CS-XXX: Graduate Programming Languages

Lecture 7 — Lambda Calculus

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Where we are

- Done: Syntax, semantics, and equivalence
  - For a language with little more than loops and global variables
- Now: Didn't IMP leave some things out?
  - In particular: scope, functions, and data structures
  - (Not to mention threads, I/O, exceptions, strings, ...)

Time for a new model...

Data + Code

Higher-order functions work well for scope *and* data structures

- Scope: not all memory available to all code

```
let x = 1
let add3 y =
   let z = 2 in
   x + y + z
let seven = add3 4
```

- Data: Function closures store data. Example: Association "list"

```
let empty = (fun k -> raise Empty)
let cons k v lst = (fun k' -> if k'=k then v else lst k')
let lookup k lst = lst k
```

(Later: Objects do both too)

Adding data structures

Extending IMP with data structures is not too hard:

$$\begin{array}{rcl} e & ::= & c \mid x \mid e+e \mid e*e \mid (e,e) \mid e.1 \mid e.2 \\ v & ::= & c \mid (v,v) \\ H & ::= & \cdot \mid H, x \mapsto v \end{array}$$

$\boxed{H \; ; e \Downarrow v}$ all old rules plus:

$$\frac{H \; ; e_1 \Downarrow v_1 \quad H \; ; e_2 \Downarrow v_2}{H \; ; (e_1, e_2) \Downarrow (v_1, v_2)} \qquad \frac{H \; ; e \Downarrow (v_1, v_2)}{H \; ; e.1 \Downarrow v_1} \qquad \frac{H \; ; e \Downarrow (v_1, v_2)}{H \; ; e.2 \Downarrow v_2}$$

Notice:

- We allow pairs of values, not just pairs of integers
- We now have *stuck* programs (e.g., $c.1$)
  - What would C++ do? Scheme? ML? Java? Perl?
  - Division also causes stuckness

What about functions

But adding functions (or objects) does not work well:

$$\begin{array}{rcl} e & ::= & \ldots \mid \texttt{fun}\ x \texttt{ -> } s \\ v & ::= & \ldots \mid \texttt{fun}\ x \texttt{ -> } s \\ s & ::= & \ldots \mid e(e) \end{array}$$

$\boxed{H \; ; e \Downarrow v}$ $\boxed{H \; ; s \to H' \; ; s'}$

Additions:

$$\frac{}{H \; ; \texttt{fun}\ x \texttt{ -> } s \Downarrow \texttt{fun}\ x \texttt{ -> } s} \qquad \frac{H \; ; e_1 \Downarrow \texttt{fun}\ x \texttt{ -> } s \quad H \; ; e_2 \Downarrow v}{H \; ; e_1(e_2) \to H \; ; x := v; s}$$

Does this match "the semantics we want" for function calls?

What about functions

But adding functions (or objects) does not work well:

$$\begin{array}{rcl} e & ::= & \ldots \mid \texttt{fun}\ x \texttt{ -> } s \\ v & ::= & \ldots \mid \texttt{fun}\ x \texttt{ -> } s \\ s & ::= & \ldots \mid e(e) \end{array}$$

$$\frac{}{H \; ; \texttt{fun}\ x \texttt{ -> } s \Downarrow \texttt{fun}\ x \texttt{ -> } s} \qquad \frac{H \; ; e_1 \Downarrow \texttt{fun}\ x \texttt{ -> } s \quad H \; ; e_2 \Downarrow v}{H \; ; e_1(e_2) \to H \; ; x := v; s}$$

NO: Consider $\texttt{x} := 1; (\texttt{fun x -> y} := \texttt{x})(2); \texttt{ans} := \texttt{x}$.

Scope matters; variable name does not. That is:

- Local variables should "be local"
- Choice of local-variable names should have only local ramifications

Another try

$$\frac{H \; ; \; e_1 \Downarrow \texttt{fun}\ x \ \texttt{->}\ s \qquad H \; ; \; e_2 \Downarrow v \qquad y \text{ ``fresh''}}{H \; ; \; e_1(e_2) \to H \; ; \; y := x; x := v; s; x := y}$$

- "fresh" is not very IMP-like but okay (think malloc)
- not a good match to how functions are implemented
- yuck: the way we want to think about something as fundamental as a call?
- NO: wrong model for most functional and OO languages
  - (Even wrong for C if s calls another function that accesses the global variable x)

The wrong model

$$\frac{H \; ; \; e_1 \Downarrow \texttt{fun}\ x \ \texttt{->}\ s \qquad H \; ; \; e_2 \Downarrow v \qquad y \text{ ``fresh''}}{H \; ; \; e_1(e_2) \to H \; ; \; y := x; x := v; s; x := y}$$

$\texttt{f}_1 := (\texttt{fun x -> f}_2 := (\texttt{fun z -> ans} := \texttt{x} + \texttt{z}));$
$\texttt{f}_1(2);$
$\texttt{x} := 3;$
$\texttt{f}_2(4)$

"Should" set ans to 6:

- $\texttt{f}_1(2)$ should assign to $\texttt{f}_2$ a function that adds 2 to its argument and stores result in ans

"Actually" sets ans to 7:

- $\texttt{f}_2(2)$ assigns to $\texttt{f}_2$ a function that adds *the current value of* x to its argument

Punch line

Cannot properly model local scope via a global heap of integers.

- Functions are not syntactic sugar for assignments to globals

So let's build a new model that focuses on this essential concept

- (can add back IMP features later)

Or just borrow a model from Alonzo Church

And drop mutation, conditionals, integers (!), and loops (!)

The Lambda Calculus

The Lambda Calculus:

$$\begin{array}{rcl} e & ::= & \lambda x.\ e \mid x \mid e\ e \\ v & ::= & \lambda x.\ e \end{array}$$

You *apply* a function by *substituting* the argument for the *bound variable*

- (There is an equivalent *environment* definition not unlike heap-copying; see future homework)

Example Substitutions

$$\begin{array}{rcl} e & ::= & \lambda x.\ e \mid x \mid e\ e \\ v & ::= & \lambda x.\ e \end{array}$$

Substitution is the key operation we were missing:

$$(\lambda x.\ x)(\lambda y.\ y) \to (\lambda y.\ y)$$

$$(\lambda x.\ \lambda y.\ y\ x)(\lambda z.\ z) \to (\lambda y.\ y\ \lambda z.\ z)$$

$$(\lambda x.\ x\ x)(\lambda x.\ x\ x) \to (\lambda x.\ x\ x)(\lambda x.\ x\ x)$$

After substitution, the bound variable is gone, so its "name" was irrelevant. (Good!)

A Programming Language

Given substitution ($e_1[e_2/x] = e_3$), we can give a semantics:

$\boxed{e \to e'}$

$$\frac{e[v/x] = e'}{(\lambda x.\ e)\ v \to e'} \qquad \frac{e_1 \to e_1'}{e_1\ e_2 \to e_1'\ e_2} \qquad \frac{e_2 \to e_2'}{v\ e_2 \to v\ e_2'}$$

A small-step, *call-by-value (CBV)*, left-to-right semantics

- Terminates when the "whole program" is some $\lambda x.\ e$

But (also) gets stuck when there's a *free variable* "at top-level"

- Won't "cheat" like we did with $H(x)$ in IMP because scope is what we are interested in

This is the "heart" of functional languages like OCaml

- But "real" implementations do not substitute; they do something *equivalent*

Roadmap

- Motivation for a new model (done)
- CBV lambda calculus using substitution (done)
- Notes on concrete syntax
- Simple Lambda encodings (it is Turing complete!)
- Other reduction strategies
- Defining substitution

Concrete-Syntax Notes

We (and OCaml) resolve concrete-syntax ambiguities as follows:

1. $\lambda x.\ e_1\ e_2$ is $(\lambda x.\ e_1\ e_2)$, not $(\lambda x.\ e_1)\ e_2$
2. $e_1\ e_2\ e_3$ is $(e_1\ e_2)\ e_3$, not $e_1\ (e_2\ e_3)$
   - Convince yourself application is not associative

More generally:

1. Function bodies extend to an unmatched right parenthesis
   Example: $(\lambda x.\ y(\lambda z.\ z)w)q$
2. Application associates to the left
   Example: $e_1\ e_2\ e_3\ e_4$ is $(((e_1\ e_2)\ e_3)\ e_4)$

- Like in IMP, assume we really have ASTs (with non-leaves labeled λ or "application")
- Rules may seem strange at first, but it is the most convenient concrete syntax
  - Based on 70 years experience

Lambda Encodings

Fairly crazy: we left out constants, conditionals, primitives, and data structures

In fact, we are *Turing complete* and can *encode* whatever we need (just like assembly language can)

Motivation for encodings:

- Fun and mind-expanding
- Shows we are not oversimplifying the model ("numbers are syntactic sugar")
- Can show languages are *too expressive* (e.g., unlimited C++ template instantiation)

Encodings are also just "(re)definition via translation"

Encoding booleans

The "Boolean ADT"

- There are two booleans and one conditional expression.
- The conditional takes 3 arguments (e.g., via currying). If the first is one boolean it evaluates to the second. If it is the other boolean it evaluates to the third.

Any set of three expressions meeting this specification is a proper encoding of booleans

Here is one of an infinite number of encodings:

"true"	$\lambda x.\ \lambda y.\ x$
"false"	$\lambda x.\ \lambda y.\ y$
"if"	$\lambda b.\ \lambda t.\ \lambda f.\ b\ t\ f$

Example: "if" "true" $v_1\ v_2 \rightarrow^* v_1$

Evaluation Order Matters

Careful: With CBV we need to "thunk"...

$$\text{“if” “true”}\ (\lambda x.\ x)\ \underbrace{((\lambda x.\ x\ x)(\lambda x.\ x\ x))}_{\text{an infinite loop}}$$

diverges, but

$$\text{“if” “true”}\ (\lambda x.\ x)\ \underbrace{(\lambda z.\ ((\lambda x.\ x\ x)(\lambda x.\ x\ x))\ z))}_{\text{a value that when called diverges}}$$

does not

Encoding Pairs

The "pair ADT":

- There is 1 constructor (taking 2 arguments) and 2 selectors
- 1st selector returns the 1st arg passed to the constructor
- 2nd selector returns the 2nd arg passed to the constructor

"mkpair"	$\lambda x.\ \lambda y.\ \lambda z.\ z\ x\ y$
"fst"	$\lambda p.\ p(\lambda x.\ \lambda y.\ x)$
"snd"	$\lambda p.\ p(\lambda x.\ \lambda y.\ y)$

Example:

$$\text{“snd” (“fst” (“mkpair” (“mkpair”}\ v_1\ v_2)\ v_3)) \rightarrow^* v_2$$

Reusing Lambdas

Is it weird that the encodings of Booleans and pairs both used $\lambda x.\ \lambda y.\ x$ and $\lambda x.\ \lambda y.\ y$ for different purposes?

Is it weird that the same bit-pattern in binary code can represent an int, a float, an instruction, or a pointer?

Von Neumann: Bits can represent (all) code and data

Church (?): Lambdas can represent (all) code and data

Beware the "Turing tarpit"

Encoding Lists

Rather than start from scratch, notice that booleans and pairs are enough to encode lists:

- Empty list is "mkpair" "false" "false"
- Non-empty list is $\lambda h.\ \lambda t.$ "mkpair" "true" ("mkpair" $h\ t$)
- Is-empty is ...
- Head is ...
- Tail is ...

Note:

- Not too far from how lists are implemented
- Taking "tail" ("tail" "empty") will produce some lambda
  - Just like, without page-protection hardware, `null->tail->tail` would produce some bit-pattern

Encoding Recursion

Some programs diverge, but can we write *useful* loops? Yes!

- Write a function that takes an f and calls it in place of recursion
  - Example (in enriched language):

$$\lambda f.\ \lambda x.\ \textbf{if}\ (x = 0)\ \textbf{then}\ 1\ \textbf{else}\ (x * f(x - 1))$$

- Then apply "fix" to it to get a recursive function:
  - "fix" $\lambda f.\ \lambda x.\ \textbf{if}\ (x = 0)\ \textbf{then}\ 1\ \textbf{else}\ (x * f(x - 1))$
- "fix" $\lambda f.\ e$ reduces to *something roughly equivalent to* $e[(\text{"fix"}\lambda f.\ e)/f]$, which is "unrolling the recursion once" (and further unrollings will happen as necessary)
- The details, especially for CBV, are icky; the point is it is possible and you define "fix" only once
- Not on exam:
  "fix" $\lambda g.\ (\lambda x.\ g\ (\lambda y.\ x\ x\ y))(\lambda x.\ g\ (\lambda y.\ x\ x\ y))$

Encoding Arithmetic Over Natural Numbers

How about arithmetic?

- Focus on non-negative numbers, addition, is-zero, etc.

How I would do it based on what we have so far:

- Lists of booleans for binary numbers
  - Zero can be the empty list
  - Use fix to implement adders, etc.
  - Like in hardware except fixed-width avoids recursion
- Or just use list length for a unary encoding
  - Addition is list append

But instead everybody always teaches Church numerals. Why?

- Tradition? Some sense of professional obligation?
- Better reason: You do not need fix: Basic arithmetic is often encodable in languages where all programs terminate
- In any case, we will show some basics "just for fun"

Church Numerals

"0"	$\lambda s.\ \lambda z.\ z$
"1"	$\lambda s.\ \lambda z.\ s\ z$
"2"	$\lambda s.\ \lambda z.\ s\ (s\ z)$
"3"	$\lambda s.\ \lambda z.\ s\ (s\ (s\ z))$
...	

- Numbers encoded with two-argument functions
- The "number i" composes the first argument i times, starting with the second argument
  - z stands for "zero" and s for "successor" (think unary)
- The trick is implementing arithmetic by cleverly passing the right arguments for s and z

Church Numerals

"0"	$\lambda s.\ \lambda z.\ z$
"1"	$\lambda s.\ \lambda z.\ s\ z$
"2"	$\lambda s.\ \lambda z.\ s\ (s\ z)$
"3"	$\lambda s.\ \lambda z.\ s\ (s\ (s\ z))$
"successor"	$\lambda n.\ \lambda s.\ \lambda z.\ s\ (n\ s\ z)$

successor: take "a number" and return "a number" that (when called) applies s one more time

Church Numerals

"0"	$\lambda s.\ \lambda z.\ z$
"1"	$\lambda s.\ \lambda z.\ s\ z$
"2"	$\lambda s.\ \lambda z.\ s\ (s\ z)$
"3"	$\lambda s.\ \lambda z.\ s\ (s\ (s\ z))$
"successor"	$\lambda n.\ \lambda s.\ \lambda z.\ s\ (n\ s\ z)$
"plus"	$\lambda n.\ \lambda m.\ \lambda s.\ \lambda z.\ n\ s\ (m\ s\ z)$

plus: take two "numbers" and return a "number" that uses one number as the zero argument for the other

Church Numerals

"0"	$\lambda s.\ \lambda z.\ z$
"1"	$\lambda s.\ \lambda z.\ s\ z$
"2"	$\lambda s.\ \lambda z.\ s\ (s\ z)$
"3"	$\lambda s.\ \lambda z.\ s\ (s\ (s\ z))$
"successor"	$\lambda n.\ \lambda s.\ \lambda z.\ s\ (n\ s\ z)$
"plus"	$\lambda n.\ \lambda m.\ \lambda s.\ \lambda z.\ n\ s\ (m\ s\ z)$
"times"	$\lambda n.\ \lambda m.\ m$ ("plus" n) "zero"

times: take two "numbers" m and n and pass to m a function that adds n to its argument (so this will happen m times) and "zero" (where to start the m iterations of addition)

Church Numerals

"0"	$\lambda s.\ \lambda z.\ z$
"1"	$\lambda s.\ \lambda z.\ s\ z$
"2"	$\lambda s.\ \lambda z.\ s\ (s\ z)$
"3"	$\lambda s.\ \lambda z.\ s\ (s\ (s\ z))$
"successor"	$\lambda n.\ \lambda s.\ \lambda z.\ s\ (n\ s\ z)$
"plus"	$\lambda n.\ \lambda m.\ \lambda s.\ \lambda z.\ n\ s\ (m\ s\ z)$
"times"	$\lambda n.\ \lambda m.\ m$ ("plus" n) "zero"
"isZero"	$\lambda n.\ n\ (\lambda x.$ "false") "true"

isZero: an easy one, see how the two arguments will lead to the correct answer

Church Numerals

"0"	$\lambda s.\ \lambda z.\ z$
"1"	$\lambda s.\ \lambda z.\ s\ z$
"2"	$\lambda s.\ \lambda z.\ s\ (s\ z)$
"3"	$\lambda s.\ \lambda z.\ s\ (s\ (s\ z))$
"successor"	$\lambda n.\ \lambda s.\ \lambda z.\ s\ (n\ s\ z)$
"plus"	$\lambda n.\ \lambda m.\ \lambda s.\ \lambda z.\ n\ s\ (m\ s\ z)$
"times"	$\lambda n.\ \lambda m.\ m$ ("plus" n) "zero"
"isZero"	$\lambda n.\ n\ (\lambda x.$ "false") "true"
"predecessor"	(with 0 sticky) the hard one; see Wikipedia
"minus"	similar to times with pred instead of plus
"isEqual"	subtract and test for zero

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Then start type systems

- Later take a break from types to consider first-class continuations and related topics