Dan Grossman; Graduate Programming Languages; Lecture 6 Supplement

In class we sketched several proofs, but proof sketches invariably skip steps and have small errors. Here are the proofs more carefully laid out, as one might do on a homework assignment.

Theorem: $H\ ;\ e * 2 \Downarrow c$ if and only if $H\ ;\ e + e \Downarrow c$.

Proof: (Does not use induction)

- First assume $H\ ;\ e * 2 \Downarrow c$ and show $H\ ;\ e + e \Downarrow c$. Any derivation of $H\ ;\ e * 2 \Downarrow c$ must end with the MULT rule, which means there must exist derivations of $H\ ;\ e \Downarrow c'$ and $H\ ;\ 2 \Downarrow 2$, and c must be $2c'$. That is, there must be a derivation that looks like this:

$$\dfrac{\dfrac{\vdots}{H\ ;\ e \Downarrow c'} \qquad \dfrac{}{H\ ;\ 2 \Downarrow 2}}{H\ ;\ e * 2 \Downarrow 2c'}$$

  So given that there exists a derivation of $H\ ;\ e \Downarrow c'$, we can use ADD to derive:

$$\dfrac{H\ ;\ e \Downarrow c' \qquad H\ ;\ e \Downarrow c'}{H\ ;\ e + e \Downarrow c' + c'}$$

  Math provides $c' + c' = 2c'$, so the conclusion of this derivation is what we need.

- Now assume $H\ ;\ e + e \Downarrow c$ and show $H\ ;\ e * 2 \Downarrow c$. Any derivation of $H\ ;\ e + e \Downarrow c$ must end with the ADD rule, which means there exists a derivation that looks like this (where $c = c_1 + c_2$):

$$\dfrac{\dfrac{\vdots}{H\ ;\ e \Downarrow c_1} \qquad \dfrac{\vdots}{H\ ;\ e \Downarrow c_2}}{H\ ;\ e + e \Downarrow c_1 + c_2}$$

  In fact, we earlier proved determinacy (there is at most one c such that $H\ ;\ e \Downarrow c$), so the derivation must have this form (where $c = c_1 + c_1$):

$$\dfrac{\dfrac{\vdots}{H\ ;\ e \Downarrow c_1} \qquad \dfrac{\vdots}{H\ ;\ e \Downarrow c_1}}{H\ ;\ e + e \Downarrow c_1 + c_1}$$

  So given that there exists a derivation of $H\ ;\ e \Downarrow c_1$, we can use MULT to derive:

$$\dfrac{H\ ;\ e \Downarrow c_1 \qquad \dfrac{}{H\ ;\ 2 \Downarrow 2}}{H\ ;\ e * 2 \Downarrow 2c_1}$$

  Math provides $c_1 + c_1 = 2c_1$, so the conclusion of this derivation is what we need.

$$C ::= [\cdot] \mid C + e \mid e + C \mid C * e \mid e * C$$

Formal definition of "filling the hole":

$$\begin{array}{rcl} ([\cdot])[e] & = & e \\ (C + e_1)[e] & = & C[e] + e_1 \\ (e_1 + C)[e] & = & e_1 + C[e] \\ (C * e_1)[e] & = & C[e] * e_1 \\ (e_1 * C)[e] & = & e_1 * C[e] \end{array}$$

Theorem: $H \ ; \ C[e * 2] \Downarrow c$ if and only if $H \ ; \ C[e + e] \Downarrow c$.

Proof: By induction on (the height of) the structure of C:

- If the height is 0, then C is $[\cdot]$, so $C[e * 2] = e * 2$ and $C[e + e] = e + e$. So the previous theorem is exactly what we need.
- If the height is greater than 0, then C has one of four forms:
  - If C is $C' + e'$ for some C' and e', then $C[e * 2]$ is $C'[e * 2] + e'$ and $C[e + e]$ is $C'[e + e] + e'$. Since C' is shorter than C, induction ensures that for any constant c', $H \ ; \ C'[e * 2] \Downarrow c'$ if and only if $H \ ; \ C'[e + e] \Downarrow c'$.

    Assume $H \ ; \ C'[e * 2] + e' \Downarrow c$ and show $H \ ; \ C'[e + e] + e' \Downarrow c$: Any derivation of $H \ ; \ C'[e * 2] + e' \Downarrow c$ must end with ADD, i.e., it looks like this (where $c = c' + c''$):

    $$\dfrac{\dfrac{\vdots}{H \ ; \ C'[e * 2] \Downarrow c'} \qquad \dfrac{\vdots}{H \ ; \ e' \Downarrow c''}}{H \ ; \ C'[e * 2] + e' \Downarrow c}$$

    As argued above, the existence of a derivation of $H \ ; \ C'[e * 2] \Downarrow c'$ ensures the existence of a derivation of $H \ ; \ C'[e + e] \Downarrow c'$. So using ADD and the existence of a derivation of $H \ ; \ e' \Downarrow c''$, we can derive:

    $$\dfrac{H \ ; \ C'[e + e] \Downarrow c' \qquad H \ ; \ e' \Downarrow c''}{H \ ; \ C'[e + e] + e' \Downarrow c}$$

    Now assume $H \ ; \ C'[e + e] + e' \Downarrow c$ and show $H \ ; \ C'[e * 2] + e' \Downarrow c$: Any derivation of $H \ ; \ C'[e + e] + e' \Downarrow c$ must end with ADD, i.e., it looks like this (where $c = c' + c''$):

    $$\dfrac{\dfrac{\vdots}{H \ ; \ C'[e + e] \Downarrow c'} \qquad \dfrac{\vdots}{H \ ; \ e' \Downarrow c''}}{H \ ; \ C'[e + e] + e' \Downarrow c}$$

    As argued above, the existence of a derivation of $H \ ; \ C'[e + e] \Downarrow c'$ ensures the existence of a derivation of $H \ ; \ C'[e * 2] \Downarrow c'$. So using ADD and the existence of a derivation of $H \ ; \ e' \Downarrow c''$, we can derive:

    $$\dfrac{H \ ; \ C'[e * 2] \Downarrow c' \qquad H \ ; \ e' \Downarrow c''}{H \ ; \ C'[e * 2] + e' \Downarrow c}$$

  - The other 3 cases are similar. (Try them out.)

Theorem: The two semantics below are equivalent, i.e., $H\ ;\ e \Downarrow c$ if and only if $H;\ e \rightarrow^* c$.

$$\frac{}{H\ ;\ c \Downarrow c}\ \textsc{const} \qquad \frac{}{H\ ;\ x \Downarrow H(x)}\ \textsc{var} \qquad \frac{H\ ;\ e_1 \Downarrow c_1 \qquad H\ ;\ e_2 \Downarrow c_2}{H\ ;\ e_1 + e_2 \Downarrow c_1{+}c_2}\ \textsc{add}$$

$$\frac{}{H;\ x \rightarrow H(x)}\ \textsc{svar} \qquad \frac{}{H;\ c_1 + c_2 \rightarrow c_1{+}c_2}\ \textsc{sadd} \qquad \frac{H;\ e_1 \rightarrow e_1'}{H;\ e_1 + e_2 \rightarrow e_1' + e_2}\ \textsc{sleft} \qquad \frac{H;\ e_2 \rightarrow e_2'}{H;\ e_1 + e_2 \rightarrow e_1 + e_2'}\ \textsc{sright}$$

Proof: We prove the two directions separately.

First assume $H\ ;\ e \Downarrow c$; show $\exists n.\ H;\ e \rightarrow^n c$. By induction on the height h of derivation of $H\ ;\ e \Downarrow c$:

- $h = 1$: Then the derivation must end with CONST or VAR. For CONST, e is c and trivially $H;\ e \rightarrow^0 c$. For VAR, e is some x where $H(x) = c$, so using SVAR, $H;\ e \rightarrow^1 c$.
- $h > 1$: Then the derivation must end with ADD, so e is some $e_1 + e_2$ where $H\ ;\ e_1 \Downarrow c_1$, $H\ ;\ e_2 \Downarrow c_2$, and c is $c_1{+}c_2$. By induction $\exists n_1, n_2.\ H;\ e_1 \rightarrow^{n_1} c_1$ and $H;\ e_2 \rightarrow^{n_2} c_2$. Therefore, using the lemma below, $H;\ e_1 + e_2 \rightarrow^{n_1} c_1 + e_2$ and $H;\ c_1 + e_2 \rightarrow^{n_2} c_1 + c_2$, so ADD lets us derive $H;\ e_1 + e_2 \rightarrow^{n_1+n_2+1} c$.

Lemma: If $H;\ e \rightarrow^n e'$, then $H;\ e_1 + e \rightarrow^n e_1 + e'$ and $H;\ e + e_2 \rightarrow^n e' + e_2$.

Proof: By induction on n. If $n = 0$, the result is trivial because $e = e'$. If $n > 0$, then there exists some e'' such that $H;\ e \rightarrow^{n-1} e''$ and $H;\ e'' \rightarrow^1 e'$. So by induction $H;\ e_1 + e \rightarrow^{n-1} e_1 + e''$ and $H;\ e + e_2 \rightarrow^{n-1} e'' + e_2$. Using SRIGHT and SLEFT respectively, $H;\ e'' \rightarrow^1 e'$ ensures $H;\ e_1 + e'' \rightarrow^1 e_1 + e'$ and $H;\ e'' + e_2 \rightarrow^1 e' + e_2$. So with the inductive hypotheses, $H;\ e_1 + e \rightarrow^n e_1 + e'$ and $H;\ e + e_2 \rightarrow^n e' + e_2$.

Now assume $\exists n.\ H;\ e \rightarrow^n c$; show $H\ ;\ e \Downarrow c$. By induction on n:

- $n = 0$: e is c and CONST lets us derive $H\ ;\ c \Downarrow c$.
- $n > 0$: So $\exists e'.\ H;\ e \rightarrow e'$ and $H;\ e' \rightarrow^{n-1} c$. By induction $H\ ;\ e' \Downarrow c$. So this lemma suffices: If $H;\ e \rightarrow e'$ and $H\ ;\ e' \Downarrow c$, then $H\ ;\ e \Downarrow c$. Prove the lemma by induction on height h of derivation of $H;\ e \rightarrow e'$:
  - $h = 1$: Then the derivation ends with SVAR or SADD. For SVAR, e is some x and $e' = H(x) = c$. So with VAR we can derive $H\ ;\ x \Downarrow H(x)$, i.e., $H\ ;\ e \Downarrow c$. For SADD, e is some $c_1 + c_2$ and $e' = c = c_1{+}c_2$. So with ADD, we can derive $H\ ;\ c_1 + c_2 \Downarrow c_1{+}c_2$, i.e., $H\ ;\ e \Downarrow c$. (Note the $h = 1$ case may look a little weird because in fact in this case $n = 1$, i.e., e' must be a constant.)
  - $h > 1$: Then the derivation ends with SLEFT or SRIGHT. For SLEFT, the assumed derivations end like this:

    $$\frac{H;\ e_1 \rightarrow e_1'}{H;\ e_1 + e_2 \rightarrow e_1' + e_2} \qquad \frac{H\ ;\ e_1' \Downarrow c_1 \qquad H\ ;\ e_2 \Downarrow c_2}{H\ ;\ e_1' + e_2 \Downarrow c_1{+}c_2}$$

    Using $H;\ e_1 \rightarrow e_1'$, $H\ ;\ e_1' \Downarrow c_1$, and the induction hypothesis, $H\ ;\ e_1 \Downarrow c_1$. Using this fact, $H\ ;\ e_2 \Downarrow c_2$, and ADD, we can derive $H\ ;\ e_1 + e_2 \Downarrow c_1{+}c_2$.
    For SRIGHT, the assumed derivations end like this:

    $$\frac{H;\ e_2 \rightarrow e_2'}{H;\ e_1 + e_2 \rightarrow e_1 + e_2'} \qquad \frac{H\ ;\ e_1 \Downarrow c_1 \qquad H\ ;\ e_2' \Downarrow c_2}{H\ ;\ e_1 + e_2' \Downarrow c_1{+}c_2}$$

    Using $H;\ e_2 \rightarrow e_2'$, $H\ ;\ e_2' \Downarrow c_2$, and the induction hypothesis, $H\ ;\ e_2 \Downarrow c_2$. Using this fact, $H\ ;\ e_1 \Downarrow c_1$, and ADD, we can derive $H\ ;\ e_1 + e_2 \Downarrow c_1{+}c_2$.