEC equally among all of the streams. For unequal assignment, the result was an allocation with an expected PSNR

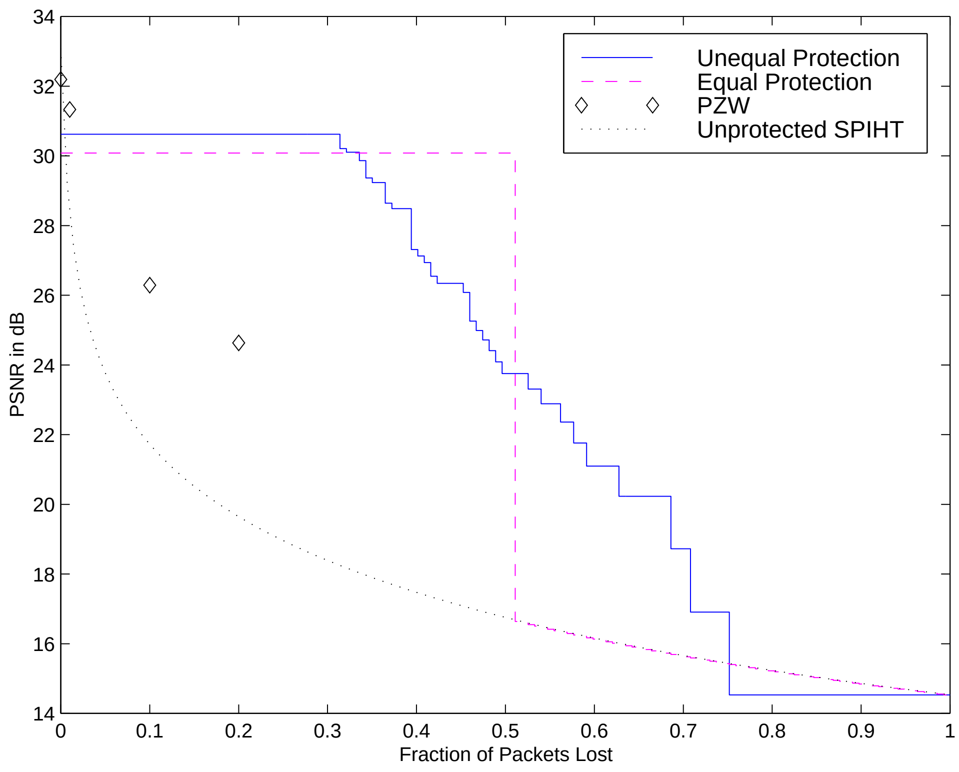


Figure 5: Effect of packet loss on PSNR for unequal erasure protection, equal erasure protection, Rogers and Cosman's PZW [14], and unprotected SPIHT. The two erasure protection results are from an exponential packet loss model with a mean loss rate of 20%.

of 29.42 dB. For equal assignment, the result was an allocation with an expected PSNR of 28.94 dB, or 0.48 dB lower than the unequal assignment result.

As shown in Figure 5, under good channel conditions (erasure rates of up to 32%, which occur 80% of the time) unequal erasure protection yields a PSNR that is 0.66 dB higher than equal erasure protection. This is because more bytes are used for data and fewer for FEC. Equal erasure protection does surpass unequal erasure protection when loss rates are 33% to 51%, but those occur with only 11% probability. In addition, unequal erasure protection degrades gracefully whereas equal erasure protection has a sharp transition at loss rates near 51%. As expected, both of these cases substantially outperform not using any protection on the data, except when the loss rate is very low.

At those low loss rates, e.g. below 1-2%, unprotected SPIHT will often survive with a significant prefix of the transmitted data remaining intact, and the more-robust PZW coder [14] will perform slightly better. On the other hand, their performance degrades rapidly as losses increase, while the addition of FEC allows protected data to survive at larger loss rates. We also note that the protected data is affected only by the number of lost packets, but the reconstruction quality of unprotected SPIHT, and to a lesser extent PZW, depends upon which packets are lost.

We display results of using unequal erasure protection in Figure 6. It shows the graceful degradation of the image transmitted over a lossy packet network with loss rates of 30%, 40%, 50%, and 60%. Notice that the image quality remains high at a loss rate of 40% and the image is still recognizable at a loss rate of 50%.

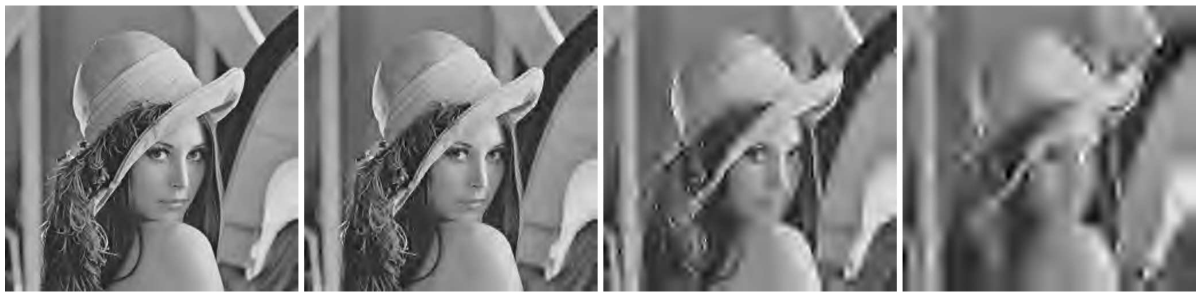


Figure 6: Image quality at 0.2 bpp total rate for unequal erasure protection on a channel with an exponential loss model that averages 20%. Loss rates from left to right: 30%, 40%, 50%, and 60%.

Finally, we show how the algorithm assigns data and FEC to the different data streams for the Lenna image compressed with the SPIHT algorithm in Figure 7. Stream 1 is the first stream (most important data from the SPIHT algorithm) and it has an assignment of 24% data and 76% FEC. Stream 47 is the last stream (least important data) with 70% data and only 30% FEC. As expected, the amount of FEC decreases with increasing stream number, as required by our algorithm.

4 Conclusion

We have developed an algorithm that determines the amount of FEC to assign to data from SPIHT to provide graceful degradation in the case of packet loss. Our algorithm is modular and can input any compression scheme that produces a progressive bitstream. Future work will include allowing more constraints such as specifying a minimum usable PSNR, speeding up the FEC assignment algorithm so that it can be recalculated in real-time, and applying it to other progressive compression algorithms for images and video.

The authors would like to thank Gary W. Garrabrant for helpful discussions about lossy packet networks, and Professor William Pearlman for the SPIHT source code. Demo programs and data files can be found at http://isdl.ee.washington.edu/compression/amohr/uep/.

References

[1] A. Said and W. A. Pearlman, "A new, fast, and efficient image codec based on set partitioning in hierarchical trees," *IEEE Transactions on Circuits and Systems for Video Technology*, vol. 6, pp. 243–250, June 1996.

[2] J. M. Shapiro, "Embedded image coding using zerotrees of wavelet coefficients," *IEEE Transactions on Signal Processing*, vol. 41, pp. 3445–3462, Dec. 1993.

[3] W. B. Pennebaker and J. L. Mitchell, *JPEG: Still Image Data Compression Standard.* New York: Van Nostrand Reinhold, 1993.

[4] A. Gersho and R. M. Gray, *Vector Quantization and Signal Compression.* Boston: Kluwer Academic Publishers, 1992.

[5] M. Antonini, M. Barlaud, P. Mathieu, and I. Daubechies, "Image coding using wavelet transform," *IEEE Transactions on Image Processing*, vol. 1, pp. 205–220, Apr. 1992.

[6] K. Tzou, "Progressive image transmission: a review and comparison of techniques," *Optical Engineering*, vol. 26, pp. 581–589, July 1987.

[7] C. E. Shannon, "A mathematical theory of communication," *Bell Systems Technical Journal*, vol. 27, pp. 379–423, 1948.

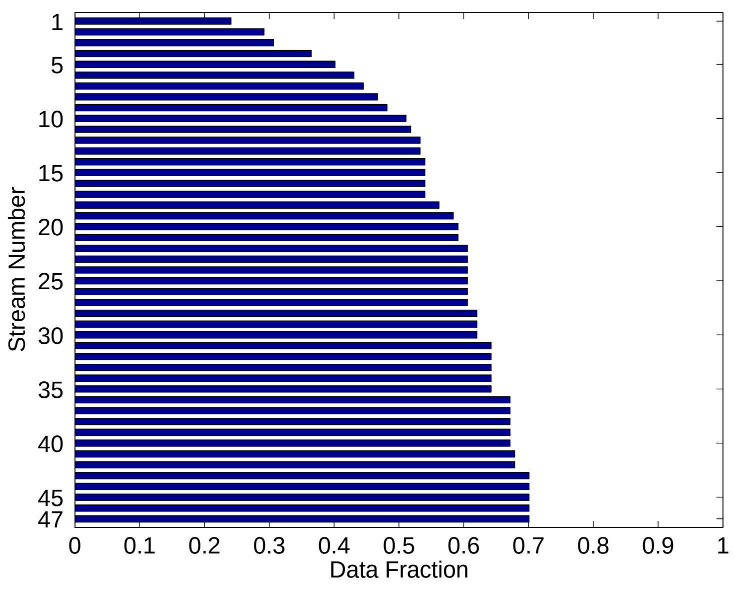


Figure 7: Data fraction for each stream. Note that the FEC fraction is (1 - Data Fraction). Stream 1 is the first stream (most important data) and stream 47 is the last stream (least important data).

[8] N. Farvardin, "A study of vector quantization for noisy channels," *IEEE Transactions on Information Theory*, vol. 36, pp. 799–809, July 1990.

[9] R.-Y. Wang, E. A. Riskin, and R. Ladner, "Codebook organization to enhance maximum a priori detection of progressive transmission of vector quantized images over noisy channels," *IEEE Transactions on Image Processing*, vol. 5, pp. 37–48, Jan. 1996.

[10] J. Wen and J. D. Villasenor, "Reversible variable length codes for efficient and robust image and video coding," in *Proceedings Data Compression Conference*, pp. 471–480, March-April 1998.

[11] P. G. Sherwood and K. Zeger, "Progressive image coding for noisy channels," *IEEE Signal Processing Letters*, vol. 4, pp. 189–191, July 1997.

[12] J. Hagenauer, "Rate-compatible punctured convolutional codes (RCPC Codes) and their applications," *IEEE Transactions on Communications*, vol. 36, pp. 389–400, Apr. 1988.

[13] P. G. Sherwood and K. Zeger, "Error protection for progressive image transmission over memoryless and fading channels," *IEEE Transactions on Communications*, 1998. Accepted for publication.

[14] J. K. Rogers and P. C. Cosman, "Robust wavelet zerotree image compression with fixed-length packetization," in *Proceedings Data Compression Conference*, pp. 418–427, March-April 1998.

[15] P. C. Cosman, J. K. Rogers, P. G. Sherwood, and K. Zeger, "Combined forward error control and packetized zerotree wavelet encoding for transmission of images over varying channels," *IEEE Transactions on Image Processing*, 1998. Submitted for publication.

[16] A. Albanese, J. Blömer, J. Edmonds, M. Luby, and M. Sudan, "Priority encoding transmission," *IEEE Transactions on Information Theory*, vol. 42, pp. 1737–1744, Nov. 1996.

[17] L. Rizzo, "Effective erasure codes for reliable computer communication protocols," *ACM Computer Communication Review*, vol. 27, pp. 24–36, Apr. 1997.

[18] C. Leicher, "Hierarchical encoding of MPEG sequences using priority encoding transmission (PET)," Tech. Rep. TR-94-058, ICSI, Nov. 1994.

[19] G. M. Davis, J. M. Danskin, and X. Song, "Joint source and channel coding for Internet image transmission," in *Proceedings of ICIP*, vol. 1, pp. 21–24, Sept. 1996.

[20] Y. Shoham and A. Gersho, "Efficient bit allocation for an arbitrary set of quantizers," *IEEE Transactions on Acoustics Speech and Signal Processing*, vol. 36, pp. 1445 – 1453, Sept. 1988.