# Harmonic maps on amenable groups and a diffusive lower bound for random walks

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**Abstract**

We prove that on any infinite, connected, locally finite, transitive graph $G$, the probability of the random walk being within $\varepsilon\sqrt{t}$ of the origin after $t$ steps is at most $O(\varepsilon)$. A similar statement holds for finite graphs, up to the relaxation time of the walk. Our approach uses non-constant equivariant harmonic mappings taking values in a Hilbert space. For the special case of discrete, amenable groups, we present a more explicit proof of the Mok-Korevaar-Schoen theorem on existence of such harmonic maps by constructing them from the heat flow on a Følner set.

## 1 Introduction

Let $G$ be a $d$-regular, transitive graph (i.e., with transitive automorphism group), let $\{X_t\}$ denote the symmetric simple random walk on $G$ with $X_0$ arbitrary, and let $\mathsf{dist}$ be the path metric on $G$. In the case when $G$ is the Cayley graph of a finitely-generated, amenable group, Èrshler [È05] showed that $\mathbb{E}\left[\mathsf{dist}(X_0, X_t)^2\right] \geqslant Ct/d$ for all times $t \geqslant 1$, where $C \geqslant 1$ is some absolute constant.

Our first theorem concerns a more precise analysis of the random walk behavior, as well as an extension to general transitive, amenable graphs. Recall that a graph $G$ is **amenable** if there exists a sequence of finite subsets $\{S_j\}$ of the vertices such that $|S_j \triangle N(S_j)|/|S_j| \to 0$, where $N(S_j)$ denotes the neighborhood of $S_j$ in $G$.

**Theorem 1.1.** *Suppose $G$ is an infinite, connected, and amenable transitive $d$-regular graph. Then the simple random walk on $G$ satisfies*

$$\mathbb{E}\left[\mathsf{dist}(X_0, X_t)^2\right] \geqslant t/d,$$

*for every $t \geqslant 0$. Moreover, there exists a universal constant $C \geqslant 1$ such that for every $\varepsilon > 0$ and $t \geqslant \varepsilon^{-8} d$, we have*

$$\mathbb{P}\left[\mathsf{dist}(X_0, X_t) \leqslant \varepsilon\sqrt{t/d}\right] \leqslant C\varepsilon. \tag{1}$$

We also prove a version for finite graphs which holds up to the relaxation time of the random walk.

**Theorem 1.2.** *Suppose $G$ is a finite, connected, transitive $d$-regular graph and $\lambda$ denotes the second-largest eigenvalue of the transition matrix $P$ of the random walk on $G$. Then for every $t \leqslant (1-\lambda)^{-1}$,*

$$\mathbb{E}\left[\mathsf{dist}(X_0, X_t)^2\right] \geqslant \frac{t}{2d}.$$

*Moreover, there exists a universal constant $C \geqslant 1$ such that for every $\varepsilon > 0$ and any $t$ such that $(1-\lambda)^{-1} \geqslant t \geqslant \varepsilon^{-8} d$,*

$$\mathbb{P}\left[\mathsf{dist}(X_0, X_t) \leqslant \varepsilon\sqrt{t/d}\right] \leqslant C\varepsilon. \tag{2}$$

We remark that, in both cases, the dependence on $d$ is necessary (see Remark 1).

The proof of Theorem 1.1 is based on the existence of non-constant, equivariant harmonic maps on amenable groups. To this end, we first restrict ourselves to the setting of groups. Let $\Gamma$ be a group with finite generating set $S \subseteq \Gamma$, and let $G$ be the corresponding Cayley graph. Suppose that $\mathcal{H}$ is some Hilbert space on which $\Gamma$ acts by isometries, and we have a non-constant equivariant harmonic map $\Psi : \Gamma \to \mathcal{H}$, i.e. such that $g\Psi(h) = \Psi(gh)$ and $\Psi(h) = |S|^{-1}\sum_{s\in S}\Psi(hs)$ hold for every $h \in \Gamma$. Èrshler [È05] observed that this can be used to lower bound $\mathbb{E}\left[\mathsf{dist}(X_0, X_t)^2\right]$, as follows.

We may normalize $\Psi$ so that, if $e \in \Gamma$ is the identity,

$$\frac{1}{|S|}\sum_{s\in S}\|\Psi(e) - \Psi(s)\|^2 = 1. \tag{3}$$

By equivariance, this implies that $\Psi$ is $\sqrt{|S|}$-Lipschitz, hence

$$\mathbb{E}\left[\mathsf{dist}(X_0, X_t)^2\right] \geqslant \frac{1}{|S|}\mathbb{E}\,\|\Psi(X_0) - \Psi(X_t)\|^2.$$

But since $\Psi$ is harmonic, $\Psi(X_t)$ is a martingale, thus

$$\mathbb{E}\,\|\Psi(X_0) - \Psi(X_t)\|^2 = \sum_{j=0}^{t-1}\mathbb{E}\,\|\Psi(X_j) - \Psi(X_{j+1})\|^2 = t,$$

where in the final line we have used equivariance and (3).

By results of Mok [Mok95] and Korevaar-Schoen [KS97], if $\Gamma$ is amenable, then it always admits such an equivariant harmonic map. On the other hand, if $\Gamma$ is not amenable, then $G$ has spectral radius $\rho < 1$ [Kes59], hence $\mathbb{E}[\mathsf{dist}(X_0, X_t)^2] \geqslant Ct^2$, for some constant $C = C(\rho) > 0$ (see, e.g. [Woe00, Prop. 8.2]). Thus the preceding discussion shows that $\mathbb{E}[\mathsf{dist}(X_0, X_t)^2]$ grows at least linearly in $t$, for any infinite group $\Gamma$.

In Section 2, we exhibit a general method for proving escape lower bounds. For any function $\psi \in \ell^2(\Gamma)$, we have

$$\mathbb{E}\left[\mathsf{dist}(X_0, X_t)^2\right] \geqslant \frac{1}{d}\left(t - t^2\frac{\|(I-P)\psi\|^2}{2\langle\psi, (I-P)\psi\rangle}\right), \tag{4}$$

where $P$ is the transition kernel of the random walk on $G$. For finite groups, we choose $\psi$ to be the eigenfunction corresponding to the second-largest eigenvalue of $P$.

For infinite amenable groups, we show that one can reproduce the $\mathbb{E}[\mathsf{dist}(X_0, X_t)^2] \geqslant t/|S|$ bound by taking a sequence of functions $\{\psi_n\}$ to be a truncated heat flow from some sets $A_n \subseteq \Gamma$, i.e. $\psi_n = \sum_{i=0}^{n} P^i \mathbf{1}_{A_n}$, where $\{A_n\}$ forms an appropriate Følner sequence in $G$. These lower bounds, and indeed all the results in our paper, are proved for amenable, transitive graphs (and even quasi-transitive graphs), and more general forms of stochastic processes.

The existence of non-constant equivariant harmonic maps on groups without property (T) is proved in [Mok95, KS97] (see also [Kle07, App. A] for an exposition in the case of discrete groups, based on [FM05]). In Section 3, inspired by the preceding escape lower bounds, we give a more explicit construction of these harmonic maps (for the case of amenable groups), simple enough to describe here. Define $\Psi_n : \Gamma \to \ell^2(\Gamma)$ by

$$\Psi_n(x) = g \mapsto \frac{\psi_n(gx)}{\sqrt{2\langle \psi_n, (I-P)\psi_n \rangle}}.$$

We show that, after applying an appropriate affine isometry to each $\Psi_n$, there is a subsequence of $\{\Psi_n\}$ which converges pointwise to a non-constant, equivariant harmonic map. Our construction works for any infinite, transitive, amenable graph (see Theorem 3.1).

**Theorem 1.3.** *Let $G = (V, E)$ be any infinite, connected, amenable, transitive graph. Then there exists a Hilbert space $\mathcal{H}$ and an $\mathcal{H}$-valued, non-constant equivariant harmonic mapping on $G$.*

One can use such mappings to obtain more detailed information on the random walk. Virág [Vir05] showed that, in the setting of Cayley graphs of amenable groups, one has $\mathbb{E}[\mathsf{dist}(X_0, X_t)] \geqslant C\sqrt{t}/|S|^{3/2}$ for some $C > 0$. This can be proved by analyzing the process $\Psi(X_t)$ via the BDG martingale inequalities (see, e.g. [KW92, Thm. 5.6.1]).[1] In Section 4, we perform a more intricate analysis of the escape behavior of such martingales, allowing us to obtain the diffusive lower bounds (1) and (2) on the random walk. In particular, we prove the following diffusive estimate for martingales taking values in a Hilbert space.

**Theorem 1.4.** *Let $\mathcal{H}$ be a Hilbert space, and let $\{M_t\}$ be any $\mathcal{H}$-valued martingale with respect to the filtration $\{\mathcal{F}_t\}$. Suppose $\mathbb{E}\left[\|M_{t+1} - M_t\|_{\mathcal{H}}^2 \,|\, \mathcal{F}_t\right] = 1$ for all $t \geqslant 0$, and that there exists a common random variable $A$ which stochastically dominates $\|M_{t+1} - M_t\|_{\mathcal{H}}$ conditioned on $\mathcal{F}_t$, for every $t \geqslant 0$. Then then for every $\varepsilon \geqslant 0$, and $t \geqslant \varepsilon^{-8}(\mathbb{E}A^3)^2$,*

$$\mathbb{P}\left(\|M_t\|_{\mathcal{H}} \leqslant \varepsilon\sqrt{t}\right) \leqslant O(\varepsilon).$$

The proof of this estimate is made more delicate by the fact that there is no central limit theorem for $\mathcal{H}$-valued martingales, even when $\|M_{t+1} - M_t\|_{\mathcal{H}} = 1$ for all $t$ (see Remark 4).

## 1.1 Related work

We recall some previous results on the rate of escape of random walks on groups. A large amount of work has been devoted to classifying situations where the rate of escape $\mathbb{E}[\mathsf{dist}(X_0, X_t)]$ is linear; we refer to the survey of Vershik [Ver00] and to the forthcoming book [Lyo09]. Èrshler has given examples where the rate can be asymptotic to $t^{1-2^{-k}}$ for any $k \in \mathbb{N}$ [Èrs01]. Following seminal work of Varopoulos [Var85], Hebisch and Saloff-Coste [HSC93] obtained precise heat kernel estimates for symmetric bounded-range random walks on groups of polynomial growth. In particular, Theorem 5.1 in [HSC93] implies our Theorem 1.1 for groups of polynomial growth. However, for groups of super-polynomial growth, it seems that existing heat-kernel bounds (see, e.g., Theorem 4.1 in [HSC93]) are not powerful enough to imply Theorem 1.1. Diaconis and Saloff-Coste show that on

[1] In fact, Virag proceeds by explicitly bounding $\mathbb{E}\left[\|M_0 - M_t\|^4\right] \leqslant O(|S|^2 t^2)$ when $\{M_t\}$ is any Hilbert space-valued martingale with $\mathbb{E}\left[\|M_{t+1} - M_t\|^2 \,|\, \mathcal{F}_t\right] \leqslant 1$ and $\mathbb{E}\left[\|M_{t+1} - M_t\|^4 \,|\, \mathcal{F}_t\right] \leqslant |S|^2$, for all $t \geqslant 0$.

finite groups satisfying a certain "moderate growth" condition, the random walk mixes in $O(D^2)$ steps, where $D$ is the diameter of the group in the word metric. A sequence of works [ANP09, NP08, NP09] have related the rate of escape of random walks to questions in geometric group theory, notably to estimates of Hilbert compression exponents of groups. Our argument for finite groups was motivated by the work of the first author with Y. Makarychev [LM09] on effective, finitary versions of Gromov's polynomial growth theorem. Another substantial work in this direction is the recent preprint of Shalom and Tao [ST09], written independently of the present paper. Constructions of nearly harmonic functions play a key role there as well.

## 2 Escape rate of random walks

In the present section, we will consider a finite or infinite symmetric, stochastic matrix $\{P(x,y)\}_{x,y\in V}$ for some index set $V$. We write $\mathsf{Aut}(P)$ for the set of all bijections $\sigma : V \to V$ whose diagonal action preserves $P$, i.e. $P(x,y) = P(\sigma x, \sigma y)$ for all $x, y \in V$. For the most part, we will be concerned with matrices $P$ for which $\mathsf{Aut}(P)$ acts transitively on $V$. A primary example is given by taking $P$ to be the transition matrix of the simple random walk on a finite or infinite vertex-transitive graph $G$.

**Theorem 2.1.** *Let $V$ be an at most countable index set, and consider any symmetric, stochastic matrix $\{P(x,y)\}_{x,y\in V}$. Suppose that $\Gamma \leqslant \mathsf{Aut}(P)$ is a closed, unimodular subgroup which acts transitively on $V$, and let $G = (V,E)$ be any graph on which $\Gamma$ acts by automorphisms. If $\mathsf{dist}$ is the path metric on $G$, and $\psi \in \ell^2(V)$, then*

$$\mathbb{E}\left[\mathsf{dist}(X_0, X_t)^2\right] \geqslant p_* \frac{\langle \psi, (I-P^t)\psi\rangle}{\langle \psi, (I-P)\psi\rangle} \geqslant p_* \left( t - t^2 \frac{\|(I-P)\psi\|^2}{2\langle \psi, (I-P)\psi\rangle}\right). \tag{5}$$

*where $\{X_t\}$ denotes the random walk with transition kernel $P$ started at any $X_0 = x_0 \in V$, and*

$$p_* = \min\{P(x,y) : \{x,y\} \in E\}.$$

*Proof.* Since $\Gamma$ is unimodular, we can choose the Haar measure $\mu$ on $\Gamma$ to be normalized so that $\mu(\Gamma_x) = 1$ for every $x \in V$, where $\Gamma_x$ is the stabilizer subgroup of $x$. Define $\Psi : V \to L^2(\Gamma, \mu)$ by $\Psi(x) = \sigma \mapsto \psi(\sigma x)$.

In this case, for every $z \in V$,

$$\begin{aligned}
\sum_{y\in V} P(z,y)\|\Psi(y)-\Psi(z)\|^2 &= \sum_{y\in V} P(z,y) \int |\psi(\sigma z) - \psi(\sigma y)|^2 \, d\mu(\sigma) \\
&= \sum_{y\in V} \int P(\sigma z, \sigma y) |\psi(\sigma z) - \psi(\sigma y)|^2 \, d\mu(\sigma) \\
&= \sum_{x,y\in V} P(x,y) |\psi(x) - \psi(y)|^2 \\
&= 2\langle \psi, (I-P)\psi\rangle.
\end{aligned} \tag{6}$$

Thus for $\{x,y\} \in E$, we have $\|\Psi(x) - \Psi(y)\|^2 \leqslant \frac{2\langle\psi,(I-P)\psi\rangle}{p_*}$, which implies that

$$\|\Psi\|_{\mathrm{Lip}} \leqslant \sqrt{\frac{2\langle \psi, (I-P)\psi\rangle}{p_*}}, \tag{7}$$

where $\Psi$ is considered as a map from $(V, \mathsf{dist})$ to $L^2(\Gamma, \mu)$, and we use $\|\Psi\|_{\mathrm{Lip}}$ to denote the infimal number $L$ such that $\Psi$ is $L$-Lipschitz.

So, for any $x_0 \in V$, we have

$$\begin{aligned}
\|\Psi\|_{\mathrm{Lip}}^2 \, \mathbb{E}\left[\mathsf{dist}(X_0, X_t)^2 \,|\, X_0 = x_0\right] &\geqslant \mathbb{E}\left[\|\Psi(X_0) - \Psi(X_t)\|^2 \,|\, X_0 = x_0\right] \\
&= \int \mathbb{E}\left[|\psi(\sigma X_0) - \psi(\sigma X_t)|^2 \,|\, X_0 = x_0\right] \, d\mu(\sigma) \\
&= \sum_{x \in V} \mathbb{E}\left[|\psi(X_0) - \psi(X_t)|^2 \,|\, X_0 = x\right] \\
&= 2\langle \psi, (I - P^t)\psi \rangle \\
&= 2\sum_{i=0}^{t-1} \langle \psi, (I - P)P^i \psi \rangle. \qquad (8)
\end{aligned}$$

To finish, we use the fact that $I - P$ is self-adjoint to compare adjacent terms via

$$|\langle \psi, (I-P)P^i\psi\rangle - \langle \psi, (I-P)P^{i+1}\psi\rangle| = |\langle (I-P)\psi, P^i(I-P)\psi\rangle| \leqslant \|(I-P)\psi\|^2.$$

where the final inequality follows because $P$ is stochastic, and hence a contraction. From this, we infer that $\langle \psi, (I-P)P^i\psi\rangle \geqslant \langle \psi, (I-P)\psi\rangle - i\|(I-P)\psi\|^2$, whence

$$\sum_{i=0}^{t-1} \langle \psi, (I-P)P^i\psi\rangle \geqslant t\langle \psi, (I-P)\psi\rangle - \frac{t^2}{2}\|(I-P)\psi\|^2.$$

Combining the preceding line with (7) and (8) yields

$$\frac{1}{p_*}\mathbb{E}\left[\mathsf{dist}(X_0, X_t)^2\right] \geqslant \frac{\langle \psi, (I-P^t)\psi\rangle}{\langle \psi, (I-P)\psi\rangle} \geqslant t - t^2 \frac{\|(I-P)\psi\|^2}{2\langle \psi, (I-P)\psi\rangle}.$$

□

We now demonstrate circumstances in which an appropriate $\psi \in \ell^2(V)$ exists. Corollaries 2.2, 2.7, and Conjecture 2.3 all assume the notation of Theorem 2.1.

**Corollary 2.2** (The finite case)**.** *Let $V$ be a finite index set and suppose that $\mathsf{Aut}(P)$ acts transitively on $V$. If $\lambda < 1$ is the second-largest eigenvalue of $P$, then*

$$\mathbb{E}\left[\mathsf{dist}(X_0, X_t)^2\right] \geqslant p_*(1 + \lambda + \lambda^2 + \cdots + \lambda^{t-1}), \qquad (9)$$

*In particular,*

$$\mathbb{E}\left[\mathsf{dist}(X_0, X_t)^2\right] \geqslant p_* t/2.$$

*for $t \leqslant (1-\lambda)^{-1}$.*

*Proof.* Let $\psi : V \to \mathbb{R}$ satisfy $P\psi = \lambda\psi$. By Theorem 2.1,

$$\mathbb{E}\left[\mathsf{dist}(X_0, X_t)^2\right] \geqslant p_* \frac{\langle \psi, (I-P^t)\psi\rangle}{\langle \psi, (I-P)\psi\rangle} = p_* \frac{1-\lambda^t}{1-\lambda} = p_*(1 + \lambda + \lambda^2 + \cdots + \lambda^{t-1}).$$

To complete the proof, use the inequality $\lambda^j \geqslant (1-t^{-1})^j \geqslant 1 - j/t$. □

**Remark 1** (Weighted graphs)**.** In particular, if $P$ is irreducible and $p_* = \min\{P(x,y) : P(x,y) > 0\}$ then, the conclusion is that $\mathbb{E}\left[\mathsf{dist}(X_0, X_t)^2\right] \geqslant p_* t/2$ for $t \leqslant (1-\lambda)^{-1}$. Thus if $P$ is the simple random walk on a $d$-regular graph, the conclusion is $\mathbb{E}\left[\mathsf{dist}(X_0, X_t)^2\right] \geqslant t/(2d)$.

To see that the asymptotic dependence on $d$ is tight, one can consider a cycle of length $n$, together with $d-2$ self loops at each vertex, for $d \geqslant 2$. In this case, $\mathbb{E}\left[\mathsf{dist}(X_0, X_t)^2\right] \leqslant 2t/d$ for all $t \geqslant 0$.

**Remark 2** (After the relaxation time)**.** The quantity $(1-\lambda)^{-1}$ is called the *relaxation time* of the random walk specified by $P$, and the bound (9) degrades after this time. It is interesting to consider what happens between the relaxation time and the mixing time which is always at most $O(\log|V|)(1-\lambda)^{-1}$. One might conjecture that $\mathbb{E}\left[\mathsf{dist}(X_0, X_t)^2\right]$ continues to have a linear lower bound until the mixing time. Towards this end, we pose the following conjecture.

**Conjecture 2.3.** *There exists a constant $\varepsilon_0 > 0$ such that the following holds. For every finite, connected, $d$-regular transitive graph $G = (V, E)$ with diameter $D$,*

$$\mathbb{E}\left[\mathsf{dist}(X_0, X_t)^2\right] \geqslant \varepsilon_0 t/d$$

*for $t \leqslant \varepsilon_0 D^2$, where $\{X_t\}$ is the simple random walk on $G$.*

## 2.1 Infinite amenable graphs

We now analyze the case of infinite, amenable graphs. The following theorem will play a role in a number of arguments. The transitive version is due to Soardi and Woess [SW90], and the extension to quasi-transitive actions is from [Sal92]. See also a different proof in [BLPS99, Thm. 3.4].

We recall that for a graph $G = (V, E)$, we say that a group $\Gamma \leqslant \mathsf{Aut}(G)$ is *quasi-transitive* if $|\Gamma\backslash V| < \infty$, where $\Gamma\backslash V$ denotes the set of $\Gamma$-orbits of $V$.

**Theorem 2.4.** *Let $G$ be a graph and $\Gamma \leqslant \mathsf{Aut}(G)$ a closed, quasi-transitive subgroup. Then $G$ is amenable if and only if $\Gamma$ is amenable and unimodular.*

For the next lemma, we recall that if $P$ is transient or null-recurrent, then we have the pointwise limit,

$$P^i f \to 0 \quad \text{for every } f \in \ell^2(V). \tag{10}$$

(This is usually proved for finitely supported $f$, see e.g. [GS92, Thm. 6.4.17] or [LPW09, Thm. 21.17]. The general case follows by approximation using the contraction property of $P$).

**Lemma 2.5.** *Suppose that $P$ satisfies* (10) *and, for some $\theta \in (0, \frac{1}{2})$, there exists an $f \in \ell^2(V)$ with $\|f\| = 1$ and $\|Pf - f\| \leqslant \theta$. Then there exists a $\varphi \in \ell^2(V)$ such that*

$$\frac{\|(I-P)\varphi\|^2}{\langle \varphi, (I-P)\varphi\rangle} \leqslant 32\,\theta. \tag{11}$$

*Proof.* Given $f \in \ell^2(V)$ and $k \in \mathbb{N}$, we define $\varphi_k \in \ell^2(V)$ by

$$\varphi_k = \sum_{i=0}^{k-1} P^i f.$$

First, using $(I-P)\varphi_k = (I-P^k)f$ and the fact that $P$ is a contraction, we have

$$\|(I-P)\varphi_k\|^2 \leqslant 4\|f\|^2. \tag{12}$$

On the other hand,

$$\begin{aligned} \langle \varphi_k, (I-P)\varphi_k \rangle &= \langle \varphi_k, (I-P^k)f \rangle \\ &= \left\langle (I-P^k)\sum_{i=0}^{k-1} P^i f, f \right\rangle \\ &= \langle 2\varphi_k - \varphi_{2k}, f \rangle, \end{aligned}$$

where in the second line we have used the fact that $I-P^k$ is self-adjoint. Combining this with (12) yields

$$\frac{\|(I-P)\varphi_k\|^2}{\langle \varphi_k, (I-P)\varphi_k \rangle} \leqslant \frac{4\|f\|^2}{\langle 2\varphi_k - \varphi_{2k}, f \rangle}. \tag{13}$$

The following claim will conclude the proof.

**Claim**: There exists a $k \in \mathbb{N}$ such that

$$\langle 2\varphi_k - \varphi_{2k}, f \rangle \geqslant \frac{1}{8\theta}. \tag{14}$$

It remains to prove the claim. By assumption, $f$ satisfies $\|f\| = 1$, and $\|Pf - f\| \leqslant \theta$. Since $P$ is a contraction, we have $\|P^j f - P^{j-1} f\| \leqslant \theta$ for every $j \geqslant 1$, and thus by the triangle inequality, $\|P^j f - f\| \leqslant j\theta$ for every $j \geqslant 1$. It follows by Cauchy-Schwarz that $\langle f, (I-P^j)f \rangle \leqslant j\theta$, therefore

$$\langle f, P^j f \rangle \geqslant 1 - j\theta.$$

Thus for every $j \geqslant 1$,

$$\langle \varphi_{2^j}, f \rangle \geqslant 2^j(1 - 2^j\theta).$$

Fix $\ell \in \mathbb{N}$ so that $2^\ell\theta \leqslant \frac{1}{2} \leqslant 2^{\ell+1}\theta$, yielding

$$\langle \varphi_{2^\ell}, f \rangle \geqslant \frac{1}{8\theta}. \tag{15}$$

Now, let $a_m = \langle \varphi_{2^m}, f \rangle$, and write, for some $N \geqslant 1$,

$$a_\ell - \frac{a_N}{2^{N-\ell}} = \sum_{m=\ell}^{N-1} \frac{2a_m - a_{m+1}}{2^{m-\ell+1}}.$$

By (10), we have $\langle P^i f, f \rangle \to 0$ as $i \to \infty$, hence $\lim_{N\to\infty} \frac{a_N}{2^N} = 0$. Using (15) and taking $N \to \infty$ on both sides above yields

$$\frac{1}{8\theta} \leqslant a_\ell = \sum_{m=\ell}^{\infty} \frac{2a_m - a_{m+1}}{2^{m-\ell+1}}.$$

Since $\sum_{m=\ell}^{\infty} \frac{1}{2^{m-\ell+1}} = 1$, there must exist some $m \geqslant \ell$ with $2a_m - a_{m+1} \geqslant \frac{1}{8\theta}$. This establishes the claim (14) for $k = 2^m$ and, in view of (13), completes the proof of the lemma. □

We arrive at the following corollaries.

**Corollary 2.6.** *If $V$ is infinite, $P$ satisfies* (10), *and* $\Gamma \leqslant \mathsf{Aut}(P)$ *is a closed, amenable, unimodular subgroup, which acts transitively on $V$, then*

$$\inf_{\varphi \in \ell^2(V)} \frac{\|(I-P)\varphi\|^2}{\langle \varphi, (I-P)\varphi \rangle} = 0. \tag{16}$$

*Proof.* This follows from Lemma 2.5 using the fact that, under the stated assumptions, for every $\theta > 0$, there exists an $f \in \ell^2(V)$ with $\|f\| = 1$ and $\|Pf - f\| \leqslant \theta$.

This is a standard fact that can be proved as in [Woe00, Thm. 12.10]. In general, for every $\theta > 0$, one considers, for some $\varepsilon = \varepsilon(\theta) > 0$, the graph $G_\varepsilon$ with vertices $V$ and an edge $\{x, y\}$ whenever $P(x, y) \geqslant \varepsilon$. Since $\Gamma \leqslant \mathsf{Aut}(G_\varepsilon)$, Theorem 2.4 implies that $G_\varepsilon$ is amenable, and then one can take $f$ to be the (normalized) indicator of a suitable Følner set in $G_\varepsilon$. □

The following is an immediate consequence of Theorem 2.1 combined with the preceding result.

**Corollary 2.7** (The amenable case)**.** *Under the assumptions of Theorem 2.1, the following holds. If $V$ is a countably infinite index set, $P$ satisfies* (10), *and* $\Gamma \leqslant \mathsf{Aut}(P)$ *is a closed, amenable, unimodular subgroup which acts transitively on $V$, then*

$$\mathbb{E}\left[\mathsf{dist}(X_0, X_t)^2\right] \geqslant p_* t. \tag{17}$$

**Corollary 2.8** (Infinite amenable graphs)**.** *If $G = (V, E)$ is an infinite, transitive, connected, amenable graph with degree $d$ and $\{X_t\}$ is the simple random walk, then*

$$\mathbb{E}\left[\mathsf{dist}(X_0, X_t)^2\right] \geqslant t/d.$$

*Proof.* It is a standard fact that the transition kernel of the simple random walk on an infinite, connected, locally finite graph satisfies (10). Indeed, if $P$ is recurrent it must be null-recurrent, since it is irreducible and has an infinite stationary measure; see, e.g., [GS92, Thm. 6.4.6]. By Theorem 2.4, when $G$ is amenable, $\mathsf{Aut}(G)$ is amenable and unimodular, thus Theorem 2.7 applies with $p_* = 1/d$. □

**Corollary 2.9** (The nearly amenable case, for small times)**.** *Under the assumptions of Theorem 2.1, the following holds. If $\rho = \rho(P)$ is the spectral radius of $P$, then for all times $t \leqslant (32(1-\rho))^{-1}$,*

$$\mathbb{E}\left[\mathsf{dist}(X_0, X_t)^2\right] \geqslant \frac{p_* t}{2}.$$

*Proof.* Since $P$ is self-adjoint and positive, we have $\rho = \|P\|_{2\to 2} = \sup_{\|f\|=1} \langle Pf, f\rangle$. It follows that

$$\inf_{\|f\|=1} \|f - Pf\|^2 \leqslant \inf_{\|f\|=1} 1 + \rho^2 - 2\langle f, Pf\rangle = (1-\rho)^2.$$

Combining this with Lemma 2.5 yields the claimed result. □

Compare the preceding bound with the finite case (Corollary 2.2).

**Remark 3** (Asymptotic rate of escape)**.** The constant $p_*$ in (5) is not tight. To do slightly better, one can argue as follows. Fix $x, y \in V$ with $L = \mathsf{dist}(x,y)$, and let $x = v_0, v_1, \ldots, v_L = y$ be a shortest path from $x$ to $y$ in $G$. In this case, the triangle inequality yields

$$2\|\Psi(x)-\Psi(y)\| \leqslant \|\Psi(v_0)-\Psi(v_1)\|+\sum_{i=1}^{L-1} \left(\|\Psi(v_{i-1}) - \Psi(v_i)\| + \|\Psi(v_i) - \Psi(v_{i+1})\|\right)+\|\Psi(v_{L-1})-\Psi(v_L)\|.$$

But for every $i \in \{1, 2, \ldots, L-1\}$, there are two terms involving $v_i$, and for such $i$, we can bound

$$\|\Psi(v_i) - \Psi(v_{i-1})\|^2 + \|\Psi(v_i) - \Psi(v_{i+1})\|^2 \leqslant \frac{2\langle \psi, (I-P)\psi\rangle}{p_*}$$

as in (6). In this way, we gain a factor of 2 for such terms. Letting $\alpha$ denote the right-hand side of the preceding inequality, we have

$$\|\Psi(x) - \Psi(y)\| \leqslant \sqrt{\alpha}\left(1 + \frac{L-1}{\sqrt{2}}\right) \leqslant \sqrt{\frac{\alpha}{2}}(L+1).$$

Thus for all $x, y \in V$, we have $\|\Psi(x) - \Psi(y)\|^2 \leqslant [\mathsf{dist}(x,y)+1]^2 \frac{\langle \psi,(I-P)\psi\rangle}{p_*}$. Plugging this improvement into the proof of Theorem 2.1 yields

$$\mathbb{E}\left[(\mathsf{dist}(X_0, X_t)+1)^2\right] \geqslant 2p_* \frac{\langle \psi, (I-P^t)\psi\rangle}{\langle \psi, (I-P)\psi\rangle}, \tag{18}$$

which is asymptotically tight since, on the one hand, the simple random walk on $\mathbb{Z}$ satisfies $\mathbb{E}\left[\mathsf{dist}(X_0,X_t)^2\right] = t$, while plugging (18) into Corollary 2.7 yields $\mathbb{E}\left[(\mathsf{dist}(X_0,X_t)+1)^2\right] \geqslant t$.

The dependence on $p_*$ is easily seen to be tight for the simple random walk on $\mathbb{Z}$ with a $1-2p_*$ holding probability added to every vertex, as in Remark 1.

# 3 Equivariant harmonic maps

Let $V$ be a countably infinite index set, and let $\{P(x,y)\}_{x,y\in V}$ be a stochastic, symmetric matrix. If $\mathcal{H}$ is a Hilbert space, a mapping $\Psi : V \to \mathcal{H}$ is called *$P$-harmonic* if, for all $x \in V$,

$$\Psi(x) = \sum_{y \in V} P(x,y)\Psi(y).$$

Suppose furthermore that we have a group $\Gamma$ acting on $V$. We say that $\Psi$ is *$\Gamma$-equivariant* if there exists an affine isometric action $\pi$ of $\Gamma$ on $\mathcal{H}$, such that for every $g \in \Gamma$, $\pi(g)\Psi(x) = \Psi(gx)$ for all $x \in V$. If we wish to emphasize the particular action $\pi$, we will say that $\Psi$ is *$\Gamma$-equivariant with respect to $\pi$.*

**Theorem 3.1.** *For $P$ as above, let $\Gamma \leqslant \mathsf{Aut}(P)$ be a closed, amenable, unimodular subgroup which acts transitively on $V$. Suppose there exists a connected graph $G = (V, E)$ on which $\Gamma$ acts by automorphisms, and that for $x \in V$,*

$$\sum_{y \in V} P(x,y)\,\mathsf{dist}(x,y)^2 < \infty, \tag{19}$$

*where* dist *is the path metric on $G$. Suppose also that*

$$p_* = \min\{P(x,y) : \{x,y\} \in E\} > 0.$$

*Then there exists a Hilbert space $\mathcal{H}$, and a non-constant $\Gamma$-equivariant $P$-harmonic mapping from $V$ into $\mathcal{H}$.*

*Proof.* It is a standard result that since $G$ is connected, $P$ satisfies (10). Let $\{\psi_j\} \subseteq \ell^2(V)$ be a sequence of functions satisfying

$$\frac{\langle (I-P)\psi_j, (I-P)\psi_j\rangle}{\langle \psi_j, (I-P)\psi_j\rangle} \to 0. \tag{20}$$

The existence of such a sequence is the content of Corollary 2.6.

Define $\Psi_j : V \to L^2(\Gamma, \mu)$ by

$$\Psi_j(x) = \sigma \mapsto \frac{\psi_j(\sigma^{-1}x)}{\sqrt{2\langle \psi_j, (I-P)\psi_j\rangle}}.$$

Since $\Gamma$ is unimodular, we can choose the Haar measure $\mu$ on $\Gamma$ to be normalized $\mu(\Gamma_x) = 1$ for all $x \in V$, where $\Gamma_x$ is the stabilizer subgroup of $x$. Now, observe that for every $x \in V$,

$$\begin{aligned}
\sum_{y\in V} P(x,y)\|\Psi_j(x) - \Psi_j(y)\|^2_{L^2(\Gamma,\mu)} &= \frac{\sum_{y\in V} P(x,y) \int |\psi_j(\sigma x) - \psi_j(\sigma y)|^2 \, d\mu(\sigma)}{2\langle \psi_j, (I-P)\psi_j\rangle} \\
&= \mu(\Gamma_x) \frac{\sum_{u,y\in V} P(u,y)|\psi_j(u) - \psi_j(y)|^2}{2\langle \psi_j, (I-P)\psi_j\rangle} \\
&= 1.
\end{aligned} \tag{21}$$

Next, for every $x \in V$, we have

$$\begin{aligned}
\left\| \Psi_j(x) - \sum_{y\in V} P(x,y)\Psi_j(y) \right\|^2_{L^2(\Gamma,\mu)} &= \frac{\int \left| \psi_j(\sigma x) - \sum_{y\in V} P(x,y)\psi_j(\sigma y)\right|^2 d\mu(\sigma)}{2\langle \psi_j, (I-P)\psi_j\rangle} \\
&= \mu(\Gamma_x) \frac{\sum_{u\in V} \left| \psi_j(u) - \sum_{y\in V} P(u,y)\psi_j(y)\right|^2}{2\langle \psi_j, (I-P)\psi_j\rangle} \\
&= \frac{\langle (I-P)\psi_j, (I-P)\psi_j\rangle}{2\,\langle \psi_j, (I-P)\psi_j\rangle}.
\end{aligned}$$

In particular, from (20),

$$\lim_{j\to\infty} \left\| \Psi_j(x) - \sum_{y\in V} P(x,y)\Psi_j(y) \right\|^2_{L^2(\Gamma,\mu)} = 0, \tag{22}$$

where the limit is uniform in $x \in V$.

Define an affine isometric action $\pi_0$ of $\Gamma$ on $L^2(\Gamma,\mu)$ as follows: For $\gamma \in \Gamma, h \in L^2(\Gamma,\mu)$, $[\pi_0(\gamma)h](\sigma) = h(\gamma^{-1}\sigma)$ for all $\sigma \in \Gamma$. Notice that each $\Psi_j$ is $\Gamma$-equivariant since for $\gamma \in \Gamma, x \in V$, we have

$$(\pi_0(\gamma)[\Psi_j(x)])(\sigma) = [\Psi_j(x)](\gamma^{-1}\sigma) = \frac{\psi_j(\sigma^{-1}\gamma x)}{\sqrt{2\langle \psi_j,(I-P)\psi_j\rangle}} = [\Psi_j(\gamma x)](\sigma).$$

Now, order arbitrarily the points of $V = \{x_1, x_2, \dots\}$ and fix a sequence of subspaces $\{W_j\}_{j=1}^{\infty}$ of $L^2(\Gamma,\mu)$ with $W_j \subseteq W_{j+1}$ for each $j = 1,2,\dots$, and $\mathsf{dim}(W_j) = j$. For each such $j$, define an affine isometry $T_j : L^2(\Gamma,\mu) \to L^2(\Gamma,\mu)$ which satisfies $T_j\Psi_j(x_1) = 0$ and, for every $r = 1,2,\dots,j$, $\{T_j\Psi_j(x_k)\}_{k=1}^{r} \subseteq W_r$. Put $\widehat{\Psi}_j = T_j\Psi_j$, and define an action $\pi_j$ of $\Gamma$ on $L^2(\Gamma,\mu)$ by $\pi_j = T_j\pi_0T_j^{-1}$. It is straightforward to check that each $\widehat{\Psi}_j$ is $\Gamma$-equivariant with respect $\pi_j$.

Using (21), one observes that for all $j \in \mathbb{N}$, the map $\Psi_j$ is $\sqrt{1/p_*}$-Lipschitz on $(V,\mathsf{dist})$, and thus the same holds for $\widehat{\Psi}_j$. We now pass to a subsequence $\{\alpha_j\}$ along which $\widehat{\Psi}_{\alpha_j}(x)$ converges pointwise for every $x \in V$. To see that this is possible, notice that by construction, for every fixed $x \in V$, there is a finite-dimensional subspace $W \subseteq L^2(\Gamma,\mu)$ such that $\widehat{\Psi}_j(x) \subseteq W$ for every $j \in \mathbb{N}$. Hence by the Lipschitz property of $\widehat{\Psi}_j$, the sequence $\{\widehat{\Psi}_j(x)\}_{j=1}^{\infty}$ lies in a compact set.

From (22), we see that $\widehat{\Psi}$ is $P$-harmonic. Furthermore, since the $\widehat{\Psi}_j$'s are uniformly Lipschitz, and we have the estimate (19), we see that (21) holds for $\widehat{\Psi}$ as well, showing that $\widehat{\Psi}$ is non-constant. We are thus left to show that $\widehat{\Psi}$ is $\Gamma$-equivariant.

Toward this end, we define an action $\pi$ of $\Gamma$ on $L^2(\Gamma,\mu)$ as follows: On the image of $\widehat{\Psi}$, set $\pi(\gamma)\widehat{\Psi}(x) = \widehat{\Psi}(\gamma x)$. Extend $\pi(\gamma)$ linearly to the span of $\{\widehat{\Psi}(x)\}_{x\in V}$, and have $\pi(\gamma)$ act as the identity outside the span. To see that such a linear extension exists, observe that

$$\pi(\gamma)\widehat{\Psi}(x) = \widehat{\Psi}(\gamma x) = \lim_{j\to\infty} \pi_{\alpha_j}(\gamma)\widehat{\Psi}_{\alpha_j}(x).$$

From this expression, it also follows immediately that $\pi$ acts by affine isometries, since each $\pi_{\alpha_j}$ does. Thus $\widehat{\Psi}$ is $\Gamma$-equivariant with respect to $\pi$, completing our construction. $\square$

### 3.1 Quasi-transitive graphs

Only for the present section, we allow $P$ to be a non-symmetric kernel on the state space $V$. We recall that $\Gamma$ is said to *act quasi-transitively on a set* $V$ if $|\Gamma \backslash V| < \infty$, where $\Gamma\backslash V$ denotes the set of $\Gamma$-orbits of $V$. We prove an analog of Theorem 3.1 in the quasi-transitive setting, under the assumption that kernel $P$ is reversible.

**Corollary 3.2** (Quasi-transitive actions)**.** *Let $\Gamma \leqslant \mathsf{Aut}(P)$ be a closed, amenable, unimodular subgroup which acts* quasi-transitively *on $V$. Suppose also that $P$ is the kernel of a reversible Markov chain, and there exists a connected graph $G = (V,E)$ on which $\Gamma$ acts by automorphisms, and that for $x \in V$,*

$$\sum_{y\in V} P(x,y)\,\mathsf{dist}(x,y)^2 < \infty, \tag{23}$$

*where* $\mathsf{dist}$ *is the path metric on $G$. Suppose also that*

$$p_* = \min\{P(x,y) : \{x,y\} \in E\} > 0.$$

*Then there exists a Hilbert space $\mathcal{H}$, and a non-constant $\Gamma$-equivariant $P$-harmonic mapping from $V$ into $\mathcal{H}$.*

*Proof.* Let $x_0, x_1, \ldots, x_L \in V$ be a complete set of representatives of the orbits of $\Gamma$. Let $P_0$ be the transition kernel of the $P$-random walk restricted to states in $V_0$. Since $P$ is reversible, we see that $P_0$ is symmetric. We have $\Gamma \leqslant \mathsf{Aut}(P_0)$, with $\Gamma$ acting transitively on the orbit $V_0$.

Letting $D = \max_{i \neq j} \mathsf{dist}(x_i, x_j)$, we define a new graph $G_0 = (V_0, E_0)$ by having an edge $\{x, y\} \in E_0$ whenever

1. $\{x, y\} \in E$ and $x, y \in V_0$, or
2. there exists a path $x = v_0, v_1, \ldots, v_k = y$ in $G$ with $v_1, \ldots, v_{k-1} \notin V_0$ and $k \leqslant 2D$.

Let $\mathsf{dist}_0$ denote the path metric on $G_0$. It is clear that $\Gamma$ acts on $G_0$ by automorphisms, and also that $p_*(G_0) = \min\{P_0(x, y) : \{x, y\} \in E_0\} \geqslant (p_*)^{2D} > 0$.

Now, since every point $x \notin V_0$ has $\mathsf{dist}(x, V_0) \leqslant D$, we see that actually $\mathsf{dist}(x, y) \approx \mathsf{dist}_0(x, y)$ for all $x, y \in V_0$ (up to a multiplicative constant depending on $D$). Furthermore, this implies that for any $x \in V$ there exists $y \in V_0$ with $\sum_{i=0}^{D} P^i(x, y) \geqslant (p_*)^D$, hence (23) implies that for every $x \in V_0$,

$$\sum_{y \in V_0} P_0(x, y)\, \mathsf{dist}(x, y)^2 < \infty$$

(the number of $P$-steps taking before returning to $V_0$ is dominated by a geometric random variable), which implies the same for $\mathsf{dist}_0$.

Thus we can apply Theorem 3.1 to obtain a Hilbert space $\mathcal{H}$ and a non-constant $\Gamma$-equivariant $P_0$-harmonic map $\Psi_0 : V_0 \to \mathcal{H}$. We extend this to a mapping $\Psi : V \to \mathcal{H}$ by defining $\Psi(x) = \mathbb{E}\left[\Psi_0(W_0(x))\right]$ where $W_0(x)$ is the first element of $V_0$ encountered in the $P$-random walk started at $x$. Note that $\Psi|_{V_0} = \Psi_0$, and $\Psi$ is again $\Gamma$-equivariant. To finish the proof, it suffices to check that $\Psi$ is $P$-harmonic.

From the definition of $\Psi$, this is immediately clear for $x \notin V_0$. Since $\Psi_0$ is $P_0$-harmonic, it suffices to check that for $x \in V_0$,

$$\sum_{y \in V} P(x, y)\Psi(y) = \sum_{y \in V_0} P_0(x, y)\Psi_0(y),$$

but both sides are precisely $\mathbb{E}\left[\Psi_0(W_0(Z))\right]$, where $Z$ is the random vertex arising from one step of the $P$-walk started at $x$. □

**Corollary 3.3** (Harmonic functions on quasi-transitive graphs)**.** *If $G = (V, E)$ is an infinite, connected, amenable graph and $\Gamma \leqslant \mathsf{Aut}(G)$ is a quasi-transitive subgroup, then $G$ admits a non-constant $\Gamma$-equivariant harmonic mapping into some Hilbert space.*

Now let $G = (V, E)$ be an infinite, connected, quasi-transitive, amenable graph. The preceding construction of harmonic functions also gives escape lower bounds for the random walk on $G$. By Theorem 2.4, when $G$ is amenable, $\Gamma = \mathsf{Aut}(G)$ is amenable and unimodular. Let $R \subseteq V$ be a complete set of representatives from $\Gamma \backslash V$. Let $\mu$ be the Haar measure on $\Gamma$. For $r \in R$, let $\mu_r = \mu(\Gamma_r)$, and normalize $\mu$ so that $\sum_{r \in R} \deg(r)/\mu_r = 1$.

**Corollary 3.4** (Random walks on quasi-transitive graphs)**.** *Let* $\mathsf{dist}$ *be the path metric on* $G$*, and let* $X_0$ *have the distribution* $\mathbb{P}[X_0 = r] = \deg(r)/\mu_r$ *for* $r \in R$*. Then,*

$$\mathbb{E}\left[\mathsf{dist}(X_0, X_t)^2\right] \geqslant \frac{t}{\max\left\{\mu_r \cdot \deg(r) : r \in R\right\}}, \tag{24}$$

*where* $\{X_t\}$ *denotes the simple random walk on* $G$*.*

*Proof.* Let $\Psi : V \to \mathcal{H}$ be the harmonic map guaranteed by Corollary 3.3 normalized so that

$$\sum_{r \in R} \frac{1}{\mu_r} \sum_{x:\{x,r\} \in E} \|\Psi(x) - \Psi(r)\|^2 = 1. \tag{25}$$

We have

$$\|\Psi\|_{\mathrm{Lip}} \leqslant \max_{r \in R} \sqrt{\mu_r \cdot \deg(r)}.$$

For every $r, \hat{r} \in R$, [BLPS99, Cor. 3.5] (with $f(x,y) = 1$ for $\{x,y\} \in E$ such that $x \in \Gamma r$ and $y \in \Gamma \hat{r}$ and $f(x,y) = 0$ otherwise) implies that

$$\frac{1}{\mu_r} \# \left\{x \in \Gamma \hat{r} : \{r, x\} \in E\right\} = \frac{1}{\mu_{\hat{r}}} \# \left\{x \in \Gamma r : \{\hat{r}, x\} \in E\right\}.$$

Thus if we use the notation $[x]$ to denote the unique $r \in R$ such that $x \in \Gamma r$, then $[X_i]$ and $[X_0]$ are identically distributed for every $i \geqslant 0$. (This is also a special case of [LS99], Theorem 3.1). It follows that,

$$\begin{aligned}
\|\Psi\|_{\mathrm{Lip}}^2 \, \mathbb{E}\left[\mathsf{dist}(X_0, X_t)^2\right] &\geqslant \mathbb{E}\, \|\Psi(X_0) - \Psi(X_t)\|^2 \\
&= \sum_{i=1}^{t} \mathbb{E}\, \|\Psi(X_i) - \Psi(X_{i-1})\|^2 \\
&= \sum_{i=1}^{t} \mathbb{E}\, \|\Psi([X_i]) - \Psi([X_{i-1}])\|^2 \\
&= t \cdot \mathbb{E}\, \|\Psi([X_0]) - \Psi([X_1])\|^2 \\
&= t,
\end{aligned}$$

where in the second line we have used the fact that $\{\Psi(X_t)\}$ is a martingale, in the third line we have used equivariance of $\Psi$, in the fourth line we have used the fact that $[X_i]$ has the same distribution for all $i \geqslant 0$, and in the final line we have used (25). □

# 4 Small ball estimates

We now prove small ball estimates for the random walk on finite and infinite, amenable graphs. Using the harmonic maps of the preceding section, it suffices to study small ball estimates for Hilbert space-valued martingales. Our bounds are proved in two steps: First, we use a known reduction to the case of $\mathbb{R}^2$-valued martingales. Since there is no central limit theorem in this setting (see Remark 4), we argue by explicitly bounding the characteristic function.

## 4.1 Martingale dimension reduction

The following theorem is a special case of one due to Kallenberg and Sztencel [KS91], who prove it in the more difficult setting of continuous-time martingales. We provide a proof of the discrete case for the reader's convenience. A similar exposition appears in [KW92, Prop. 5.8.3].

**Theorem 4.1.** *Let $\mathcal{H}$ be a Hilbert space, and let $\{N_t\}$ be any $\mathcal{H}$-valued martingale. Then there exists an $\mathbb{R}^2$-valued martingale $\{M_t\}$ such that for any time $t \geqslant 0$, $\|M_t\| = \|N_t\|_{\mathcal{H}}$ and $\|M_{t+1} - M_t\| = \|N_{t+1} - N_t\|_{\mathcal{H}}$.*

*Proof.* We prove the claim by induction on $t$. The case $t = 1$ is trivial. Suppose now we can construct $\{M_t\}_{t\leqslant n}$ successfully based on $\{N_t\}_{t\leqslant n}$. We wish to specify the value of $M_{n+1}$ given $\{N_t\}_{t\leqslant n+1}$ and $\{M_t\}_{t\leqslant n}$ such that the required conditions hold.

In the generic case, there exist two distinct points in $x_1, x_2 \in \mathbb{R}^2$ satisfying

$$\|N_{n+1}\| = \|x_i\| \ , \text{ and } \langle N_n, N_{n+1}\rangle = \langle M_n, x_i\rangle, \tag{26}$$

for $i = 1, 2$. (One can see them as the intersections of a circle and a line.) Denoting these two points by $M^{(1)}_{n+1}$ and $M^{(2)}_{n+1}$, we now let $M_{n+1}$ be $M^{(1)}_{n+1}$ (resp., $M^{(2)}_{n+1}$) with probability $\frac{1}{2}$. It is clear that $\|N_{n+1}\| = \|M_{n+1}\|$. Recalling (26) and using the induction hypothesis $\|N_n\| = \|M_n\|$, we also infer that

$$\begin{aligned}\|M_{n+1} - M_n\|^2 &= \|M_{n+1}\|^2 - 2\langle M_{n+1}, M_n\rangle + \|M_n\|^2\\ &= \|N_{n+1}\|^2 - 2\langle N_{n+1}, N_n\rangle + \|N_n\|^2\\ &= \|N_{n+1} - N_n\|^2 \ .\end{aligned}$$

It remains to prove that $\mathbb{E}[M_{n+1} \mid M_1, \ldots, M_n] = M_n$. To this end, it suffices to show

$$\mathbb{E}[\langle M_{n+1}, M_n^{\perp}\rangle \mid M_1, \ldots, M_n] = 0, \tag{27}$$

$$\mathbb{E}[\langle M_{n+1}, M_n\rangle \mid M_1, \ldots, M_n] = \|M_n\|^2, \tag{28}$$

where $M_n^{\perp}$ is a unit vector with $\langle M_n^{\perp}, M_n\rangle = 0$. Equality (27) follows by our uniform random choice of $M_n$ over $\{M^{(1)}_{n+1}, M^{(2)}_{n+1}\}$. Since $\{N_t\}$ is a martingale, we have that $\mathbb{E}\{\langle N_{n+1}, N_n\rangle \mid N_1, \ldots, N_n\} = \|N_n\|^2$. Combined with (26) and our choice of $M_{n+1}$, we obtain (28) as required.

In the degenerate case when $N_n$ and $N_{n+1}$ are proportional, there is a unique solution to (26), and we just let $M_{n+1}$ be that unique point. In the case when $N_n = 0$, there are infinitely many solutions, one can pick out two symmetric ones and let $M_{n+1}$ be uniformly random over those two points. □

## 4.2 Escape lower bounds for martingales

For the entirety of this section, let $\{M_t\}$ be an $\mathbb{R}^2$-valued martingale on a probability space $(\Omega, \mu, \mathcal{F})$ with respect to the filtration $\{\mathcal{F}_t\}$, and assume that $M_0 = (0, 0)$. Also let $\xi_t = M_t - M_{t-1}$ be the corresponding difference sequence. We write $\mathbb{E}_j(\cdot) = \mathbb{E}[\cdot \mid \mathcal{F}_j]$, and suppose that $\mathbb{E}_{j-1}(\|\xi_j\|^2) = 1$ for all $j \in \mathbb{N}$.

In this section, we use the asymptotic notation $A \lesssim B$ to denote $A = O(B)$ and $A \approx B$ to denote the conjunction of $A \lesssim B$ and $A \gtrsim B$.

**Remark 4** (No CLT for $\|M_t\|$)**.** Consider the following martingale $\{M_t\}$ in $\mathbb{R}^2$. We put $M_0 = (0,0)$, and $M_1$ is a uniformly random point on the unit sphere. Writing $M_t^\perp$ for a unit vector perpendicular to $M_t$, we define, for $t \geqslant 1$, $M_{t+1} = M_t \pm M_t^\perp$ each with probability $\frac{1}{2}$. It is easy to check that $\|M_t\|^2 = t$ with probability one. Thus there is no CLT for $\|M_t\|$, or for $\langle u, M_t\rangle$ for any $u \in \mathbb{R}^2$.

In order to prove a small ball estimate on $\|M_t\|$, write

$$\begin{aligned}\mathbb{P}(\|M_t\| \leqslant \varepsilon\sqrt{t}) &\leqslant e^{1/4} \cdot \mathbb{E}\left[e^{-\|M_t\|^2/(4\varepsilon^2 t)}\right] \\ &= \frac{e^{1/4}}{\pi}\,\varepsilon^2 \int_{\mathbb{R}^2} e^{-\varepsilon^2\|s\|^2}\mathbb{E}\left[e^{i\langle s, M_t/\sqrt{t}\rangle}\right] ds, \end{aligned} \tag{29}$$

where in the second line we have taken Fourier transforms and used Parseval's formula (cf. [Kat76, §VI]).

**Lemma 4.2.** *There exists a constant $C > 0$ such that the following holds. Suppose there exists a random variable $A$ such that for each $j = 1, 2, \ldots, t$, $\|\xi_j\|$ conditioned on $\mathcal{F}_{j-1}$ is stochastically dominated by $A$. Then for any $\varepsilon \in (0,1)$,*

$$\int_{\mathbb{R}^2} e^{-\varepsilon^2\|s\|^2}\mathbb{E}\left[e^{i\langle s, M_t/\sqrt{t}\rangle}\right] ds \leqslant C\left(\frac{1}{\varepsilon^6 t} + \frac{1}{\varepsilon^5}\mathbb{E}\left[A^2 \wedge t^{-1/2}A^3\right]\right) + \frac{\pi\sqrt{2}}{\varepsilon}.$$

*Proof.* For $x \in \mathbb{R}$, let $\rho(x) = e^{ix} - 1 - ix + \frac{1}{2}x^2$. Then we have the standard estimate (see, e.g. [Wil91, §18.3]),

$$|\rho(x)| \leqslant |x|^2 \wedge |x|^3. \tag{30}$$

Let $S_j = \frac{1}{2t}\mathbb{E}_{j-1}\langle s, \xi_j\rangle^2$, and put

$$Z = \exp\left(i\langle s, M_t/\sqrt{t}\rangle + \sum_{j=1}^{t} S_j\right) - 1$$

and

$$Z_j = \exp\left(i\langle s, M_{j-1}/\sqrt{t}\rangle + \sum_{k=1}^{j} S_k\right)\left(e^{i\langle s, \xi_j/\sqrt{t}\rangle} - e^{-S_j}\right)$$

so that

$$Z = \sum_{j=1}^{t} Z_j.$$

Now, for each $j = 1, 2, \ldots, t$,

$$\begin{aligned} |\mathbb{E}_{j-1} Z_j| &= \exp\left(\sum_{k=1}^{j} S_k\right) \left|\mathbb{E}_{j-1}\left[e^{i\langle s, \xi_j/\sqrt{t}\rangle} - e^{-S_j}\right]\right| \\ &\leqslant \exp\left(\sum_{k=1}^{j} S_k\right) \mathbb{E}_{j-1}\left(\left|\rho(\langle s, \xi_j/\sqrt{t}\rangle)\right| + \left|e^{-S_j} - 1 + S_j\right|\right) \\ &\leqslant \exp\left(\sum_{k=1}^{j} S_k\right) \mathbb{E}_{j-1}\left(|\langle s, \xi_j/\sqrt{t}\rangle|^2 \wedge |\langle s, \xi_j/\sqrt{t}\rangle|^3 + \frac{1}{2}\left[\mathbb{E}_{j-1}\frac{\|s\|^2\|\xi_j\|^2}{2t}\right]^2\right) \\ &= \exp\left(\sum_{k=1}^{j} S_k\right) \left(\mathbb{E}_{j-1}\left[|\langle s, \xi_j/\sqrt{t}\rangle|^2 \wedge |\langle s, \xi_j/\sqrt{t}\rangle|^3\right] + \frac{\|s\|^4}{8t^2}\right) \end{aligned}$$

where in the second line we have used $\mathbb{E}_{j-1}\xi_j = 0$, in the third line (30), and the fact that for $x \geqslant 0$, one has the bound $e^{-x} - 1 + x \leqslant \frac{1}{2}x^2$, and in the last line $\mathbb{E}_{j-1}\|\xi_j\|^2 = 1$.

Therefore, for $j = 1, 2, \ldots, t$,

$$\begin{aligned} &\int_{\mathbb{R}^2} \exp\left(-\varepsilon^2\|s\|^2 - \sum_{k=1}^{j} S_k\right) |\mathbb{E}_{j-1} Z_j| \, ds \\ &\qquad \leqslant \int_{\mathbb{R}^2} e^{-\varepsilon^2\|s\|^2} \left(\mathbb{E}_{j-1}\left[|\langle s, \xi_j/\sqrt{t}\rangle|^2 \wedge |\langle s, \xi_j/\sqrt{t}\rangle|^3\right] + \frac{\|s\|^4}{8t^2}\right) ds \\ &\qquad \lesssim \frac{1}{\varepsilon^6 t^2} + \frac{1}{t}\int_{\mathbb{R}^2} e^{-\varepsilon^2\|s\|^2} \left(\|s\|^2 + \|s\|^3\right) \mathbb{E}_{j-1}\left[\|\xi_j\|^2 \wedge t^{-1/2}\|\xi_j\|^3\right] ds \\ &\qquad \lesssim \frac{1}{\varepsilon^6 t^2} + \frac{1}{\varepsilon^5 t}\left[A^2 \wedge t^{-1/2} A^3\right]. \end{aligned}$$

So we have,

$$\begin{aligned} \mathbb{E}\int_{\mathbb{R}^2} \exp\left(-\varepsilon^2\|s\|^2 - \sum_{k=1}^{t} S_k\right) Z \, ds &= \sum_{j=1}^{t} \mathbb{E}\int_{\mathbb{R}^2} \exp\left(-\varepsilon^2\|s\|^2 - \sum_{k=1}^{t} S_k\right) Z_j \, ds \\ &= \sum_{j=1}^{t} \mathbb{E}\mathbb{E}_{j-1}\int_{\mathbb{R}^2} \exp\left(-\varepsilon^2\|s\|^2 - \sum_{k=1}^{t} S_k\right) Z_j \, ds \\ &\leqslant \sum_{j=1}^{t} \mathbb{E}\int_{\mathbb{R}^2} \exp\left(-\varepsilon^2\|s\|^2 - \sum_{k=1}^{j} S_k\right) \mathbb{E}_{j-1} Z_j \, ds \\ &\leqslant \sum_{j=1}^{t} \mathbb{E}\int_{\mathbb{R}^2} \exp\left(-\varepsilon^2\|s\|^2 - \sum_{k=1}^{j} S_k\right) |\mathbb{E}_{j-1} Z_j| \, ds \\ &\lesssim \frac{1}{\varepsilon^6 t} + \frac{1}{\varepsilon^5}\mathbb{E}\left[A^2 \wedge t^{-1/2} A^3\right]. \end{aligned}$$

In particular, there exists a constant $C \geqslant 0$ such that

$$\int_{\mathbb{R}^2} e^{-\varepsilon^2\|s\|^2} \mathbb{E}\left[e^{i\langle s, M_t/\sqrt{t}\rangle}\right] ds \leqslant C\left(\frac{1}{\varepsilon^6 t} + \frac{1}{\varepsilon^5}\mathbb{E}\left[A^2 \wedge t^{-1/2} A^3\right]\right) + \mathbb{E}\int_{\mathbb{R}^2} e^{-\varepsilon^2\|s\|^2} \exp\left(-\sum_{k=1}^{t} S_k\right) ds.$$

The next lemma completes the proof.

**Lemma 4.3.** *For every* $\varepsilon \in (0,1)$,

$$\int_{\mathbb{R}^2} e^{-\varepsilon^2\|s\|^2} \exp\left(-\sum_{k=1}^t S_k\right) ds \leqslant \frac{\pi\sqrt{2}}{\varepsilon(1+2\varepsilon^2)^{1/2}} \leqslant \frac{\pi\sqrt{2}}{\varepsilon}.$$

*Proof.* Let $\mathfrak{M}_k = \frac{1}{2t}\mathbb{E}_{k-1}\left(\xi_k\xi_k^\top\right)$, and observe that $S_k = \frac{1}{2t}\mathbb{E}_{k-1}\langle s, \xi_k\rangle^2 = s^\top \mathfrak{M}_k s$. By assumption, $\mathrm{tr}(\mathfrak{M}_k) = \frac{1}{2t}\mathbb{E}_{k-1}\|\xi_k\|^2 = \frac{1}{2t}$. Let $\mathfrak{M} = \sum_{k=1}^t \mathfrak{M}_k$ so that $\mathrm{tr}(\mathfrak{M}) = 1/2$, and

$$\int_{\mathbb{R}^2} e^{-\varepsilon^2\|s\|^2} \exp\left(-\sum_{k=1}^t S_k\right) ds = \int_{\mathbb{R}^2} e^{-\varepsilon^2\|s\|^2} e^{-s^\top \mathfrak{M} s}\, ds.$$

Since $\mathfrak{M}$ is symmetric, real, and positive semi-definite, there exists a unitary matrix $U$ such that $U^\top \mathfrak{M} U$ is diagonal with entries $a, b \geqslant 0$ and $a + b = 1/2$. Hence,

$$\begin{aligned}
\int_{\mathbb{R}^2} e^{-\varepsilon^2\|s\|^2} e^{-s^\top \mathfrak{M} s}\, ds &= \int_{\mathbb{R}^2} e^{-\varepsilon^2\|s\|^2} e^{-s^\top U^\top \mathfrak{M} U s}\, ds \\
&= \int_0^\infty r e^{-\varepsilon^2 r^2} \int_{-\pi}^{\pi} e^{-r^2(a\cos^2\theta + b\sin^2\theta)}\, d\theta\, dr \\
&\leqslant \int_0^\infty r e^{-\varepsilon^2 r^2} \left(\int_{-\pi}^{\pi} e^{-\frac{1}{2}r^2\cos^2\theta}\, d\theta\right)^{2a} \left(\int_{-\pi}^{\pi} e^{-\frac{1}{2}r^2\sin^2\theta}\, d\theta\right)^{2b} dr \\
&= \int_0^\infty r e^{-\varepsilon^2 r^2} \int_{-\pi}^{\pi} e^{-\frac{1}{2}r^2\cos^2\theta}\, d\theta\, dr \\
&= \int_{-\pi}^{\pi} \int_0^\infty r e^{-(\varepsilon^2 + \frac{1}{2}\cos^2\theta) r^2}\, dr\, d\theta \\
&= \int_{-\pi}^{\pi} \frac{1}{2\varepsilon^2 + \cos^2\theta}\, d\theta \\
&= \frac{\pi\sqrt{2}}{\varepsilon(1+2\varepsilon^2)^{1/2}}.
\end{aligned}$$

where in the third line we have used Hölder's inequality with exponents $\frac{1}{2a}$ and $\frac{1}{2b}$, and in the final line we have used the fact that $\frac{1}{p\sqrt{p^2+1}} \tan^{-1}\left(\frac{p\tan\theta}{\sqrt{p^2+1}}\right)$ is the antiderivative of $\frac{1}{p^2+\cos^2\theta}$ for all $p \neq 0$. ☐

☐

Combining Theorem 4.1 with (29) and Lemma 4.2 yields the following.

**Theorem 4.4.** *Let* $\mathcal{H}$ *be a Hilbert space, and let* $\{M_t\}$ *be any* $\mathcal{H}$*-valued martingale with respect to the filtration* $\{\mathcal{F}_t\}$*. If* $\mathbb{E}\left[\|M_{t+1} - M_t\|_{\mathcal{H}}^2 \,|\, \mathcal{F}_t\right] = 1$ *for all* $t \geqslant 0$*, and if there exists a common random variable* $A$ *which stochastically dominates* $\|M_{t+1} - M_t\|_{\mathcal{H}}$ *conditioned on* $\mathcal{F}_t$*, for every* $\varepsilon > 0, t \geqslant 0$*, then*

$$\mathbb{P}\left(\|M_t\|_{\mathcal{H}} \leqslant \varepsilon\sqrt{t}\right) \lesssim \left(\frac{1}{\varepsilon^5 t} + \frac{1}{\varepsilon^4}\mathbb{E}\left[A^2 \wedge t^{-1/2}A^3\right] + 1\right)\varepsilon.$$

**Corollary 4.5.** *For every* $\varepsilon > 0$ *and* $t \geqslant \varepsilon^{-8}(\mathbb{E}A^3)^2$,

$$\mathbb{P}\left(\|M_t\|_{\mathcal{H}} \leqslant \varepsilon\sqrt{t}\right) \leqslant O(\varepsilon).$$

### 4.3 Diffusive lower bounds for random walks

Finally, we put together the tools of Section 3, Theorem 4.1, and Lemma 4.2 to prove our diffusive lower bounds.

**Theorem 4.6.** *Let $V$ be a countably infinite index set, and let $\{P(x,y)\}_{x,y\in V}$ be a stochastic, symmetric matrix. Let $\Gamma \leqslant \mathsf{Aut}(P)$ be a closed, amenable, unimodular subgroup which acts transitively on $V$. Suppose there exists a connected graph $G=(V,E)$ on which $\Gamma$ acts by automorphisms, and that for $x \in V$,*

$$\sum_{y\in V} P(x,y)\,\mathsf{dist}(x,y)^2 \langle \infty, \tag{31}$$

*where $\mathsf{dist}$ is the path metric on $G$. Suppose also that*

$$p_* = \min\{P(x,y) : \{x,y\} \in E\} > 0.$$

*Then for every $t \geqslant \varepsilon^{-8}(p_*)^{-3}K^2$,*

$$\mathbb{P}\left[\mathsf{dist}(X_0,X_t) \leqslant \varepsilon\sqrt{p_* t}\right] \leqslant O(\varepsilon),$$

*where $\{X_t\}$ denotes the random walk with transition kernel $P$, and*

$$K = \sum_{y\in V} P(x,y)\,\mathsf{dist}(x,y)^3.$$

*If $P(x,y)$ is supported only on pairs with $\mathsf{dist}(x,y)=1$, then the preceding conclusion holds for $t \geqslant \varepsilon^{-8}/p_*$.*

*Proof.* Let $\mathcal{H}$ and $\Psi : V \to \mathcal{H}$ be the Hilbert space and non-constant $\Gamma$-equivariant harmonic mapping guaranteed by Theorem 3.1. Let $\|\cdot\| = \|\cdot\|_{\mathcal{H}}$, and normalize $\Psi$ so that for every $x \in V$,

$$\sum_{y\in V} P(x,y)\|\Psi(x)-\Psi(y)\|_{\mathcal{H}}^2 = 1. \tag{32}$$

Then $M_t = \Psi(X_t)$ is an $\mathcal{H}$-valued martingale with $\mathbb{E}\left[\|M_{t+1}-M_t\|^2 \mid \mathcal{F}_t\right] = 1$ for every $t \geqslant 0$.

Furthermore, from (32), we see that $\Psi$ is $\sqrt{1/p_*}$-Lipschitz as a mapping from $(V,\mathsf{dist})$ to $\mathcal{H}$. Therefore, conditioned on $\mathcal{F}_t$, we have $\|M_{t+1}-M_t\|$ stochastically dominated by the random variable $A$ which equals $\sqrt{1/p_*}\,\mathsf{dist}(x,y)$ with probability $P(x,y)$, as $y$ varies over $V$. Using Corollary 4.5, we have

$$\mathbb{P}\left[\mathsf{dist}(X_0,X_t) \leqslant \varepsilon\sqrt{p_* t}\right] \leqslant \mathbb{P}\left[\|M_0 - M_t\| \leqslant \varepsilon\sqrt{t}\right] \leqslant O(\varepsilon),$$

for $t \geqslant \varepsilon^{-8}(p_*)^{-3}K^2$.

The final assertion follows because, under the assumption that $P(x,y) > 0 \implies \mathsf{dist}(x,y)=1$, we can take $A$ to satisfy $\mathbb{E}A^2 = 1$ and $\|A\|_\infty \leqslant \sqrt{1/p_*}$, yielding $(\mathbb{E}A^3)^2 \leqslant 1/p_*$. □

Finally, we have a version for finite graphs.

**Theorem 4.7.** *Let $V$ be a finite index set and suppose that $\mathsf{Aut}(P)$ acts transitively on $V$, and on the graph $G = (V, E)$ by automorphisms. If*

$$p_* = \min\{P(x,y) : \{x,y\} \in E\} > 0,$$

*and $\lambda < 1$ is the second-largest eigenvalue of $P$, then for every $(1-\lambda)^{-1} \geqslant t \geqslant \varepsilon^{-8}(p_*)^{-3}K^2$,*

$$\mathbb{P}\left[\mathsf{dist}(X_0, X_t) \leqslant \varepsilon\sqrt{p_* t}\right] \leqslant O(\varepsilon),$$

*where $\{X_t\}$ denotes the random walk with transition kernel $P$, $\mathsf{dist}$ denotes the path metric on $G$, and*

$$K = \sum_{y \in V} P(x,y)\,\mathsf{dist}(x,y)^3.$$

*If $P(x,y)$ is supported only on pairs with $\mathsf{dist}(x,y) = 1$, then the preceding conclusion holds for $(1-\lambda)^{-1} \geqslant t \geqslant \varepsilon^{-8}/p_*$.*

*Proof.* Let $\psi : V \to \mathbb{R}$ be such that $P\psi = \lambda\psi$, and define $\Psi : V \to \ell^2(\mathsf{Aut}(P))$ by

$$\Psi(x) = \frac{(\psi(\sigma x))_{\sigma \in \mathsf{Aut}(P)}}{\sqrt{2\langle \psi, (I-P)\psi \rangle}}.$$

An argument as in (7) shows that $\|\Psi\|_{\mathrm{Lip}} \leqslant \sqrt{1/p_*}$.

Now, observe that $\{\lambda^{-t}\Psi(X_t)\}$ is a martingale. This follows from the fact that $\lambda^{-t}\psi(X_t)$ is a martingale, which one easily checks:

$$\mathbb{E}\left[\lambda^{-t-1}\psi(X_{t+1}) \,|\, X_t\right] = \lambda^{-t-1}(P\psi)(X_t) = \lambda^{-t}\psi(X_t).$$

Thus for $t \leqslant (1-\lambda)^{-1}$, the mapping $x \mapsto \lambda^{-t}\Psi(x)$ is $O(\sqrt{1/p_*})$-Lipschitz. Hence the same argument as in Theorem 4.6 applies. □

The preceding theorems immediately yield Theorems 1.1 and 1.2 as corollaries.

Given Remark 4, it is impossible for our approach to prove, e.g. that there exists a constant $\delta > 0$ such that $\mathbb{P}(\mathsf{dist}(X_0, X_t) \geqslant 100\sqrt{t}) \geqslant \delta$ (because such a theorem is false in the martingale setting). This suggests the following intriguing question.

**Question 4.8.** *Is is true that there exists a constant $\delta > 0$ such that for every infinite, d-regular, transitive graph $G$, and every $B \geqslant 0$,*

$$\mathbb{P}\left[\sqrt{\frac{d}{t}}\,\mathsf{dist}(X_0, X_t) \geqslant B\right] \geqslant \mathbb{P}(\delta|g| \geqslant B),$$

*where $g$ is an $N(0,1)$ random variable and $\{X_t\}$ is the random walk on $G$?*


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